Analytical Mechanics

Fabian Hassler

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Prof. Dr. Fabian Hassler

Institute for Quantum Information

RWTH Aachen

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Please report any typographical or other errors to F. Hassler mitteilen.

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Chapter 1

Newtonian Mechanics

We will introduce the fundamentals of Newtonian mechanics in this chapter. Most of this material you have already encountered in Experimental Physics 1. We will briefly summarize the basics again in Section 1.1. We will then discuss the Galilean principle of relativity, conserved quantities, and the concept of fictitious forces in detail.

1.1 Fundamentals

Space and Time: After fixing the units of length (measuring rods) and time (clocks), as well as a reference frame, we characterize *events* by $(t, \boldsymbol{x}) \in \mathbb{R}^{1+3}$, where t denotes the time coordinate and $\boldsymbol{x} = (x_1, x_2, x_3)$ the Cartesian coordinates of a three-dimensional Euclidean space. A meaning absolute and independent of the reference frame have the quantities

- $|t_1 - t_2|$: time interval between two arbitrary events $(t_1, \boldsymbol{x}_1)$, $(t_2, \boldsymbol{x}_2)$

$$(\Longrightarrow \text{simultaneity is absolute}); \tag{1.1}$$

- if $t_1 = t_2$:

$$|\boldsymbol{x}_1 - \boldsymbol{x}_2|\text{: spatial distance between two simultaneous events.} \tag{1.2}$$

The general coordinate transformations that leave these quantities invariant are

$$\begin{aligned} t' &= \lambda t + a\,, & &(\lambda = \pm 1; a \in \mathbb{R})\,, \\ \boldsymbol{x}' &= R(t)\boldsymbol{x} + \boldsymbol{b}(t)\,, & &(R(t) \in \mathrm{O}(3); \boldsymbol{b}(t) \in \mathbb{R}^3)\,. \end{aligned} \tag{1.3}$$

Here $R(t) \in \mathrm{O}(3)$ is an orthogonal 3×3 matrix depending on t. These transformations allow us in particular to: (i) reverse the direction of time ($\lambda = -1$), (ii) shift the origin of time ($a \neq 0$) and (iii) rotate and translate the spatial reference frame (in a time-dependent manner) $(R(t), \boldsymbol{b}(t))$.

It is easy to verify that the transformations (1.3) form a group. This is essentially a consequence of the fact that the transformations are characterized by the invariants (1.1) and (1.2).

Inertial Systems: From the standpoint of mechanics, however, not all of these coordinate systems are equivalent. The special class of *inertial frames* is distinguished by the *law of inertia*. In an inertial frame, a *free particle*[1] with coordinates $\boldsymbol{x}$ satisfies the equations of motion

$$\ddot{\boldsymbol{x}} = 0\,, \qquad \left(\dot{f}(t) \equiv \frac{d}{dt} f(t)\right); \tag{1.4}$$

i.e., it moves uniformly along the trajectory

$$\boldsymbol{x}(t) = \boldsymbol{x}(0) + \dot{\boldsymbol{x}}(0)\, t\,. \tag{1.5}$$

In an inertial frame, therefore, the trajectory of a free particle $t \mapsto (t, \boldsymbol{x}(t))$ is a straight line in $\mathbb{R}^4$. The subgroup of coordinate transformations (1.3) that map inertial frames to inertial frames are then the transformations preserving straight lines,

$$\begin{aligned} t' &= \lambda t + a\,, & &(\lambda = \pm 1; a \in \mathbb{R})\,, \\ \boldsymbol{x}' &= R\boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{b}\,, & &(R \in \mathrm{O}(3); \boldsymbol{v},\, \boldsymbol{b} \in \mathbb{R}^3)\,. \end{aligned} \tag{1.6}$$

These transformations in turn form a group, the *Galilei group* or group of *Galilean transformations*.

We postulate the existence of an inertial frame (and hence infinitely many inertial frames moving uniformly relative to one another). In practice, different reference frames are used as inertial frames. For laboratory experiments on small time and length scales, one may approximate the Earth as an inertial frame. On the other hand, when considering planetary orbits in our solar system, its center of mass with axes oriented toward fixed stars (with high precision) is an inertial frame. On even larger time and length scales, one would then have to turn, for example, to the center of mass of our Milky Way.

Unless stated otherwise, $(t, \boldsymbol{x})$ will henceforth always denote the coordinates of an event in an (arbitrary) inertial frame.

Center-of-mass theorem (concept of mass): Every point particle possesses an invariant mass $m > 0$, which (after fixing a unit of mass) is characterized by the *center-of-mass theorem*. For a closed (isolated) system of N particles with coordinates $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$, the center-of-mass theorem holds:

$$\sum_{i=1}^{N} m_i \ddot{\boldsymbol{x}}_i = \frac{d}{dt} \sum_{i=1}^{N} \boldsymbol{p}_i = 0\,. \tag{1.7}$$

[1] A free particle is a particle that is not subject to any (external) force.

The definition of mass m_i is independent of the choice of inertial frame, since $\ddot{\boldsymbol{x}}_i' = R\ddot{\boldsymbol{x}}_i$. The quantity $\boldsymbol{p}_i = m_i\dot{\boldsymbol{x}}_i$ is called the *momentum* of the i-th particle. The above equation is then the conservation law of total momentum $\boldsymbol{P} = \sum_{i=1}^N \boldsymbol{p}_i$. Defining the center of mass of the N-particle system by

$$\boldsymbol{X} = \frac{1}{M}\sum_{i=1}^N m_i\boldsymbol{x}_i\,, \qquad M = \sum_{i=1}^N m_i\,, \tag{1.8}$$

the center-of-mass theorem can also be written as $\ddot{\boldsymbol{X}} = 0$. The center of mass of a closed system therefore moves along an inertial trajectory

$$\boldsymbol{X}(t) = \boldsymbol{X}(0) + \frac{\boldsymbol{P}}{M}t\,. \tag{1.9}$$

In scattering processes, particles can be created or annihilated. However, the total momentum $\boldsymbol{P}$ must be conserved in every inertial frame. Let us now consider two inertial frames related by $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{v}t$: the momentum of the i-th particle in the primed inertial frame is then $\boldsymbol{p}_i' = \boldsymbol{p}_i + m_i\boldsymbol{v}$. Since both $\boldsymbol{P}$ and $\boldsymbol{P}'$ are constant, it follows from the relation $\boldsymbol{P}' = \boldsymbol{P} + M\boldsymbol{v}$ that the total mass M must also be constant. In particular, the mass is additive if several particles combine into a single one.

1.2 The Newtonian Equations of Motion

We consider a system of N particles whose position coordinates are described by $\boldsymbol{x}_i(t)$. Newton's principle of determinacy requires that the trajectories $\boldsymbol{x}_i(t)$ be uniquely determined for all t if the positions $\boldsymbol{x}_i(t_0)$ and the velocities $\dot{\boldsymbol{x}}_i(t_0)$ are given at some time t_0. In particular, the accelerations $\ddot{\boldsymbol{x}}_i(t_0)$ are then determined, i.e. there exist functions $\boldsymbol{F}_i$ such that

$$m_i\ddot{\boldsymbol{x}}_i(t) = \underbrace{\boldsymbol{F}_i\big(\boldsymbol{x}_1(t),\ldots,\boldsymbol{x}_N(t),\dot{\boldsymbol{x}}_1(t),\ldots,\dot{\boldsymbol{x}}_N(t),t\big)}_{\text{force law of the system}}\,. \tag{1.10}$$

These are the so-called *Newtonian equations of motion*. Conversely, it follows from the existence and uniqueness theorem for ordinary differential equations that (under weak assumptions on $\boldsymbol{F}_i$) the trajectories $\boldsymbol{x}_i(t)$ are uniquely determined by (1.10) and the above initial data to $t = t_0$ at least for a small time interval around t_0. We call a system of N particles that obeys a force law of the form (1.10) a *mechanical system*.

In physics we usually investigate force laws of the form $\boldsymbol{F}_i = \boldsymbol{F}_i^{(\mathrm{in})} + \boldsymbol{F}_i^{(\mathrm{ex})}$ with an *external force* $\boldsymbol{F}_i^{(\mathrm{ex})} = \boldsymbol{F}^{(\mathrm{ex})}(\boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t)$ independent of $\boldsymbol{x}_j, \dot{\boldsymbol{x}}_j$ with $j \neq i$ and an *internal force* $\boldsymbol{F}_i^{(\mathrm{in})}$. The internal force $\boldsymbol{F}_i^{(\mathrm{in})}$ acting on the i-th particle depends only on the positions of the particles,

$$\boldsymbol{F}_i^{(\mathrm{in})} = \boldsymbol{F}_i^{(\mathrm{in})}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)\,. \tag{1.11}$$

Since it does not depend on the velocities, it can be measured statically. In most cases $\boldsymbol{F}_i^{(\mathrm{in})}$ is simply a superposition of two-body forces

$$\boldsymbol{F}_i^{(\mathrm{in})} = \sum_{\substack{k=1 \\ k\neq i}}^{N} \boldsymbol{F}_{ik}\,, \tag{1.12}$$

where $\boldsymbol{F}_{ik} = \boldsymbol{F}_{ik}(\boldsymbol{x}_i, \boldsymbol{x}_k)$ is the force exerted by particle k on particle i. For a (closed) two-particle system one then has, by (1.7), simply

$$\boldsymbol{F}_{ik} + \boldsymbol{F}_{ki} = 0 \tag{1.13}$$

(action = reaction).

Examples of closed mechanical systems (with $\boldsymbol{F}^{(\mathrm{ex})} = 0$) are the solar system, whose force law is given by

$$m_i \ddot{\boldsymbol{x}}_i = -G \sum_{\substack{k=1 \\ k\neq i}}^{N} m_i m_k \frac{\boldsymbol{x}_i - \boldsymbol{x}_k}{|\boldsymbol{x}_i - \boldsymbol{x}_k|^3} \tag{1.14}$$

or a system of charged particles (with charge e_i), for which the force law

$$m_i \ddot{\boldsymbol{x}}_i = \sum_{\substack{k=1 \\ k\neq i}}^{N} e_i e_k \frac{\boldsymbol{x}_i - \boldsymbol{x}_k}{|\boldsymbol{x}_i - \boldsymbol{x}_k|^3} \tag{1.15}$$

holds.

Examples of non-closed mechanical systems are a charged particle in an external electromagnetic field (given by $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$)

$$m\ddot{\boldsymbol{x}} = e\boldsymbol{E}(\boldsymbol{x}, t) + \frac{e}{c}\dot{\boldsymbol{x}} \times \boldsymbol{B}(\boldsymbol{x}, t)\,, \tag{1.16}$$

or a one-dimensional oscillator with frictional force and driving force $k(t)$

$$m\ddot{x} = -fx - r\dot{x} + k(t)\,. \tag{1.17}$$

In both cases the particle is subjected only to an external force.

The description of these systems as mechanical systems relies on approximations. For example, in (1.16) $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$ are prescribed, external, electromagnetic fields, and the influence of the charged particle on them has been ignored. In (1.17) $-r\dot{x}$ is a summary description of the friction, without consideration of the dynamics of the damping medium.

Since friction is an effective description of a complicated interaction of a particle with a medium, we will hardly consider it at all in the present lecture. You will find that friction overall plays a subordinate role in the standard lectures of the curriculum. Thus you (probably) will first encounter it again in statistical mechanics with irreversibility and friction.

1.3 The Galilean Relativity Principle

The Galilean (or classical) relativity principle states that *the laws of nature are the same in all inertial systems*, i.e. they are invariant under Galilean transformations.[2] The principle therefore renders all inertial systems equivalent. For mechanics, the relativity principle requires that the equations of motion of a closed system be invariant under Galilean transformations (1.6). We now wish to show that then $\boldsymbol{F}_{ik}$ must be of the form

$$\boldsymbol{F}_{ik} = f_{ik}(|\boldsymbol{x}_i - \boldsymbol{x}_k|)\frac{\boldsymbol{x}_i - \boldsymbol{x}_k}{|\boldsymbol{x}_i - \boldsymbol{x}_k|} \tag{1.18}$$

where $f_{ik}(r)$ is an arbitrary scalar function of one variable.[3] This means that the force acts along the line connecting the two particles and depends in magnitude only on the distance between the two particles.

To prove this, consider a mechanical two-*species* system whose force law in an inertial frame is of the form

$$m_1 \frac{d^2\boldsymbol{x}_1}{dt^2} = \boldsymbol{F}_{12}(\boldsymbol{x}_1, \boldsymbol{x}_2) \tag{1.19}$$

Let $\boldsymbol{x}'$ be the coordinates in another inertial frame obtained from the unprimed inertial frame by a (time-independent) Galilean transformation with $t = t'$

$$\boldsymbol{x}'_i(t) = R\boldsymbol{x}_i(t) + \boldsymbol{b}\,. \tag{1.20}$$

The relativity principle then requires that in the primed inertial frame the same equation of motion holds; i.e.

$$m_1 \ddot{\boldsymbol{x}}'_1 = \boldsymbol{F}_{12}(\boldsymbol{x}'_1, \boldsymbol{x}'_2) \tag{1.21}$$

should be equivalent to (1.19) with $\boldsymbol{F}_{12}$ the same function. For the left-hand side we have $m_1\ddot{\boldsymbol{x}}'_1 = m_1 R\ddot{\boldsymbol{x}}_1$ from (1.20).

Invariance under a pure translation ($R = 1$, $\boldsymbol{b} \neq \boldsymbol{0}$) thus leads to the condition $\boldsymbol{F}_{12}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{F}_{12}(\boldsymbol{x}_1 + \boldsymbol{b}, \boldsymbol{x}_2 + \boldsymbol{b})$. This immediately implies that $\boldsymbol{F}_{12}$ can depend only on $\boldsymbol{x}_1 - \boldsymbol{x}_2$.[4] We thus obtain that $\boldsymbol{F}_{12}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{F}_{12}(\boldsymbol{x}_1 - \boldsymbol{x}_2)$. Invariance under rotations $R \in \mathrm{SO}(3)$ then leads to the condition that

$$R\,\boldsymbol{F}_{12}(\boldsymbol{x}) = \boldsymbol{F}_{12}(R\,\boldsymbol{x}) \tag{1.22}$$

holds. Choosing specifically a rotation R about the axis in the direction of the line joining $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ ($R\,\boldsymbol{x} = \boldsymbol{x}$), it follows with $R\,\boldsymbol{F}_{12}(\boldsymbol{x}) = \boldsymbol{F}_{12}(\boldsymbol{x})$ the assertion

$$\boldsymbol{F}_{12}(\boldsymbol{x}) = f_{12}(\boldsymbol{x})\frac{\boldsymbol{x}}{|\boldsymbol{x}|} \tag{1.23}$$

[2] This fundamental principle will later be replaced by Einstein's relativity principle of special relativity theory.

[3] The factor $1/|\boldsymbol{x}_i - \boldsymbol{x}_k|$ could of course also have been absorbed into the definition of f_{ik}. However, we want f_{ik} to have the units of a force.

[4] This is seen, e.g., by setting $\boldsymbol{b} = -\boldsymbol{x}_2$.

concerning the direction of the force. Furthermore, (1.22) requires that $f_{12}(\boldsymbol{x}) = f_{12}(R\boldsymbol{x})$ for every rotation R, hence $f_{12}(\boldsymbol{x}) = f_{12}(|\boldsymbol{x}|)$. Thus the assertion (1.18) is proved.

The forces in (1.18) always possess a *potential*

$$\boldsymbol{F}_{ik} = -\frac{\partial}{\partial \boldsymbol{x}_i} V_{ik}(|\boldsymbol{x}_i - \boldsymbol{x}_k|)\,, \quad \text{with} \quad \frac{d}{dr} V_{ik}(r) = -f_{ik}(r)\,. \tag{1.24}$$

Consequently, for the superposition of two-body forces

$$\boldsymbol{F}_i^{(\mathrm{in})} = -\frac{\partial}{\partial \boldsymbol{x}_i} V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)\,, \qquad V = \sum_{\substack{i,k=1 \\ i<k}}^{N} V_{ik}(|\boldsymbol{x}_i - \boldsymbol{x}_k|)\,. \tag{1.25}$$

So far we have considered the "passive interpretation" of the relativity principle: if $\boldsymbol{x}(t)$ and $\boldsymbol{x}'(t')$ describe the *same* trajectory in different inertial frames, then they must satisfy the equations of motion in both frames (which must have the same form). One may, however, also use an "active interpretation": let $\boldsymbol{x}(t)$ be a solution of the equation of motion in one inertial frame, then $\boldsymbol{x}(t)$ is also a solution with respect to every other inertial frame.[5] Consider as an example the case of time reversal ($\lambda = -1$): if $\boldsymbol{x}(t)$ is a solution, then so is $\boldsymbol{x}(-t)$; the time-reversed motion is therefore always also a solution of the equations of motion.

Invariant versus form-invariant: *Form invariance* of an equation requires that the equation retain the same form in different reference frames. In this case the potential $V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ must be replaced by $V'(\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_N)$ defined by

$$V'(\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_N) = V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N), \tag{1.26}$$

since the value of the potential energy of a particle configuration obviously does not depend on the choice of coordinates. Fields that transform as (1.26) are called *scalar fields*. Starting from the potential, one finds directly by the chain rule and (1.6) the relation

$$\boldsymbol{F}'_i(\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_N) = -\frac{\partial V'}{\partial \boldsymbol{x}'_i} = -R\frac{\partial V}{\partial \boldsymbol{x}_i} = R\boldsymbol{F}_i(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \tag{1.27}$$

between the force laws in the two reference frames. Fields that transform as (1.27) are called *vector fields*. If one restricts the allowed transformations to Galilean transformations (1.20), one obtains that $\ddot{\boldsymbol{x}}' = R\ddot{\boldsymbol{x}}$, as already shown. Thus Newton's equations (1.10) are form-invariant under Galilean transformations, provided $\boldsymbol{F}$ is a vector field (both sides are simply multiplied by R in the primed frame). Form invariance therefore merely expresses the

[5] The trajectory $\boldsymbol{x}(t)$ then generally describes a different path, since this function now denotes the components with respect to a different coordinate system!

mathematical fact that an equation exhibits a certain (desirable) form in particular reference frames.

Invariance of an equation under Galilean transformations, by contrast, requires that the equation be identical in two reference frames. For the Newtonian equation this implies that $\boldsymbol{F}'(\boldsymbol{x}'_1,\ldots,\boldsymbol{x}'_N) = \boldsymbol{F}(\boldsymbol{x}'_1,\ldots,\boldsymbol{x}'_N)$ or equivalently $V'(\boldsymbol{x}'_1,\ldots,\boldsymbol{x}'_N) = V(\boldsymbol{x}'_1,\ldots,\boldsymbol{x}'_N)$, i.e. V' and V are the same function. By (1.26) this entails

$$V(R\boldsymbol{x}_1+\boldsymbol{b},\ldots,R\boldsymbol{x}_N+\boldsymbol{b}) = V(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N) \tag{1.28}$$

for a time-independent Galilean transformation $\boldsymbol{x}'_i = R\boldsymbol{x}_i + \boldsymbol{b}$. The invariance of a closed system in classical mechanics under Galilean transformations is an empirical fact with physical consequences. The force law then takes the form (1.18), and as discussed in detail in the next chapter we obtain the ten classical constants of the motion.

1.4 Conservation Laws

We now wish to make some general statements about mechanical systems. Mechanical systems are characterized by the fact that they satisfy a force law of the form (1.10).

(a) Momentum Theorem: It follows directly from (1.10), that

$$\frac{d}{dt}\underbrace{\sum_{i=1}^{N}\boldsymbol{p}_i}_{\boldsymbol{P}} = \underbrace{\sum_{i=1}^{N}\boldsymbol{F}_i}_{\boldsymbol{F}\text{: resultant force}}. \tag{1.29}$$

(b) Angular Momentum Theorem: Since the time derivative of $\boldsymbol{x}_i$ is proportional to $\boldsymbol{p}_i$, it further follows that

$$\frac{d}{dt}\underbrace{\sum_{i=1}^{N}\boldsymbol{x}_i\times\boldsymbol{p}_i}_{\boldsymbol{L}} = \underbrace{\sum_{i=1}^{N}\boldsymbol{x}_i\times\boldsymbol{F}_i}_{\boldsymbol{M}\text{: resultant torque about } \boldsymbol{x}=0}. \tag{1.30}$$

The angular momentum $\boldsymbol{L}$ can be decomposed $\boldsymbol{L} = \boldsymbol{X}\times\boldsymbol{P} + \boldsymbol{L}_{\mathrm{S}}$ into the contribution from the center-of-mass motion $\boldsymbol{X}\times\boldsymbol{P}$ and a contribution from the internal motion relative to the center of mass

$$\boldsymbol{L}_{\mathrm{S}} = \sum_{i=1}^{N}(\boldsymbol{x}_i-\boldsymbol{X})\times(\boldsymbol{p}_i - m_i\dot{\boldsymbol{X}})\,. \tag{1.31}$$

In the decomposition we have used that

$$\sum_{i=1}^{N} m_i\boldsymbol{x}_i\times\dot{\boldsymbol{X}} = \boldsymbol{X}\times\boldsymbol{P} = \boldsymbol{X}\times\sum_{i=1}^{N}\boldsymbol{p}_i = \boldsymbol{X}\times\sum_{i=1}^{N} m_i\dot{\boldsymbol{X}}\,, \tag{1.32}$$

since $\sum_{i=1}^N m_i \boldsymbol{x}_i = M\boldsymbol{X}$ and $M\dot{\boldsymbol{X}} = \mathbf{P}$.[6]

(c) Energy Theorem: Multiplying (1.10) by $\dot{\boldsymbol{x}}_i$ and summing over i leads to

$$\underbrace{\frac{d}{dt}\sum_{i=1}^N \frac{1}{2} m_i \dot{\boldsymbol{x}}_i^2}_{T} = \underbrace{\sum_{i=1}^N \boldsymbol{F}_i \cdot \dot{\boldsymbol{x}}_i}_{\text{power of the forces.}}, \tag{1.33}$$

where we have introduced the *kinetic energy* T. As with the angular momentum, the kinetic energy can also be decomposed

$$T = \frac{1}{2} M\dot{\boldsymbol{X}}^2 + T_{\mathrm{S}}\,, \qquad \text{where} \qquad T_{\mathrm{S}} = \sum_{i=1}^N \frac{1}{2} m_i \big(\dot{\boldsymbol{x}}_i - \dot{\boldsymbol{X}}\big)^2 \tag{1.34}$$

describes the kinetic energy of the internal motion relative to the center of mass. In the decomposition we have used that

$$\frac{1}{2}\sum_{i=1}^N m_i \dot{\boldsymbol{x}}_i \dot{\boldsymbol{X}} = \frac{1}{2} M\dot{\boldsymbol{X}}^2. \tag{1.35}$$

Closed Systems: Closed systems, which interact only through internal forces, are described by a potential $V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ with $\boldsymbol{F}_i^{(\mathrm{in})} = -\partial_{\boldsymbol{x}_i} V$, cf. Eq. (1.25). The invariance of the equations of motion under Galilean transformations (Galilean principle of relativity) requires that

$$V(R\boldsymbol{x}_1 + \boldsymbol{b}, \ldots, R\boldsymbol{x}_N + \boldsymbol{b}) = V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N), \qquad (R \in \mathrm{O}(3); \boldsymbol{b} \in \mathbb{R}^3). \tag{1.36}$$

Thus, for closed, Galilean-invariant systems:

(i) the net force $\boldsymbol{F} = 0$ vanishes ($\boldsymbol{P}$ is conserved),

(ii) the net torque $\boldsymbol{M} = 0$ vanishes ($\boldsymbol{L}$ is conserved),

(iii) the power reduces the potential,

$$\sum_{i=1}^N \boldsymbol{F}_i \cdot \dot{\boldsymbol{x}}_i = -\frac{dV}{dt} \tag{1.37}$$

(the energy $T + V$ is conserved).

To prove this, consider an arbitrary unit vector $\boldsymbol{e}$:

(i) It holds that

$$\boldsymbol{e} \cdot \boldsymbol{F} = -\frac{d}{d\lambda} V(\boldsymbol{x}_1 + \lambda\boldsymbol{e}, \ldots, \boldsymbol{x}_N + \lambda\boldsymbol{e})\Big|_{\lambda=0} = 0\,, \tag{1.38}$$

[6] Here M denotes the total mass, $M = \sum_{i=1}^N m_i$.

since the potential is invariant under translations. As this holds for arbitrary unit vectors $\boldsymbol{e}$, it follows that $\boldsymbol{F} = 0$.

(ii) Consider a rotation $R(\varphi)$ by an angle φ about $\boldsymbol{e}$. Then

$$\frac{d}{d\varphi} R(\varphi)\boldsymbol{x}\Big|_{\varphi=0} = \frac{d}{d\varphi}(\varphi \boldsymbol{e}) \times \boldsymbol{x}\Big|_{\varphi=0} = \boldsymbol{e} \times \boldsymbol{x}\,, \tag{1.39}$$

and therefore

$$\begin{aligned} \boldsymbol{e} \cdot \boldsymbol{M} &= \sum_{i=1}^{N} \boldsymbol{e} \cdot (\boldsymbol{x}_i \times \boldsymbol{F}_i) = \sum_{i=1}^{N} (\boldsymbol{e} \times \boldsymbol{x}_i) \cdot \boldsymbol{F}_i \\ &= -\frac{d}{d\varphi} V(R(\varphi)\boldsymbol{x}_1, \ldots, R(\varphi)\boldsymbol{x}_N)\Big|_{\varphi=0} = 0\,. \end{aligned} \tag{1.40}$$

Since $\boldsymbol{e}$ is arbitrary, it follows that $\boldsymbol{M} = 0$.

(iii) follows directly from the chain rule.

A closed system that satisfies the Galilean relativity principle therefore possesses the ten classical conserved quantities (integrals of the motion):

$$\boldsymbol{P} \quad \text{and} \quad M\boldsymbol{X} - \boldsymbol{P}t \qquad \text{(6 center-of-mass integrals)}, \tag{1.41}$$

$$\boldsymbol{L} \quad (\text{or } \boldsymbol{L}_{\mathrm{S}}) \qquad \text{(3 angular-momentum integrals)}, \tag{1.42}$$

$$T + V \quad (\text{or } T_{\mathrm{S}} + V) \qquad \text{(energy integral)}. \tag{1.43}$$

These 10 conserved quantities are related to the 10 continuous parameters $\boldsymbol{b}$, $\boldsymbol{v}$, R, and a $(3 + 3 + 3 + 1)$ of the Galilei group (see Chapter 4.5.3).

1.5 Accelerated Reference Frames

The Newtonian equations of motion take the simple form

$$m\ddot{\boldsymbol{x}} = \boldsymbol{F}, \tag{1.44}$$

which is valid in all inertial frames. Sometimes, however, it is convenient to work in coordinate systems that are not inertial frames. In such accelerated reference frames, so-called "fictitious forces" then appear.

Let $\boldsymbol{y}$ denote the coordinates of an arbitrary reference frame related to an inertial frame $\boldsymbol{x}$ by

$$\boldsymbol{x} = R(t)\boldsymbol{y} + \boldsymbol{b}(t)\,, \qquad (R(t) \in \mathrm{SO}(3); \boldsymbol{b}(t) \in \mathbb{R}^3) \tag{1.45}$$

Then

$$\dot{\boldsymbol{x}} = \dot{R}\,\boldsymbol{y} + R\,\dot{\boldsymbol{y}} + \dot{\boldsymbol{b}}\,, \qquad \ddot{\boldsymbol{x}} = \ddot{R}\,\boldsymbol{y} + 2\dot{R}\,\dot{\boldsymbol{y}} + R\,\ddot{\boldsymbol{y}} + \ddot{\boldsymbol{b}}. \tag{1.46}$$

Multiplying the equation for $\ddot{\boldsymbol{x}}$ by R^t (R^t is the transpose of R) leads to

$$m\ddot{\boldsymbol{y}} = R^t\,\boldsymbol{F} - 2m\,R^t\dot{R}\,\dot{\boldsymbol{y}} - mR^t\,\ddot{R}\,\boldsymbol{y} - m\,R^t\,\ddot{\boldsymbol{b}}\,, \tag{1.47}$$

since $R^t R = 1$.

The first term on the right-hand side is the force vector

$$\boldsymbol{K} \equiv R^t \boldsymbol{F} \tag{1.48}$$

in y coordinates. The additional terms are the fictitious forces that arise. The last term contains the acceleration of the point $\boldsymbol{y} = 0$ in y coordinates

$$\boldsymbol{a} \equiv R^t \ddot{\boldsymbol{b}}. \tag{1.49}$$

The remaining terms can be simplified by introducing the map $\Omega = R^t \dot{R}$. One easily sees that Ω is antisymmetric; for differentiating the relation $R^t R = 1$ with respect to time yields

$$R^t \dot{R} + \dot{R}^t R = \Omega + \Omega^t = 0\,. \tag{1.50}$$

Thus Ω in general has the form

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} : \qquad \Omega\,\boldsymbol{y} = \boldsymbol{\omega} \times \boldsymbol{y}, \text{ with } \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3). \tag{1.51}$$

A point at rest in the $\boldsymbol{y}$ frame has velocity $\dot{\boldsymbol{x}} = \dot{R}\,\boldsymbol{y} + \dot{\boldsymbol{b}}$ in the $\boldsymbol{x}$ frame. The components of this velocity in the $\boldsymbol{y}$ frame are therefore $R^t \dot{\boldsymbol{x}} = \boldsymbol{\omega} \times \boldsymbol{y} + R^t\,\dot{\boldsymbol{b}}$ In particular, $\boldsymbol{\omega}$ describes the $\boldsymbol{y}$ components of the (instantaneous) angular velocity of the $\boldsymbol{y}$ frame relative to the $\boldsymbol{x}$ frame; the vector $\boldsymbol{\omega}$ points along the instantaneous axis of rotation.

Further, it holds

$$R^t\,\ddot{R} = \dot{\Omega} - \dot{R}^t\,\dot{R} = \dot{\Omega} - \dot{R}^t\,R\,R^t\,\dot{R} = \dot{\Omega} + \Omega^2\,, \tag{1.52}$$

$$R^t\,\ddot{R}\,\boldsymbol{y} = \dot{\boldsymbol{\omega}} \times \boldsymbol{y} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{y})\,. \tag{1.53}$$

Altogether, (1.47) thus reads

$$m\,\ddot{\boldsymbol{y}} = \boldsymbol{K} - 2m\,(\boldsymbol{\omega} \times \dot{\boldsymbol{y}}) - m\,(\dot{\boldsymbol{\omega}} \times \boldsymbol{y}) - m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{y}) - m\boldsymbol{a}\,. \tag{1.54}$$

The terms appearing to the right of $\boldsymbol{K}$ are called fictitious forces; in particular,

$$\begin{aligned} &-2\,m\,\boldsymbol{\omega} \times \dot{\boldsymbol{y}} && \textit{is the Coriolis force and} \\ \text{and } &-m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{y}) && \textit{the centrifugal force.} \end{aligned} \tag{1.55}$$

These are the only fictitious forces in the case of uniform rotation of the $\boldsymbol{y}$ system ($\boldsymbol{\omega}$ = constant, $\boldsymbol{a} = 0$). The additional term $-m\boldsymbol{a}$, which arises for $\boldsymbol{a} \neq 0$, is called the *transport force*.

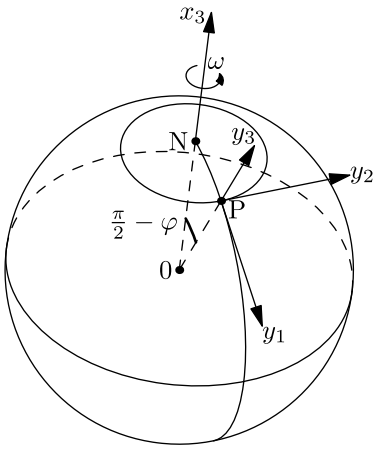


Figure 1.1: The point P lies at geographic latitude φ. The Earth's rotation takes place about the axis Earth center-North Pole ($\overrightarrow{0N}$). The $\boldsymbol{y}$ system is fixed at the location P and rotates with the angular velocity $2\pi/$ per day about the Earth's axis.

It should be noted that the above fictitious forces are proportional to the *inertial mass* m_T of the particle, whereas the gravitational force $m_S\,\boldsymbol{g}$ is proportional to the *gravitational mass* m_S.[7] Experimentally one finds that the two are equal

$$m_S = m_T\,, \qquad \text{Experiment: } m_S/m_T = 1 \pm 10^{-12}. \tag{1.56}$$

Because of the equality of gravitational and inertial mass, the equations of motion of a particle in a homogeneous gravitational field $\boldsymbol{g}$ relative to a freely falling, non-rotating ($\boldsymbol{\omega} = 0$) reference frame read

$$m_T\ddot{\boldsymbol{y}} = m_S\boldsymbol{g} - m_T\boldsymbol{g} = 0\,, \tag{1.57}$$

i.e., the weight force is completely transformed away. In an inhomogeneous gravitational field this holds only locally. The equivalence of inertial and gravitational mass is an important foundation of the general theory of relativity.

1.5.1 Example: Free fall at the Earth's surface

As an application, we consider free fall at the Earth's surface. Let P be a point on the Earth's surface whose geographic latitude is given by φ. Let $(t, \boldsymbol{x})$ be an 'inertial' reference frame fixed in space with origin at the Earth's center, whose x_3 axis points toward the North Pole N. (Here we neglect the Earth's motion about the Sun, polar motion, etc.) The rotating coordinate system is denoted by $\boldsymbol{y}$ with $\boldsymbol{x} = R(t)\boldsymbol{y} + \boldsymbol{b}(t)$. The vector $\boldsymbol{b}$ equals $\boldsymbol{b} = R\boldsymbol{p}$, where $\boldsymbol{p} = \overrightarrow{0P}$ is the position vector from the Earth's center to P (at time $t = 0$), see Figure 1.1.

[7] The inertial mass is the constant appearing in the Newtonian equations of motion $\boldsymbol{F} = m_T\ddot{\boldsymbol{x}}$. The gravitational mass describes the Newtonian gravitational force $\boldsymbol{F} = -\boldsymbol{\nabla}(Gm_{S1}m_{S2}/x_{12})$.

With (1.54) we find the equations of motion

$$\ddot{\boldsymbol{y}} = \boldsymbol{g}' - 2\,\boldsymbol{\omega} \times \dot{\boldsymbol{y}} \tag{1.58}$$

of a point mass in the vicinity of P, where

$$\boldsymbol{\omega} = (-\omega\cos\varphi, 0, \omega\sin\varphi)\,, \qquad \omega = 2\pi/\mathrm{Tag}\,,$$

and

$$\begin{aligned} \boldsymbol{g}' &= \boldsymbol{g} - \boldsymbol{\omega}\times\big(\boldsymbol{\omega}\times\boldsymbol{y}\big) \overbrace{-\,\boldsymbol{a}}^{=\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{p})} = \boldsymbol{g} - \boldsymbol{\omega}\times\big(\boldsymbol{\omega}\times(\boldsymbol{p}+\boldsymbol{y})\big) \\ &\equiv (0,0,-g)\,, \qquad\qquad (g \approx 9.81\,\mathrm{m/s^2}). \end{aligned} \tag{1.59}$$

In the last line we have defined the y_3 direction as the vertical direction along which the resultant of the gravitational acceleration, the centrifugal force, and the transport force acts at P.[8]

In components, equation (1.58) reads

$$\begin{aligned} \ddot{y}_1 &= 2\omega\sin(\varphi)\,\dot{y}_2\,, \\ \ddot{y}_2 &= -2\omega\sin(\varphi)\,\dot{y}_1 - 2\omega\cos(\varphi)\,\dot{y}_3\,, \\ \ddot{y}_3 &= -g + 2\omega\cos(\varphi)\,\dot{y}_2\,. \end{aligned} \tag{1.60}$$

We wish to describe a point mass that is released from rest ($\dot{\boldsymbol{y}} = 0$) at height h ($y_1 = y_2 = 0$, $y_3 = h > 0$) at $t = 0$. The first and third equations then yield

$$\dot{y}_1 = 2\omega\sin(\varphi) y_2 \qquad \text{and} \qquad \dot{y}_3 = -gt + 2\omega\cos(\varphi) y_2\,. \tag{1.61}$$

Substituting into the second equation leads to the differential equation

$$\ddot{y}_2 + 4\omega^2\, y_2 = 2gt\omega\cos\varphi \tag{1.62}$$

for y_2. The general solution is

$$y_2 = \frac{g\cos\varphi}{2\omega} t + A\sin(2\omega t) + B\cos(2\omega t)\,. \tag{1.63}$$

The initial condition $y_2(0) = 0$ gives $B = 0$, and $\dot{y}_2(0) = 0$ requires $A = -(2\omega)^{-2} g\cos\varphi$. We thus obtain the result

$$y_2 = \frac{g\cos\varphi}{2\omega}\left(t - \frac{1}{2\omega}\sin 2\omega t\right). \tag{1.64}$$

For $t > 0$ we have $y_2 > 0$, i.e., there occurs an *eastward deflection*. Since $\omega t \simeq$ fall time per day $\ll 1$ we expand in powers of ωt and obtain

$$y_2 = \frac{gt^2}{3}\cos(\varphi)\omega t\,. \tag{1.65}$$

[8] The gravitational acceleration is approximately the same everywhere on Earth. We have further assumed that $|\boldsymbol{y}| \ll |\boldsymbol{p}|$, so that the gravitational acceleration is independent of $\boldsymbol{y}$.

In principle, we can now also determine y_1, y_3 with the help of (1.61). However, the corrections to free fall for these components are only of order $(\omega t)^2$. The impact occurs at time T with $gT^2/2 = h$. We obtain the total eastward deflection

$$y_2(T) = \frac{1}{3}\omega\Big(\frac{8h^3}{g}\Big)^{1/2}\cos\varphi\,. \tag{1.66}$$

For example, for $\varphi = 45°$ and $h = 100$ m the eastward deflection is just $y_2 \approx 1.6$ cm.

1.6 Mechanical Similarity

To some extent it is possible to determine certain properties of the solution of Newton's equations of motion (1.10) without solving them explicitly. An important example is the case in which the forces are generated by a potential V that is a homogeneous function (of degree k) of the coordinates, i.e.

$$V(\boldsymbol{x}_1', \ldots, \boldsymbol{x}_N') = V(\lambda\boldsymbol{x}_1, \ldots, \lambda\boldsymbol{x}_N) = \lambda^k\, V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \tag{1.67}$$

with $\lambda > 0$, $\boldsymbol{x}_i' = \lambda\boldsymbol{x}_i$ and $k \in \mathbb{R}$. Differentiating this relation with respect to $\boldsymbol{x}_i$ yields

$$\lambda\frac{\partial V}{\partial \boldsymbol{x}_i'}(\boldsymbol{x}_1', \ldots, \boldsymbol{x}_N') = \lambda^k\frac{\partial V}{\partial \boldsymbol{x}_i}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N), \tag{1.68}$$

i.e. the forces $\boldsymbol{F}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = -\partial_{\boldsymbol{x}_i} V$ are homogeneous of degree $k-1$.

Given a solution $\boldsymbol{x}_i(t)$ of the equations of motion

$$m_i\ddot{\boldsymbol{x}}_i = -\frac{\partial V}{\partial \boldsymbol{x}_i}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \tag{1.69}$$

one can obtain further (scaled) solutions $\boldsymbol{x}_i'(t') = \lambda\,\boldsymbol{x}_i(t)$ by means of the transformation $\boldsymbol{x}_i' = \lambda\boldsymbol{x}_i$ and $t' = \beta t$ provided that

$$\beta = \lambda^{1-k/2} \tag{1.70}$$

is chosen. With this choice one has indeed

$$m_i\frac{d^2\boldsymbol{x}_i'}{dt'^2} = \frac{\lambda}{\beta^2}m_i\ddot{\boldsymbol{x}}_i \overset{(1.69)}{=} -\frac{\lambda}{\beta^2}\frac{\partial V}{\partial \boldsymbol{x}_i}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \overset{(1.68)}{=} -\underbrace{\frac{\lambda^{2-k}}{\beta^2}}_{=1}\frac{\partial V}{\partial \boldsymbol{x}_i'}(\boldsymbol{x}_1', \ldots, \boldsymbol{x}_N')$$

and hence $\boldsymbol{x}_i'(t')$ is also a solution of (1.69).

The change from $\boldsymbol{x}_i(t)$ to $\boldsymbol{x}_i'(t')$ corresponds to passing to new particle trajectories in which only the spatial and temporal distances are scaled by the respective factors λ and β (geometric similarity). Because the new coordinates again satisfy the equations of motion, for two trajectories (which are similar to each other) the ratio of the times on the trajectories is

$$\frac{t'}{t} = \beta = \left(\frac{l'}{l}\right)^{1-k/2} \tag{1.71}$$

where l, l' describe the spatial extent of the trajectories.[9] We also obtain the scaling relations

$$\frac{|\dot{\boldsymbol{x}}'|}{|\dot{\boldsymbol{x}}|} = \frac{\lambda}{\beta} = \left(\frac{l'}{l}\right)^{k/2}, \qquad \frac{|\boldsymbol{L}'|}{|\boldsymbol{L}|} = \frac{\lambda^2}{\beta} = \left(\frac{l'}{l}\right)^{1+k/2}, \qquad \frac{E'}{E} = \frac{\lambda^2}{\beta^2} = \left(\frac{l'}{l}\right)^{k}, \tag{1.72}$$

for the velocity $\dot{\boldsymbol{x}}$, the angular momentum $|\boldsymbol{L}|$, and the total energy $E = T + V$.

As an example we consider the gravitational attraction of two point masses ($k = -1$). In Chapter 2.2 we will show that the corresponding trajectories are ellipses about their common center of mass. Two similar trajectories then have the same eccentricity but in general a different semi-major axis a. From (1.71) we obtain

$$\frac{T'}{T} = \left(\frac{a'}{a}\right)^{3/2}, \tag{1.73}$$

i.e. the squares of the orbital periods T are proportional to the cubes of the semi-major axis (*Kepler's third law*).

Virial theorem: In the case that the potential is a homogeneous function and the motion of the particles takes place in a bounded spatial region (with bounded momentum), there exists a simple relation between the time average of the kinetic and the potential energy. The theorem is of a statistical nature, since it makes no statement about instantaneous (exact) relations among various mechanical quantities but only concerns the time average.

To obtain the relation, we rewrite the kinetic energy slightly,

$$2T = \sum_{i=1}^{N} \boldsymbol{p}_i \cdot \dot{\boldsymbol{x}}_i = \frac{d}{dt} \underbrace{\left(\sum_{i=1}^{N} \boldsymbol{p}_i \cdot \boldsymbol{x}_i \right)}_{=G} - \sum_{i=1}^{N} \boldsymbol{x}_i \cdot \underbrace{\dot{\boldsymbol{p}}_i}_{=\boldsymbol{F}_i}. \tag{1.74}$$

We now wish to time-average this expression. The time average of a function $f(t)$ is defined by

$$\overline{f} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt\, f(t). \tag{1.75}$$

It is easy to see that for a bounded function $F(t)$

$$\overline{\frac{dF}{dt}} = \lim_{\tau \to \infty} \frac{F(\tau) - F(0)}{\tau} = 0 \tag{1.76}$$

holds. By hypothesis $G(t)$ is bounded and from (1.74) we obtain the *virial theorem*

$$2\overline{T} = -\overline{\sum_{i=1}^{N} \boldsymbol{x}_i \cdot \boldsymbol{F}_i}; \tag{1.77}$$

[9] For example one could choose $l = |\boldsymbol{x}_1(0)|$ and $l' = |\boldsymbol{x}_1'(0)|$.

the right-hand side of this equation is called the *virial* of the system.

For a homogeneous potential the *Euler theorem*[10] yields

$$kV = \sum_{i=1}^{N} \boldsymbol{x}_i \cdot \frac{\partial V}{\partial \boldsymbol{x}_i}$$

and hence we find ($\boldsymbol{F}_i = -\partial_{\boldsymbol{x}_i} V$)

$$2\overline{T} = k\overline{V}. \tag{1.78}$$

For the gravitational potential ($k = -1$) this gives $2\overline{T} = -\overline{V}$ (V is negative). For an isotropic oscillator $k = 2$ and therefore $\overline{T} = \overline{V}$

[10] Proof: differentiate $V(\lambda x) = \lambda^k V(x)$ with respect to λ to obtain $\sum_i \partial_{\boldsymbol{x}_i} V \cdot \boldsymbol{x}_i = k\lambda^{k-1} V$ (by the chain rule). Then set $\lambda = 1$.

Chapter 2

The Two-Body Problem

Before we proceed to further investigate the structure of classical mechanics, it is instructive to study a few examples first. We begin with the simplest case, an isolated mechanical system of two particles. We will see that this case can in general be reduced to the problem of a single particle moving in an effective potential. This insight is the basis for the general solution (integration) of the two-particle problem.

2.1 The General Case

As we have shown in Chapter 1.3, the force between two particles acts along the line connecting them and depends in magnitude only on the distance between the two particles. The equations of motion are therefore in general of the form

$$\ddot{\boldsymbol{x}}_1 = -\frac{1}{m_1}\frac{\partial V}{\partial \boldsymbol{x}_1}(|\boldsymbol{x}_1 - \boldsymbol{x}_2|)\,, \qquad \ddot{\boldsymbol{x}}_2 = \frac{1}{m_2}\frac{\partial V}{\partial \boldsymbol{x}_1}(|\boldsymbol{x}_1 - \boldsymbol{x}_2|)\,, \tag{2.1}$$

where $V(r)$ is the potential.

For two particles with masses m_1 and m_2 interacting only through the gravitational force, the potential is for example given by $V(r) = -Gm_1m_2/r$. In this case, (2.1) also applies to extended, spherically symmetric bodies (e.g. in good approximation for celestial bodies), since outside such a body the gravitational force is exactly the same as if all the mass were concentrated at the center (*Newton's shell theorem*).

2.1.1 Relative Coordinates and Conserved Quantities

To solve the above problem, it is helpful to separate the center-of-mass motion. To this end, we introduce the center-of-mass and relative coordinates

$$\boldsymbol{X} = \frac{1}{M}(m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2)\,, \qquad (M = m_1 + m_2)\,, \tag{2.2}$$

$$\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2 \tag{2.3}$$

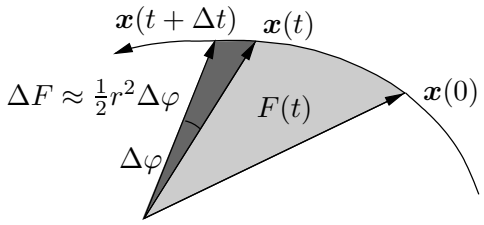


Figure 2.1: Illustration of the areal law: $F(t)$ denotes the area in the orbital plane swept out by the vector $\boldsymbol{x}(t)$ in time t. Its rate of change ΔF is given by the area $r^2\Delta\varphi/2$ of the dark gray triangle. We thus obtain for the area swept out per unit time $\dot{F} = r^2\dot{\varphi}/2$, which is proportional to the angular momentum $l = \mu r^2 \dot{\varphi}$.

With respect to these variables, the equations of motion read

$$M\ddot{\boldsymbol{X}} = 0 \tag{2.4}$$

$$\mu\ddot{\boldsymbol{x}} = -\frac{\partial V}{\partial \boldsymbol{x}}(|\boldsymbol{x}|)\,, \qquad \left(\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}\right), \tag{2.5}$$

i.e. the center-of-mass motion $\boldsymbol{X}(t)$ and the relative motion $\boldsymbol{x}(t)$ are decoupled. The center-of-mass motion $\boldsymbol{X}(t)$ is an inertial trajectory (see Chapter 1.1), and the equation for the relative motion $\boldsymbol{x}(t)$ is that *of a single* particle with reduced mass μ under the influence of the external potential $V(|\boldsymbol{x}|)$. The relative system still possesses two conserved quantities.

Angular momentum: The (relative) angular momentum

$$\boldsymbol{L} = \mu\, \boldsymbol{x} \times \dot{\boldsymbol{x}}\ (= \text{constant}) \tag{2.6}$$

is conserved since $\dot{\boldsymbol{L}} = -\boldsymbol{x} \times \partial_{\boldsymbol{x}} V(|\boldsymbol{x}|) = 0$. Thus the trajectory $\boldsymbol{x}(t)$ lies in the plane perpendicular to $\boldsymbol{L}$, the *orbital plane*. To describe the motion in the orbital plane, we use polar coordinates (r, φ) with the corresponding unit vectors $\boldsymbol{e}_r = (\cos\varphi, \sin\varphi, 0)$ and $\boldsymbol{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$ in the radial and azimuthal directions, respectively.[1] With $\boldsymbol{x} = r\boldsymbol{e}_r$ and $\dot{\boldsymbol{e}}_r = \dot{\varphi}\boldsymbol{e}_\varphi$ we find for the velocity

$$\dot{\boldsymbol{x}} = \dot{r}\, \boldsymbol{e}_r + r\, \dot{\varphi}\, \boldsymbol{e}_\varphi. \tag{2.7}$$

Hence the magnitude of the angular momentum is given by

$$l \ \equiv\ |\boldsymbol{L}| = \mu\, r^2\, \dot{\varphi} = \text{konstant}\,. \tag{2.8}$$

This is the areal law (*Second Kepler's law*): If $F(t)$ denotes the area swept out by the vector $\boldsymbol{x}(t)$ in the orbital plane, then

$$\dot{F}(t) = \frac{1}{2} r^2 \dot{\varphi} = \frac{l}{2\mu} = \text{konstant}\,, \tag{2.9}$$

[1] We have chosen the coordinate system such that $\boldsymbol{L}$ points along the (positive) x_3-axis.

see Figure 2.1.

Energy: The other conserved quantity is the relative energy

$$T + V = \frac{\mu}{2}\dot{\boldsymbol{x}}^2 + V(r) = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + V(r) = E = \text{konstant}\,. \tag{2.10}$$

The constancy of energy E follows from $\dot{E} = \dot{\boldsymbol{x}} \cdot \mu\,\ddot{\boldsymbol{x}} + \dot{\boldsymbol{x}} \cdot \partial_{\boldsymbol{x}} V(|\boldsymbol{x}|) = 0$.

The combination of the two conserved quantities allows us to solve the two-body problem. Substituting $\dot{\varphi}$ from (2.8) into the energy equation (2.10) yields

$$\frac{1}{2}\mu\dot{r}^2 + U(r) = E\,, \qquad U(r) = \frac{l^2}{2\mu r^2} + V(r)\,. \tag{2.11}$$

The quantity $l^2/2\mu r^2$ is called the *centrifugal potential*. The radial motion $r(t)$ on $0 < r < \infty$ is that of a point mass with mass μ under the influence of the effective potential $U(r)$.

The remaining differential equation (2.11) is separable with the solution

$$t(r) - t(r_0) = \pm \int_{r_0}^{r} \frac{dx}{\sqrt{\frac{2}{\mu}(E - U(x))}}\,; \tag{2.12}$$

the square root changes sign at each turning point. For the region in which the radius increases (decreases), the positive (negative) sign must be chosen. Moreover, one obtains with

$$\frac{d\varphi}{dr} = \frac{\dot{\varphi}}{\dot{r}} = \pm \frac{l}{\mu\, r^2 \sqrt{\frac{2}{\mu}(E - U(r))}}\,, \tag{2.13}$$

also

$$\varphi(r) - \varphi(r_0) = \pm \int_{r_0}^{r} \frac{l\, dx}{x^2 \sqrt{2\,\mu\,(E - U(x))}}\,. \tag{2.14}$$

The function $\varphi(r)$ describes the trajectory in the plane without reference to the temporal evolution. To find the trajectory $\boldsymbol{x}(t)$, one must solve (2.12) for r. This yields $r(t)$ and, using (2.14), also $\varphi(t)$.

Through (2.12) and (2.14), the determination of the orbit is reduced to the calculation of integrals: the problem is integrable, see also Chapter 8.2. The orbit depends on four *constants of motion* E, l, r_0 and $\varphi_0 = \varphi(r_0)$. Together with the orbital plane (direction of $\boldsymbol{L}$), these uniquely determine the orbit.[2]

[2]The connection to the "elementary" integration constants $r_0, \varphi_0, \dot{r}_0, \dot{\varphi}_0$ is obtained via the equations (2.8) and (2.10).

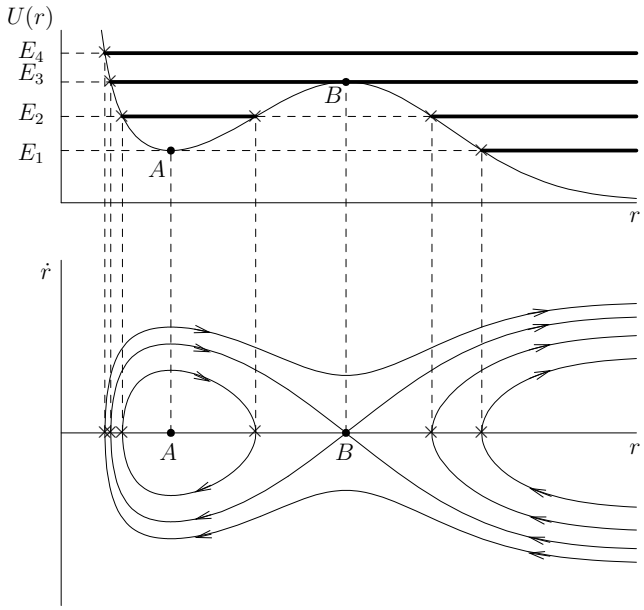


Figure 2.2: Potential and phase portrait for four different energies.

2.1.2 The different orbit types

By means of an example we illustrate possible types of motion of the relative motion. Figure 2.2 shows the motion for four different energies in the phase space with coordinates $(r, \dot{r})$ (phase portrait). The orbits always run within the region

$$E - U(r) = \frac{\mu}{2}\,\dot{r}^2 \geq 0\,. \tag{2.15}$$

At the boundaries of this region $\dot{r} = 0$; these are the turning points of the orbit (labeled $\times$ in the diagram). For a point with $\dot{r} = 0$ to be a true turning point, it must satisfy

$$\mu\ddot{r} = -\frac{dU}{dr} \neq 0 \tag{2.16}$$

. If $dU/dr = 0$ for $r = r_0$ (these are the points A and B in the diagram), there exists an equilibrium solution with $r = r_0$. The solution at A is stable (i.e. an orbit with initial conditions $(r, \dot{r})$ near A will always remain near A), whereas B is unstable. In general we distinguish *bound orbits*, which run entirely within a finite region, and *scattering orbits* with $r(t) \to \infty$ for $t \to \pm\infty$.

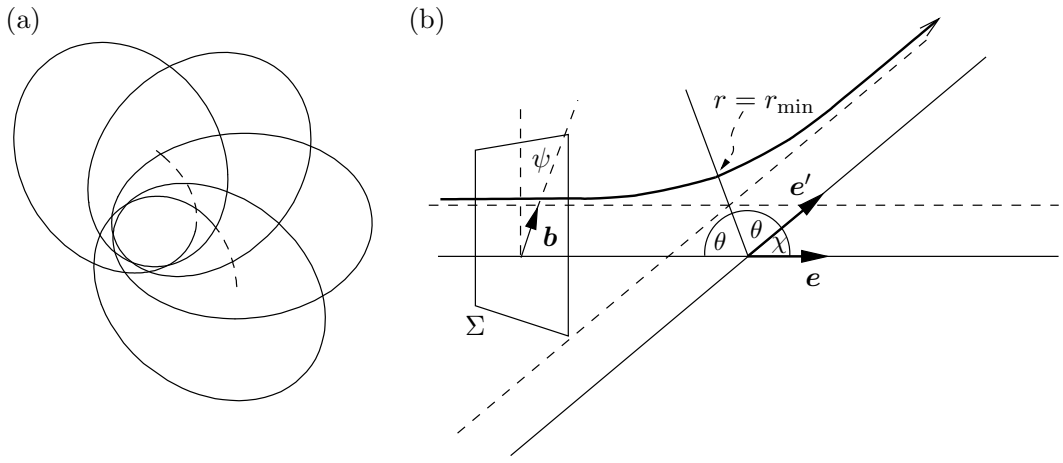


Figure 2.3: (a) Rosette orbit (b) Scattering orbit with impact parameter $\boldsymbol{b}$ in the plane Σ. The particle starts at $t \to -\infty$ at position $r \to \infty$ and approaches the scattering center in the direction $\boldsymbol{e} \perp \boldsymbol{b}$. It reaches the smallest distance from the scattering center at $r_{\min}$ before moving off for $t \to \infty$ in the direction $\boldsymbol{e}'$ (deflected by the scattering angle χ) toward $r \to \infty$.

2.1.3 Bound orbits

For bound orbits $r(t)$ is periodic with period

$$T(E) = 2 \int_{r_{\min}}^{r_{\max}} \frac{dx}{\sqrt{\frac{2}{\mu}(E - U(x))}}, \tag{2.17}$$

, where $r_{\min}$ and $r_{\max}$ are the zeros of $E - U(r)$ that bound the allowed r interval. This formula follows directly from (2.12).

If both $r_{\min}$ and $r_{\max}$ are turning points, then $T < \infty$ holds. In this case one may expand the function $E - U(x)$ locally (near $x^* = r_{\min}$ and $x^* = r_{\max}$) with $E - U(x^* + \epsilon) \approx \alpha\,\epsilon$. The resulting integral $\propto \int d\epsilon\,/\sqrt{\epsilon}$ converges. If one of the interval boundaries is not a turning point but an equilibrium point (as for $E = E_3$, where $r_{\max} = r_{\mathrm{B}}$), then $T(E_3) = \infty$ holds instead.

For a periodic orbit the azimuth φ increases by the angle (*perihelion precession*)

$$\Delta\varphi = 2 \int_{r_{\min}}^{r_{\max}} \frac{l\,dx}{x^2\sqrt{2\mu(E - U(x))}}, \tag{2.18}$$

during one period T, which follows directly from (2.14). The trajectory is generally a "rosette orbit" in the annulus $r_{\min} \le r \le r_{\max}$, which closes only when $\Delta\varphi/2\pi$ is rational; see Figure 2.3(a).

2.1.4 Scattering Orbits

We now consider the case $V(r) \to 0$ for $r \to \infty$. Then only for energies $E \geq 0$ do scattering orbits exist (with $r(t) \to \infty$ for $t \to \pm\infty$).[3] If $E > 0$, the scattering orbits have straight asymptotes (for $t \to \pm\infty$). A scattering orbit is determined by the energy $E > 0$, the direction $\boldsymbol{e}$ ($|\boldsymbol{e}| = 1$) of the incoming asymptote, and the impact parameter $\boldsymbol{b} \perp \boldsymbol{e}$, as shown in Figure 2.3(b). For $t \to -\infty$ one obtains from $V \to 0$ the asymptotic relation $\mu|\dot{\boldsymbol{x}}| \to \sqrt{2\mu E}$ and hence $\mu\dot{\boldsymbol{x}} \to \sqrt{2\mu E}\,\boldsymbol{e}$. With $l = \mu|\boldsymbol{x} \times \dot{\boldsymbol{x}}|$ we get the relation

$$l = b\sqrt{2\mu E} \tag{2.19}$$

between the impact parameter $b = |\boldsymbol{b}|$ and the angular momentum l. Therefore b and E fix the *scattering angle*

$$\chi = \pi - 2\theta \tag{2.20}$$

via

$$\theta = \int_{r_{\min}}^{\infty} \frac{l\,dx}{x^2\sqrt{2\mu(E - U(x))}} = \int_{r_{\min}}^{\infty} \frac{b\,dx}{x^2\sqrt{1 - V(x)E^{-1} - b^2x^{-2}}}\,, \tag{2.21}$$

cf. with equation (2.14).

At fixed E and $\boldsymbol{e}$, $\boldsymbol{b}$ determines the direction vector $\boldsymbol{e}'$ ($|\boldsymbol{e}'| = 1$) of the outgoing asymptote. This relation defines a mapping $\Sigma \ni \boldsymbol{b} \mapsto \boldsymbol{e}' \in S^2$. Since $\boldsymbol{e}'$ is described by the polar angle (χ, ψ) relative to $\boldsymbol{e}$, under this mapping the area element $d\sigma = b\,db\,d\psi$ of Σ is carried into the solid-angle element $d\Omega = \sin\chi\,d\chi\,d\psi$.

The *differential scattering cross section* (with the dimension of an area) is defined as the ratio

$$\frac{d\sigma}{d\Omega}(\chi) = \left|\frac{b}{\sin\chi}\frac{db}{d\chi}\right| = \frac{b}{\sin\chi}\left|\left(\frac{d\chi}{db}\right)^{-1}\right|\,, \tag{2.22}$$

, where $\chi = \chi(b)$ is given by (2.20) and (2.21). It can happen that different b_k lead to the same scattering angle χ; then (2.22) must be replaced by

$$\frac{d\sigma}{d\Omega}(\chi) = \sum_k \frac{b_k}{\sin\chi}\left|\left(\frac{d\chi}{db_k}\right)^{-1}\right| \tag{2.23}$$

. The latter case occurs, for instance, whenever $-\infty < V(0) \leq 0$. In that case one has $\chi(0) = 0$, because for $b = 0$ the trajectory passes exactly through the origin. Since in general also $\lim_{b\to\infty}\chi(b) = 0$ holds, the intermediate-value theorem implies that every (non-extremal) value $\chi(b_k)$ is attained at least twice at positions b_k.

In a *scattering experiment* one bombards the scattering center with a homogeneous current density j (number of particles per unit area and time) of incident particles with fixed E and $\boldsymbol{e}$. Then $j\,(d\sigma/d\Omega)\,d\Omega$ is the number of scattered particles per unit

[3] In the limit $r \to \infty$ one has $V(r) - U(r) = 0$.

time whose outgoing asymptotic direction lies in the solid-angle element $d\Omega$ around $\boldsymbol{e}' \equiv (\chi, \psi)$.

The total number $j\,\sigma_{\text{tot}}$ of scattered particles is therefore proportional to the *total scattering cross section*

$$\sigma_{\text{tot}} = \int_{S^2} \frac{d\sigma}{d\Omega} d\Omega. \tag{2.24}$$

, which equals the area of those impact parameters $\boldsymbol{b} \in \Sigma$ that lead to scattering (i.e. $\chi(b) \neq 0$); integrating (2.24) immediately yields the result

$$\sigma_{\text{tot}} = \pi a^2\,, \tag{2.25}$$

, where $a = \sup_{r \geq 0}\{r \mid V(r) \neq 0\} \leq \infty$ is the range of the potential.

2.2 The Kepler Problem

The Kepler problem is an important special case of the two-body problem, in which two *particles* with masses m_1 and m_2 attract each other gravitationally. In this case the potential has the form

$$V(r) = -\frac{1}{r} G m_1 m_2\,, \tag{2.26}$$

where $G = 6.67 \times 10^{-11}\,\mathrm{m}^3/\mathrm{kg\,s}^2$ denotes the Newtonian *gravitational constant*. With the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}\,, \qquad m_1 m_2 = M\mu\,, \quad (M = m_1 + m_2) \tag{2.27}$$

one finds

$$U(r) = \frac{l^2}{2\mu r^2} - \frac{GM\mu}{r}. \tag{2.28}$$

The trajectory is given by the general relation (2.14). Substitution of $x = 1/s$ (with $-dx/x^2 = ds$) leads to

$$\varphi(r) - \varphi_0 = \mp \int_{r_0^{-1}}^{r^{-1}} \frac{l\,ds}{\sqrt{2\mu(E - U(1/s))}} = \mp \int_{r_0^{-1}}^{r^{-1}} \frac{ds}{\sqrt{\alpha + 2\beta s - s^2}}\,, \tag{2.29}$$

where we have introduced the parameters $\alpha = 2\mu E l^{-2}$ and $\beta = GM\mu^2 l^{-2}$. The last integral has the antiderivative ($\alpha + \beta^2 \geq 0$)

$$\int \frac{ds}{\sqrt{\alpha + 2\beta s - s^2}} = -\arccos\Big(\frac{s - \beta}{\sqrt{\alpha + \beta^2}}\Big). \tag{2.30}$$

Thus (2.29) can be integrated explicitly, yielding the result

$$\varphi(r) = \pm \arccos\Big(\frac{r^{-1} - \beta}{\sqrt{\alpha + \beta^2}}\Big); \tag{2.31}$$

here we have chosen the free constant φ_0 precisely so that it cancels the contribution from the lower limit of integration. Solving for r gives the trajectory in polar coordinates

$$r = \frac{d}{1 + \varepsilon \cos\varphi}\,, \tag{2.32}$$

with the parameters

$$d = \frac{1}{\beta} = \frac{l^2}{GM\mu^2}\,, \text{ and} \tag{2.33}$$

$$\varepsilon = \frac{\sqrt{\alpha + \beta^2}}{\beta}\,, \quad \text{d. h.} \quad 1 - \varepsilon^2 = -\frac{\alpha}{\beta^2} = -\frac{2El^2}{G^2M^2\mu^3}\,. \tag{2.34}$$

Equation (2.32) for the trajectory defines a conic section with focus at $r = 0$; specifically, depending on the value of the eccentricity ε, a

circle	:	$\varepsilon = 0$,	i.e. $E = -G^2M^2\mu^3/2l^2$,
ellipse	:	$\varepsilon < 1$,	i.e. $E < 0$,
parabola	:	$\varepsilon = 1$,	i.e. $E = 0$,
hyperbola	:	$\varepsilon > 1$,	i.e. $E > 0$.

The angle $\varphi = 0$ corresponds here to the perihelion (r minimal). We now show that (2.32) for $\varepsilon < 1$ is precisely an ellipse.

2.2.1 Elliptical Orbits

From Figure 2.4(a) and the ellipse equation $r + \bar{r} = 2a$ one obtains the relation

$$\bar{r}^2 = (r\cos\varphi + 2a\varepsilon)^2 + r^2\sin^2\varphi = (2a - r)^2. \tag{2.35}$$

From this, after factoring, we get the result

$$4ar(1 + \varepsilon\cos\varphi) = 4a^2(1 - \varepsilon^2)\,. \tag{2.36}$$

This is exactly the trajectory (2.32) with

$$d = a(1 - \varepsilon^2). \tag{2.37}$$

That Newton's law of gravitation predicts orbits $V(r) \propto r^{-1}$ is historically significant: Johannes Kepler deduced from Tycho Brahe's observations that Mars moves on an ellipse with the Sun (or the barycenter) at a focus, and he generalized this result to the other planets (*Kepler's first law*). Newton was able to give a theoretical explanation of this finding with his theory of gravitation.

A further important feature of an ellipse is the minor semiaxis

$$b = a\sqrt{1 - \varepsilon^2} = \sqrt{ad} \tag{2.38}$$

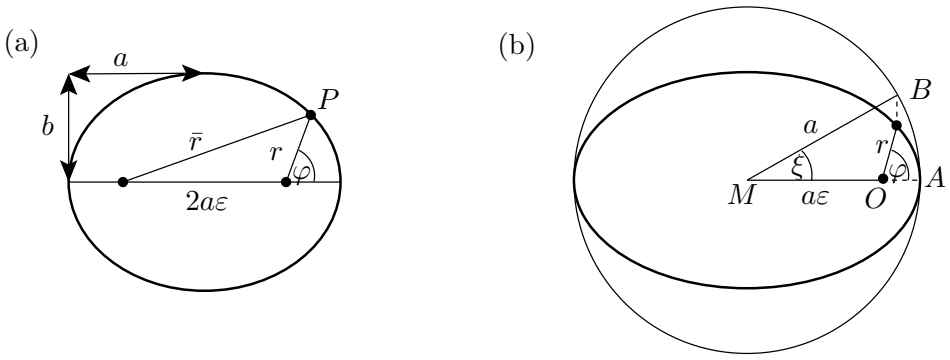


Figure 2.4: (a) Ellipse with major and minor semiaxes a and b. The two foci are separated by the distance $2a\varepsilon$ with eccentricity $\varepsilon = \sqrt{1 - b^2/a^2}$. (b) Graphical relationship between the true anomaly (azimuth) φ and the eccentric anomaly ξ. To construct the eccentric anomaly one uses the auxiliary circle of radius a. The point B on the auxiliary circle (and hence ξ) is characterized by having the same projection onto the MA axis as the point $P \equiv (r, \varphi)$ on the ellipse.

, by means of which the area of the ellipse $F = \pi ab = \pi d^{1/2} a^{3/2}$ can be determined. According to the areal law (2.9) (*Kepler's second law*),

$$F = T\dot{F} = \frac{Tl}{2\mu}\,, \tag{2.39}$$

holds, where T is the orbital period. With this relation one can solve for the orbital period

$$T = \frac{2\mu}{l} F = \frac{2\mu}{l} \pi d^{1/2}\, a^{3/2} = \frac{2\pi}{\sqrt{GM}} a^{3/2} \tag{2.40}$$

. If $m_1 \ll m_2$, then $M \approx m_2$, and the ratio $T^2 : a^3$ is the same for all planets (*Kepler's third law*).

To describe the time evolution of the orbit it is convenient to use the *eccentric anomaly* ξ (i.e., the angle measured from the ellipse center, see Figure 2.4(b)) instead of the true anomaly (azimuth) φ.[4] By projecting MB onto MA one finds the relation

$$a\cos\xi = a\varepsilon + r\cos\varphi, \tag{2.41}$$

which links the eccentric anomaly ξ to the true anomaly φ. In terms of the eccentric anomaly the *Kepler equations*

$$r = a(1 - \varepsilon\cos\xi), \qquad\qquad t = \sqrt{\frac{a^3}{GM}}(\xi - \varepsilon\sin\xi), \tag{2.42}$$

hold, where $t = 0$ corresponds to a passage through perihelion. Equations (2.41) and (2.42) together furnish an (implicit) expression for the orbit $\boldsymbol{x}(t)$.

[4] The eccentric anomaly was already introduced by Kepler.

Proof of Kepler's equations: The equation for r follows directly from (2.41) with

$$\begin{aligned} \varepsilon a\cos\xi &= \varepsilon(a\varepsilon + r\cos\varphi) \\ &= a - a(1-\varepsilon^2) + \varepsilon r\cos\varphi \\ &= a - r(1+\varepsilon\cos\varphi) + \varepsilon r\cos\varphi \\ &= a - r\,, \end{aligned} \tag{2.43}$$

, where in the second transformation the orbit 2.36 was used.

The proof of the time equation is obtained directly from the areal law $F(t) = l\,t/2\mu$, see Eq. (2.9). To this end one uses that a stretching of the ellipse by the factor $a/b = \sqrt{a/d} \geq 1$ in the direction of the minor semi-axis yields the circumcircle. The area $F(t)$ swept by the radius vector is thereby transformed into the area

$$\tilde{F}(t) = \frac{a}{b}F(t) = \sqrt{\frac{a}{d}}\,\frac{l\,t}{2\mu} = \frac{1}{2}\sqrt{GMa}\,t \tag{2.44}$$

of AOB (multiplied by a/b because of the stretching). Now the area $\tilde{F}$ of AOB is precisely the difference between the circular sector AMB and the triangle $a^2\xi/2$ with area OMB and $(a\varepsilon)(a\sin\xi)/2$. Hence we obtain

$$\tilde{F}(t) = \frac{1}{2}a^2(\xi - \varepsilon\sin\xi) \tag{2.45}$$

and by comparison with (2.44) the second equation in (2.42).

2.2.2 Hyperbolic orbits

As already shown at the beginning of the chapter, in the Kepler problem positive energies yield scattering orbits (hyperbolas). To determine the differential scattering cross section we need the relation between the scattering angle $\chi = \pi - 2\theta$ and the impact parameter $b = l/\sqrt{2\mu E}$.

As is evident from Figure 2.5, the angle θ corresponds to the limit $\theta = \pi - \lim_{r\to\infty}\varphi(r)$. According to (2.32) one has $\cos\varphi \to -1/\varepsilon$ for $r \to \infty$. Expressed in terms of the angle θ this is the condition $\cos\theta = 1/\varepsilon$ and therefore $\tan^2\theta = \varepsilon^2 - 1$. On the other hand, by (2.34) one also has

$$\varepsilon^2 - 1 = \frac{2E\overbrace{(2\mu E b^2)}^{=l^2}}{G^2M^2\mu^3}\,. \tag{2.46}$$

Thus we obtain the result

$$\tan\theta = \frac{2E}{GM\mu}b\,. \tag{2.47}$$

To determine the differential scattering cross section we require the intermediate results

$$\frac{d\chi}{db} = -2\frac{d\theta}{db} = -\frac{4E}{GM\mu}\cos^2\theta \tag{2.48}$$

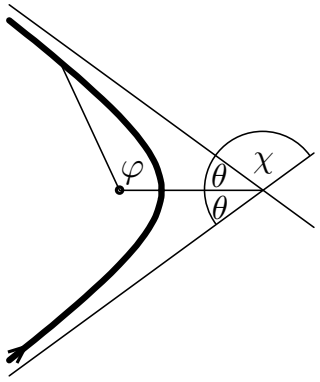


Figure 2.5: Scattering orbit (hyperbola with $\varepsilon > 1$) for the Kepler problem.

and

$$\frac{b}{\sin\chi} = \frac{b}{2\sin\theta\cos\theta} = \frac{GM\mu}{4E}\frac{1}{\cos^2\theta}\,. \tag{2.49}$$

With these we obtain the scattering cross section

$$\frac{d\sigma}{d\Omega} = \left|\frac{b}{\sin\chi}\Big(\frac{d\chi}{db}\Big)^{-1}\right| = \left(\frac{GM\mu}{4E\sin^2\frac{\chi}{2}}\right)^2\,, \tag{2.50}$$

where we have used $\cos\theta = \cos(\pi/2 - \chi/2) = \sin(\chi/2)$. For charged particles the product of the charges e_1e_2 appears in place of $GM\mu$ (*Rutherford scattering formula*). Surprisingly, the same result also holds in quantum mechanics.

2.2.3 Laplace-Runge-Lenz Vector

The Kepler problem possesses an additional constant of motion compared to the general two-body problem, namely the Laplace-Runge-Lenz (LRL) vector

$$\boldsymbol{A} = \mu\dot{\boldsymbol{x}}\times\boldsymbol{L} - GM\mu^2\frac{\boldsymbol{x}}{r}\,. \tag{2.51}$$

This conserved quantity allows a completely algebraic determination of the orbit.

To show that $\boldsymbol{A}$ is indeed conserved, we first derive a vectorial form of the areal law

$$\begin{aligned}\mu\frac{d}{dt}\frac{\boldsymbol{x}}{r} &= \mu\Big(\frac{\dot{\boldsymbol{x}}}{r} - \frac{\boldsymbol{x}}{r^2}\overbrace{\dot r}^{=\boldsymbol{x}\cdot\dot{\boldsymbol{x}}/r}\Big) = \frac{\mu}{r^3}\big(\dot{\boldsymbol{x}}\,r^2 - \boldsymbol{x}(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})\big)\\ &= -\frac{\mu}{r^3}\boldsymbol{x}\times(\boldsymbol{x}\times\dot{\boldsymbol{x}}) = -\frac{1}{r^3}\,\boldsymbol{x}\times\boldsymbol{L}.\end{aligned} \tag{2.52}$$

From this we obtain directly the result

$$\frac{d\boldsymbol{A}}{dt} = -\frac{GM\mu}{r^3}(\boldsymbol{x}\times\boldsymbol{L} - \boldsymbol{x}\times\boldsymbol{L}) = 0\,, \tag{2.53}$$

where we have used the equations of motion

$$\mu\ddot{\boldsymbol{x}} = -\frac{\partial V}{\partial \boldsymbol{x}}(|\boldsymbol{x}|) = -\frac{\boldsymbol{x}}{r^3} GM\mu \tag{2.54}$$

and the time independence of $\boldsymbol{L}$.

The LRL vector does not provide three additional conserved quantities, since it is related to the angular momentum $\boldsymbol{L}$ and the energy E. On the one hand, the LRL vector lies in the orbital plane, because it is orthogonal to $\boldsymbol{L}$ with

$$\boldsymbol{A} \cdot \boldsymbol{L} = 0, \tag{2.55}$$

as can be easily verified by explicit calculation.[5] On the other hand, the square of its length is connected with the energy by

$$\begin{aligned} \boldsymbol{A}^2 &= (GM\mu^2)^2 + \mu^2 \dot{\boldsymbol{x}}^2 \, l^2 - \frac{2GM\mu^2}{r} \overbrace{\mu \boldsymbol{x} \cdot (\dot{\boldsymbol{x}} \times \boldsymbol{L})}^{=l^2} \\ &= (GM\mu^2)^2 + 2\mu E l^2 \,, \end{aligned} \tag{2.56}$$

where we have used the formula (2.10) for E. Hence the LRL vector yields only one additional conserved quantity.

To understand the significance of this conserved quantity, we measure the azimuth φ relative to $\boldsymbol{A}$ with $\boldsymbol{x} \cdot \boldsymbol{A} = r|\boldsymbol{A}| \cos\varphi$ ($\boldsymbol{A}$ lies in the orbital plane). On the other hand, we can compute $\boldsymbol{x} \cdot \boldsymbol{A}$ directly from (2.51) and obtain the result

$$\boldsymbol{x} \cdot \boldsymbol{A} = \mu(\boldsymbol{x} \times \dot{\boldsymbol{x}}) \cdot \boldsymbol{L} - GM\mu^2 r = l^2 - GM\mu^2 r \,. \tag{2.57}$$

Thus we obtain (purely algebraically) the trajectory

$$r = \frac{l^2}{GM\mu^2 + |\boldsymbol{A}| \cos\varphi} = \frac{d}{1 + \varepsilon \cos\varphi} \tag{2.58}$$

with d as in (2.33) and the eccentricity

$$\varepsilon = \frac{|\boldsymbol{A}|}{GM\mu^2} \tag{2.59}$$

proportional to the length of $\boldsymbol{A}$. The direction of $\boldsymbol{A}$ is the additional conserved quantity. Since in the parametrization (2.58) φ corresponds to the perihelion, the vector $\boldsymbol{A}$ points from the origin to the perihelion. The conservation of $\boldsymbol{A}$ therefore corresponds to the absence of perihelion precession.

From the conservation of $\boldsymbol{A}$ we can moreover determine in a simple way the trajectory of $\boldsymbol{p}(t) = \mu\dot{\boldsymbol{x}}(t)$. Since $\boldsymbol{A}$ lies in the orbital plane, we have $\boldsymbol{L} \cdot \boldsymbol{p} = 0$. Computing

$$\begin{aligned} \boldsymbol{L} \times \boldsymbol{A} &= \boldsymbol{L} \times (\boldsymbol{p} \times \boldsymbol{L}) - GM\mu^2 \, \boldsymbol{L} \times \frac{\boldsymbol{x}}{r} \\ &= \big(l^2 \boldsymbol{p} - (\boldsymbol{L} \cdot \boldsymbol{p})\boldsymbol{L}\big) - GM\mu^2 \, \boldsymbol{L} \times \frac{\boldsymbol{x}}{r} \\ &= l^2 \boldsymbol{p} - GM\mu^2 \, \boldsymbol{L} \times \frac{\boldsymbol{x}}{r} \end{aligned}$$

[5] In fact, each term in (2.51) is separately perpendicular to $\boldsymbol{L}$.

permits us to solve for $\boldsymbol{p}$ with the result

$$\boldsymbol{p} = \frac{1}{l^2}\,\boldsymbol{L} \times \boldsymbol{A} + \frac{GM\mu^2}{l}\frac{\boldsymbol{L}}{l} \times \frac{\boldsymbol{x}}{r}. \tag{2.60}$$

Now $(\boldsymbol{L}/l) \times (\boldsymbol{x}/r)$ is simply the unit vector in the orbital plane that is perpendicular to $\boldsymbol{x}$ (in the sense of the orbital motion). Hence $\boldsymbol{p}(t)$, over the course of one revolution, describes a circle about $\boldsymbol{L} \times \boldsymbol{A}/l^2$ of radius $GM\mu^2/l$.

Chapter 3

Selected Three-Body Problems

In contrast to the two-body problem, which we treated in Chapter 2, the three-body problem is no longer integrable. However, one can find special solutions (approximately, in part).

3.1 Equilibrium Positions and Their Stability

The Sun and Jupiter are the most massive bodies in the Solar System. We wish to study the motion of a light object (an asteroid) under their influence. To do so, we assume that the Sun S and Jupiter J move on circular orbits with angular velocity $\omega = 2\pi/T_{\mathrm{J}}$ about their common center of mass 0. The distance $R = a_{\mathrm{J}}$ between the two bodies remains constant in time. The asteroid A is attracted by the gravitational force of the Sun and Jupiter but is assumed to exert no influence on the motion of S and J. Because of this approximation, this formulation is also called the *restricted three-body problem*. As units we choose

$$\omega = 1, \qquad R = 1, \qquad G = 1\ . \tag{3.1}$$

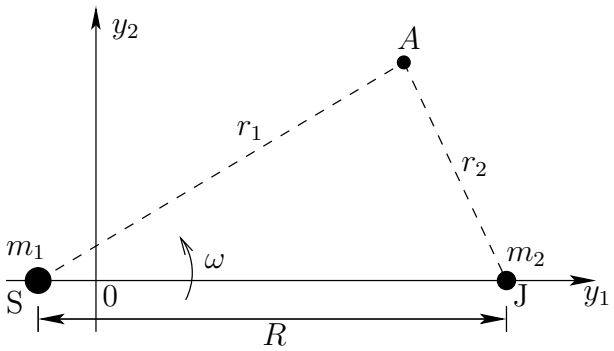


Figure 3.1: Orbit of an asteroid A under the influence of the Sun and Jupiter.

This fixes the three mechanical units (length, time, mass) as follows:

$$\begin{aligned}
&\text{Length:} && 1 \mathrel{\hat{=}} a_{\mathrm{J}} \approx 5.2\, a_{\mathrm{E}} = 7.8 \times 10^8 \text{ km},\\
&\text{Time:} && 1 \mathrel{\hat{=}} \frac{T_{\mathrm{J}}}{2\pi} \approx 1.9 \text{ years},\\
&\text{Mass:} && 1 \mathrel{\hat{=}} \frac{4\pi^2 a_{\mathrm{J}}^3}{G T_{\mathrm{J}}^2} \overset{(2.40)}{\approx} 1.0 M_{\mathrm{S}} = 2.0 \times 10^{30} \text{ kg}.
\end{aligned}$$

Since the y-system defined in the figure 3.1 is rotating, the equations of motion (1.54) must be written in an accelerated reference frame. In the center-of-mass system, Jupiter has distance $r_{\mathrm{J}} = Rm_1/(m_1+m_2)$ from the origin, while the distance of S is $r_{\mathrm{S}} = Rm_2/(m_1+m_2)$. Because S and J are in equilibrium, the gravitational force and the centrifugal force balance at their centers $(-r_{\mathrm{S}}\boldsymbol{e}_1,\, r_{\mathrm{J}}\boldsymbol{e}_1)$); i.e.,

$$\frac{G}{R^2} m_1 m_2 = \frac{m_1 m_2}{m_1+m_2} R\omega^2 \,. \tag{3.2}$$

holds. In the chosen units we obtain the relation[1]

$$m_1 + m_2 = 1\,, \tag{3.3}$$

and hence S and J have coordinates $(-m_2, 0, 0)$, $(m_1, 0, 0)$.

Since all forces on A are proportional to its mass m, this mass drops out of the equations of motion for A. We may therefore measure all forces by accelerations. With these preliminaries, the accelerations acting on body A are:

$$\begin{aligned}
&\text{Gravitational accel.:} && \boldsymbol{G} = -\frac{m_1}{r_1^3}(y_1 + m_2, y_2, y_3) - \frac{m_2}{r_2^3}(y_1 - m_1, y_2, y_3),\\
&\text{Centrifugal accel.:} && \boldsymbol{Z} = (y_1, y_2, 0),\\
&\text{Coriolis acceleration:} && \boldsymbol{C} = 2(\dot{y}_2, -\dot{y}_1, 0).
\end{aligned} \tag{3.4}$$

We seek equilibrium solutions with $\boldsymbol{y}$ = constant. In this case the Coriolis force vanishes and we obtain the conditions ($\boldsymbol{G} + \boldsymbol{Z} = 0$)

$$\begin{aligned}
&\Big(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} - 1\Big) y_1 + m_1 m_2 \Big(\frac{1}{r_1^3} - \frac{1}{r_2^3}\Big) = 0\,, \\
&\Big(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} - 1\Big) y_2 = 0\,, \qquad y_3 = 0\,.
\end{aligned} \tag{3.5}$$

For the second equation there are two possibilities. Either $y_2 = 0$ (Euler's special case) or

$$\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} = 1 \qquad \text{(Lagrange's special case)}. \tag{3.6}$$

In the *Euler case* A lies on the y_1-axis. On this axis the net acceleration $\boldsymbol{G} + \boldsymbol{Z}$ is directed as follows

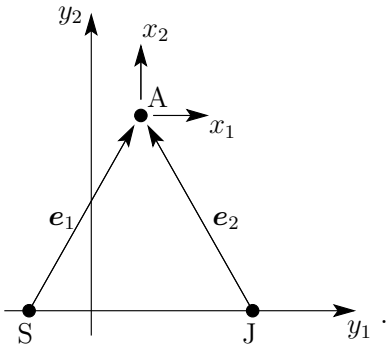
[1] This result also follows directly from Kepler's third law (2.40).

Thus there are three equilibrium positions ($\times$ in the diagram), which we shall not determine further.

In the *Lagrange case* it follows from (3.5) and (3.6)

$$r_1 = r_2 = 1\,, \tag{3.7}$$

i.e., SJA forms an equilateral triangle in the 12-plane;



Hence there are two Lagrange equilibrium positions (with $y_2 > 0$ or $y_2 < 0$), whose stability we now investigate.

For the stability analysis we consider small displacements $\boldsymbol{x} \ll 1$ from the equilibrium $\boldsymbol{x} = 0$; i.e., we set $\boldsymbol{y} = \left(\frac{1}{2} - m_2, \sqrt{3}/2, 0\right) + \boldsymbol{x}$ and linearize the gravitational force $\boldsymbol{G}$; $\boldsymbol{Z}$ is already linear. Because $\partial_{\boldsymbol{y}} r_i^{-3}\big|_{\mathrm{A}} = -3\boldsymbol{e}_i$ with

$$\boldsymbol{e}_1 = \frac{1}{2}(1, \sqrt{3}, 0)\,, \qquad\qquad \boldsymbol{e}_2 = \frac{1}{2}(-1, \sqrt{3}, 0) \tag{3.8}$$

in linear approximation

$$r_i^{-3} = 1 - 3\boldsymbol{e}_i \cdot \boldsymbol{x}. \tag{3.9}$$

Thus we obtain the linearized acceleration ($m_1 + m_2 = 1$)

$$\begin{aligned}
\boldsymbol{G} &\approx -m_1(1 - 3\boldsymbol{e}_1 \cdot \boldsymbol{x})\Big(\frac{1}{2} + x_1, \frac{\sqrt{3}}{2} + x_2, x_3\Big) \\
&\quad - m_2(1 - 3\boldsymbol{e}_2 \cdot \boldsymbol{x})\Big(-\frac{1}{2} + x_1, \frac{\sqrt{3}}{2} + x_2, x_3\Big) \\
&\approx \boldsymbol{G}^0 + \frac{3}{2} m_1 \boldsymbol{e}_1 \cdot \boldsymbol{x}\,(1, \sqrt{3}, 0) + \frac{3}{2} m_2 \boldsymbol{e}_2 \cdot \boldsymbol{x}\,(-1, \sqrt{3}, 0) - (x_1, x_2, x_3) \\
&\equiv \boldsymbol{G}^0 + (G_1, G_2, G_3)\,;
\end{aligned} \tag{3.10}$$

where $\boldsymbol{G}^0 = (\frac{m_2-m_1}{2}, -\frac{\sqrt{3}}{2}, 0)$ are the zeroth-order terms and

$$\begin{aligned} G_1 &= \Big(\frac{3}{4} - 1\Big)x_1 + \frac{3\sqrt{3}}{4}(m_1 - m_2)x_2\,, \\ G_2 &= \frac{3}{4}\sqrt{3}(m_1 - m_2)x_1 + \Big(\frac{9}{4} - 1\Big)x_2\,, \\ G_3 &= -x_3 \end{aligned} \tag{3.11}$$

are the first-order corrections in $\boldsymbol{x}$. Further,

$$\boldsymbol{Z} = \boldsymbol{Z}^0 + (x_1, x_2, 0)\,, \qquad \boldsymbol{C} = 2(\dot{x}_2, -\dot{x}_1, 0)\,, \tag{3.12}$$

where $\boldsymbol{Z}^0 = (\frac{1}{2} - m_2, \frac{\sqrt{3}}{2}, 0)$ again represents the zeroth-order contribution. By construction $\boldsymbol{G}^0 + \boldsymbol{Z}^0 = 0$ (equilibrium position), and we obtain the (linearized) equations of motion

$$\begin{aligned} \ddot{x}_1 &= \frac{3}{4}x_1 + \frac{3\sqrt{3}}{4}(m_1 - m_2)x_2 + 2\dot{x}_2, \\ \ddot{x}_2 &= \frac{3\sqrt{3}}{4}(m_1 - m_2)x_1 + \frac{9}{4}x_2 - 2\dot{x}_1, \\ \ddot{x}_3 &= -x_3\,. \end{aligned} \tag{3.13}$$

The x_3-motion is decoupled and executes harmonic oscillation with frequency $\omega = 1$ (in synchrony with the circular motion of S and J).

To solve the remaining two equations we make the exponential ansatz

$$x_k(t) = a_k\, e^{i\lambda t}\,, \qquad (k = 1, 2)\,. \tag{3.14}$$

Since the coefficients of the linear differential equation (3.13) are real, the real and imaginary parts of the complex solution (3.14) give the desired real solutions. Substituting the ansatz yields the homogeneous system

$$a_1\Big(\lambda^2 + \frac{3}{4}\Big) + a_2\Big(\frac{3\sqrt{3}}{4}(m_1 - m_2) + 2i\lambda\Big) = 0, \tag{3.15}$$

$$a_1\Big(\frac{3\sqrt{3}}{4}(m_1 - m_2) - 2i\lambda\Big) + a_2\Big(\lambda^2 + \frac{9}{4}\Big) = 0\,. \tag{3.16}$$

For a nontrivial solution $(a_1, a_2) \neq 0$ to exist, its determinant must vanish; i.e.,

$$\Big(\lambda^2 + \frac{3}{4}\Big)\Big(\lambda^2 + \frac{9}{4}\Big) = \frac{27}{16}(m_1 - m_2)^2 + 4\lambda^2\,. \tag{3.17}$$

After a few elementary manipulations using $(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2 = 1 - 4m_1m_2$ we can bring this condition into the form

$$\Big(\lambda^2 - \frac{1}{2}\Big)^2 = \frac{1}{4}(1 - 27m_1m_2) \tag{3.18}$$

For $27m_1m_2 < 1$ or, in general mass units, for

$$\frac{m_1 m_2}{(m_1+m_2)^2} < \frac{1}{27} \tag{3.19}$$

there are four distinct real solutions $\pm\lambda_1$, $\pm\lambda_2$ to (3.18) and the general solution of (3.13) is an arbitrary superposition of the four normal modes (3.14). Since then all λ_i are real, these solutions behave oscillatorily and remain bounded. At least to linear order, the equilibrium position is therefore stable if (3.19) holds. If the opposite inequality ($27m_1m_2 > 1$) holds, complex eigenfrequencies appear (including always some with $\operatorname{Im}\lambda < 0$). Hence there are exponentially growing solutions and the equilibrium position $\boldsymbol{x} = 0$ is unstable.

The stability condition (3.19) can be rewritten as a condition on the mass ratio $r = m_1/m_2$; $r > 1$, since without loss of generality $m_1 > m_2$. Inserting $m_1 = r\,m_2$ directly yields the condition $r/(1+r)^2 < 1/27$. Solving for r we obtain the result

$$r = \frac{m_1}{m_2} > \frac{1}{2}(25 + 3\sqrt{69}) \approx 25.0. \tag{3.20}$$

For the Sun-Jupiter system the inequality (3.20) is certainly satisfied ($m_\mathrm{J} \approx 10^{-3} m_\mathrm{S}$). In fact, there are numerous asteroids in the region of the Lagrange equilibrium positions, the so-called *Trojans*.[2] In the Lagrange equilibrium position relative to the Earth and Moon, the positioning of Tracking and Data Relay satellites is planned.

3.2 Motion of the Moon

The Moon M revolves about the Earth E in an orbital plane slightly inclined to the ecliptic E (the orbital plane of the Earth about the Sun S). As shown in figure 3.2, we introduce the auxiliary plane Σ. With these definitions we can distinguish the following events for the motion of the Moon:

$\mathrm{M} \in \Sigma^+$,	Full Moon (Moon in opposition to the Sun),
$\mathrm{M} \in \Sigma^-$,	New Moon (Moon in conjunction with the Sun),
$\mathrm{M} \in E \cap \Sigma^+$,	Lunar eclipse,
$\mathrm{M} \in E \cap \Sigma^-$,	Solar eclipse.

One can now distinguish two periods for the Moon's orbit about the Earth:

$$\begin{aligned} 2\pi\mu &= \text{Synodic month} \\ &= \text{Period between successive, identically directed} \\ &\quad \text{Passages through } \Sigma \text{ (from new Moon to new Moon)} \\ &= 29.53059 \text{ days;} \end{aligned}$$

[2] Without proof: For sufficiently small values of the ratio (3.19) stability also holds outside the linear approximation. The Euler equilibrium positions, by contrast, are obviously unstable.

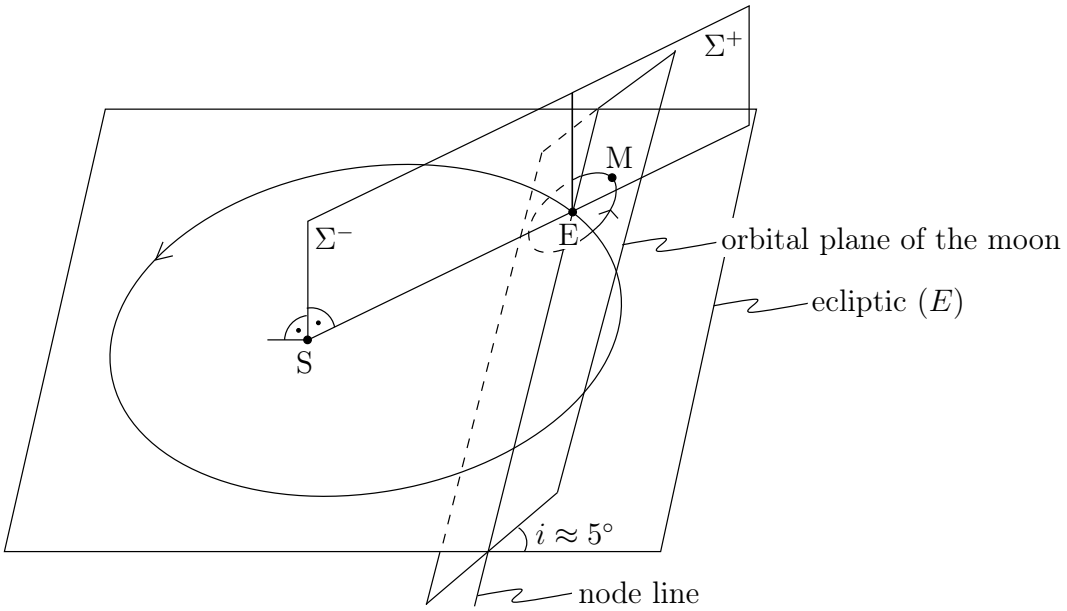


Figure 3.2: Orbit of the Moon relative to the Earth and Sun. The orbital plane of the Moon is inclined by $i \approx 5°$ to the ecliptic. We further define the (auxiliary) plane Σ, which is perpendicular to the ecliptic and connects the Sun with the Earth. We subdivide this plane into two regions: the points lying between the Earth and the Sun (conjunction) we denote by Σ^-, the points located (as seen from the Earth) in the opposite direction to the Sun (opposition) by Σ^+.

$2\pi\tilde{\mu}$ = Draconic month
= Period between successive, identically directed
Passages through the ecliptic (from ascending node to ascending node)
(i.e., from ascending to ascending)
= 27.21222 days.

The two periods satisfy the ratio

$$r = \frac{\mu}{\tilde{\mu}} = 1.08520\,. \tag{3.21}$$

This ratio can be explained almost entirely kinematically. If we project the lunar orbit onto the ecliptic, we obtain approximately (since i is small) a circular orbit. Comparing the position after one synodic month (after time $2\pi\mu$), we obtain from figure 3.3 the relation

$$(2\pi\mu)\tilde{\omega} = 2\pi + (2\pi\mu)\omega_0 \tag{3.22}$$

with $\tilde{\omega} = \tilde{\mu}^{-1}$, where the extra term on the right arises from the Earth's motion about the Sun with angular velocity $\omega_0 = J^{-1}$, $2\pi J = 1$ years $= 365.242$ days. Hence

$$\frac{\mu}{\tilde{\mu}} = 1 + \frac{\mu}{J} = 1.08085 \approx 1 + \frac{1}{12.3}\,. \tag{3.23}$$

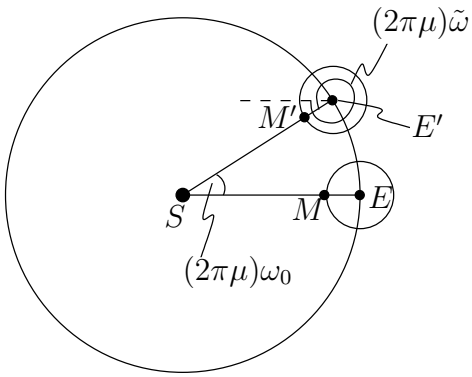


Figure 3.3: Projection of figure 3.2 onto the ecliptic. Because i is very small, the motion of the Moon (almost) describes a circular orbit.

The deviation from the observed value (3.21) suggests a slow rotation of the lunar orbital plane. We shall return to this point later.

One can also understand the kinematic ratio in (3.23) in a simple way: Because the Earth moves about the Sun, the plane Σ "runs away" from the Moon. Since the plane Σ rotates once in a year, we find that in this time the Moon traverses E times more than Σ, $N_E = N_\Sigma + 1$. By definition we have $N_E = J/\tilde{\mu}$ and $N_\Sigma = J/\mu$ and therefore

$$\frac{\mu}{\tilde{\mu}} = \frac{N_E}{N_\Sigma} = 1 + \frac{1}{N_\Sigma} = 1 + \frac{\mu}{J}. \tag{3.24}$$

3.2.1 Eclipse Cycles

The ratio (3.21) determines the cadence of solar eclipses (or lunar eclipses): Starting from an exact conjunction SME (solar eclipse), the next solar eclipse can recur only after the time

$$\Delta t = 2\pi\mu p \tag{3.25}$$

with $p \in \mathbb{Z}$, since only then $M \in \Sigma^-$. For a solar eclipse to occur, the Moon must also lie in the ecliptic. Hence one of the conditions

$$2\pi\mu p = 2\pi\tilde{\mu} q, \qquad \text{(in the same node)}, \tag{3.26}$$

or

$$2\pi\mu p = 2\pi\tilde{\mu}\left(q + \frac{1}{2}\right), \qquad \text{(in the opposite node)}, \tag{3.27}$$

must also be satisfied. Thus the determination of eclipses reduces to finding rational approximations so that[3]

$$r = \frac{\mu}{\tilde{\mu}} \approx \frac{q(+\frac{1}{2})}{p}. \tag{3.28}$$

[3] One obtains the same result for the determination of a further lunar eclipse, starting from a perfect opposition.

Because of the finite extent of S, E, and M, these conditions need only be satisfied within a "tolerance" $\Delta \approx \sin i = 0.087$,[4] i.e.,

$$\operatorname{dist}\Big(p\,r, \left\{ \begin{array}{c} \mathbb{Z} \\ \mathbb{Z}+\frac{1}{2} \end{array} \right\}\Big) < \Delta/2. \tag{3.29}$$

Because $r \approx 1 + 1/12$, see eq. (3.23), after 12 synodic months (=, a lunar year) an eclipse recurs in the same node. This periodicity is not exact; already after

$$\frac{\Delta}{|12r - 13|} \approx 3.88 \tag{3.30}$$

cycles, or the whole part thereof, the tolerance is exhausted and the periodicity breaks down.

On the other hand, starting from an exact conjunction (opposition) after time $\Delta t = 2\pi\mu(p + 1/2)$ an opposition (conjunction) recurs if

$$2\pi\mu\left(p + \frac{1}{2}\right) = 2\pi\tilde{\mu}q, \qquad \text{(in the same node)}, \tag{3.31}$$

or

$$2\pi\mu\left(p + \frac{1}{2}\right) = 2\pi\tilde{\mu}\left(q + \frac{1}{2}\right), \qquad \text{(in the opposite node)}. \tag{3.32}$$

Hence in this case we need a rational approximation

$$r = \frac{\mu}{\tilde{\mu}} \approx \frac{q(+1)}{2p+1} \tag{3.33}$$

with odd denominator.

Good rational approximations to the (irrational) number $2r$ can be found by continued fractions.[5] Among others we obtain the approximations

$$2r = 2, \frac{11}{5}, \frac{13}{6}, \frac{102}{47}, \frac{191}{88}, \frac{293}{135}, \frac{484}{223}. \tag{3.34}$$

With (3.28) we obtain the following eclipse cycles (from solar eclipse to solar eclipse, or from lunar eclipse to lunar eclipse):

q:	p:	Name:	Δt/ years	q:	p:	Name:	Δt/ years
Moon in the same node:				*Moon in the opposite node:*			
13	12	Lunar year	0.970	6	6	Semester	0.485
51	47	Octon	3.800	95	88	Tzolkinex	7.115
242	223	Saros	18.030	146	135	Tritos	10.915

[4] More information can be found at www.math.ualberta.ca/pi/issue6/page17-18.pdf.

[5] We treat $2r$, since the denominator in (3.28) is generally half-integer. The continued-fraction expansion of $2r$ is given by $[2; 5, 1, 6, 1, 1, 1, 1]$. This corresponds to the convergents $2r = 2, (11/5), 13/6, (89/41), 102/47, 191/88, 293/135, 484/223$, see de.wikipedia.org/wiki/Kettenbruch.

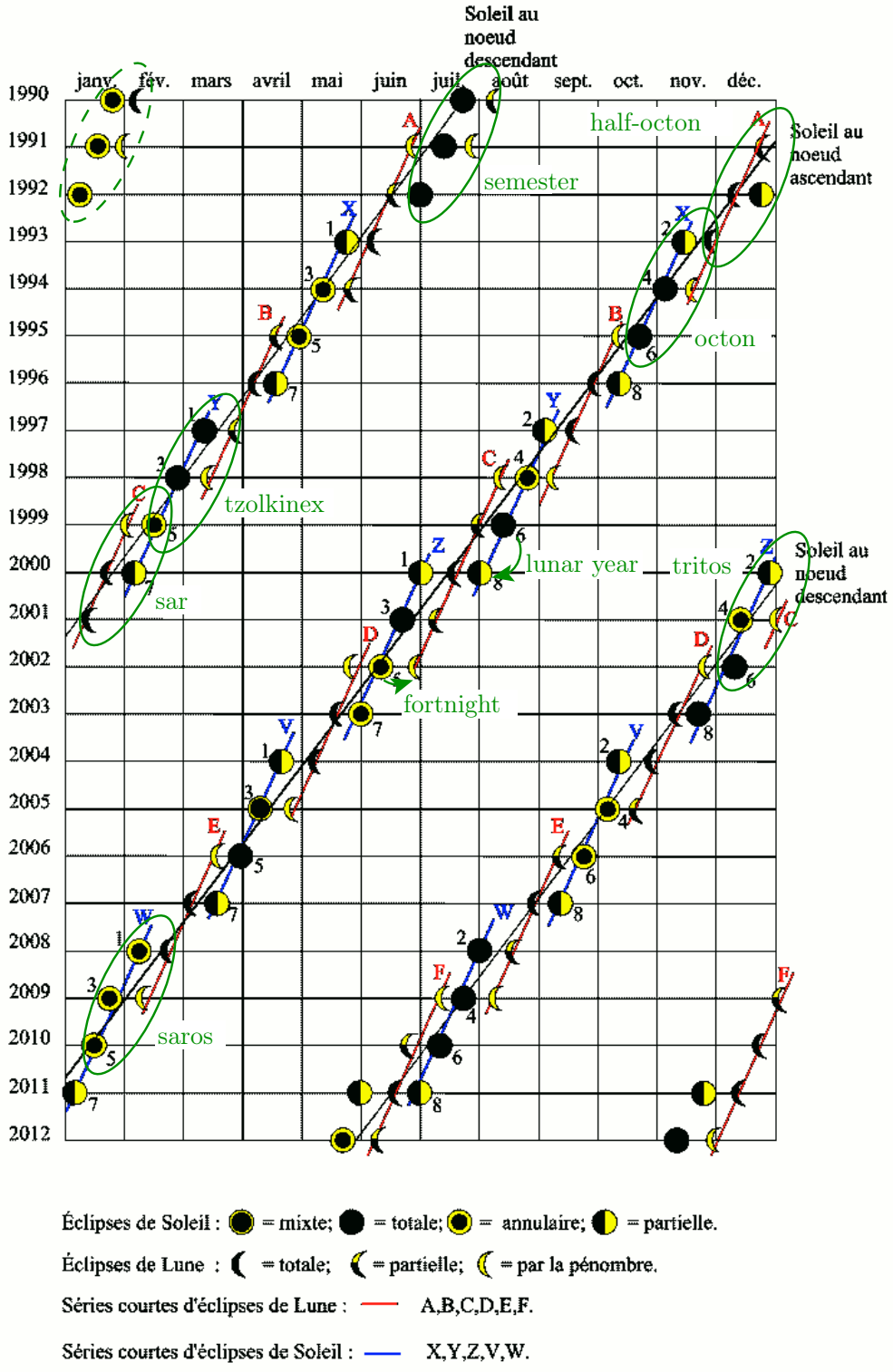


Figure 3.4: Canon of eclipses, adapted from www.imcce.fr (Patrick Rocher); starting from the situation at top left (green dashed), with three solar and two lunar eclipses, one obtains the same configuration (up to a small error) after certain times (green outlined).

Moreover, from (3.28) we obtain for the transition from a solar eclipse to a lunar eclipse (or vice versa) the cycles[6]

q:	p:	Name:	$\Delta t/$ years	q:	p:	Name:	$\Delta t/$ years
Moon in the same node:				*Moon in the opposite node:*			
121	111	Sar (half Saros)	9.015	0	0	2 weeks	0.040
				25	23	half Octon	1.900

In figure 3.4 we have illustrated the various cycles. The most famous cycle is the *Saros* (cycle) ($223r = 241.999 \approx 242$) with period[7]

$$\Delta t = (2\pi\mu)223 \equiv 1 \textbf{ Saros} \approx 18 \text{ years} + 11 \text{ days} + 8 \text{ hours}, \tag{3.35}$$

already known to the Chaldeans. The cycle lasts more than 1200 years (over 70 eclipses) before it breaks down. The total solar eclipse of 11 August 1999 was visible in Central Europe and therefore repeated on 21 August 2017 after about 8 hours (i.e., over the USA).[8] Because of the time shift, the next solar eclipse in Europe (same Saros cycle) will recur only after 3 Saros = 1 *Exeligmos* on 12 September 2053. This eclipse will graze Gibraltar and otherwise be visible only in North Africa.

3.2.2 Restricted Three-Body Problem

We now wish to determine the ratio (3.21) theoretically and as accurately as possible as a function of μ/J. To do so we treat the system S, E, M (with masses $M \gg m \gg m'$) as a restricted three-body problem, see Sec. 3.1; i.e., we regard the Moon as a body in the external gravitational field of the Sun and Earth. The Earth describes a circular orbit (radius R) about the Sun. Conversely, the Sun describes a circular orbit in the rotating frame of the Earth with origin E. We choose the units[9]

$$\omega = J = 1, \qquad m = 1, \qquad G = 1. \tag{3.36}$$

Thus the synodic month becomes

$$\mu \approx \frac{29.53 \text{ days}}{1 \text{ years}} = 0.08085\,. \tag{3.37}$$

In the two-body system Sun-Earth the gravitational and centrifugal forces must balance, as in (3.2),

$$G\frac{mM}{R^2} = \frac{mM}{M+m}R\omega^2 \approx mR\omega^2. \tag{3.38}$$

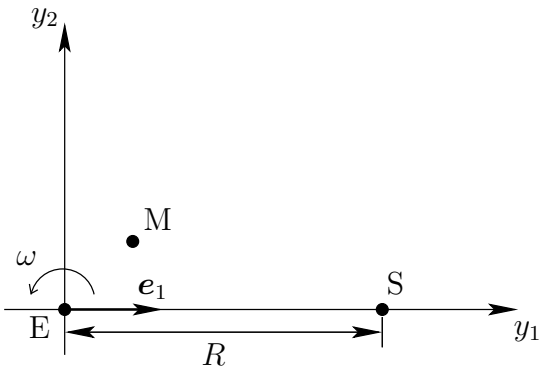
In our units we therefore obtain the relation

[6] Only approximations $r \approx q/p$ with p odd yield an entry.

[7] In this formula (and only here) 18 years refers to the time of $(18 \cdot 365 + 4)$ days. The four extra days correspond to the (minimal) number of leap years in the 18 years.

[8] The corresponding events are catalogued as solar eclipses No. 21 and 22 in the 145th Saros cycle. The discrepancy $11 + 11 = 22 \neq 21$ in the date arises because there are 5 leap years (2000, 2004, 2008, 2012, 2016) in the 18 years.

[9] With these the three mechanical units are given by 2.2×10^6 km, 58 days and 6.0×10^{24} kg.



$$\boldsymbol{e}_1 = (1, 0, 0)$$

Figure 3.5: Accelerated coordinate system for the restricted three-body problem Sun (S), Earth (E), Moon (M). The system rotates with angular velocity $\omega = J^{-1}$ about E.

$$M = R^3. \tag{3.39}$$

We choose the accelerated coordinate system $\boldsymbol{y}$, as shown in Figure 3.5. The following accelerations (forces divided by m') act on the Moon. The *gravitational accelerations* from the Earth

$$\boldsymbol{G}_E = \frac{\partial}{\partial \boldsymbol{y}} \frac{1}{|\boldsymbol{y}|} = -\frac{1}{|\boldsymbol{y}|^3}\boldsymbol{y}\,, \tag{3.40}$$

and for the Sun

$$\begin{aligned} \boldsymbol{G}_S &= M \frac{\partial}{\partial \boldsymbol{y}} \frac{1}{|R\,\boldsymbol{e}_1 - \boldsymbol{y}|} \\ &= M\Big(\frac{\boldsymbol{e}_1}{R^2} + \frac{3(\boldsymbol{e}_1 \cdot \boldsymbol{y})\boldsymbol{e}_1 - \boldsymbol{y}}{R^3} + O(y^2)\Big) \\ &= R\,\boldsymbol{e}_1 + 3\,y_1\,\boldsymbol{e}_1 - \boldsymbol{y} + \dots\,. \end{aligned} \tag{3.41}$$

The linearization at $\boldsymbol{y}/R$ is appropriate because for the lunar orbit $|\boldsymbol{y}|/R \approx 1/390$ holds. The first term naturally compensates exactly the *transport acceleration*[10] (see (1.54))

$$-\boldsymbol{a} = -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times (-R\boldsymbol{e}_1)) = -R\,\boldsymbol{e}_1\,. \tag{3.42}$$

Furthermore, the *centrifugal acceleration*

$$\boldsymbol{Z} = (y_1, y_2, 0), \tag{3.43}$$

[10]The coordinate origin, which lies in the Earth, is moved around the Sun with angular velocity ω.

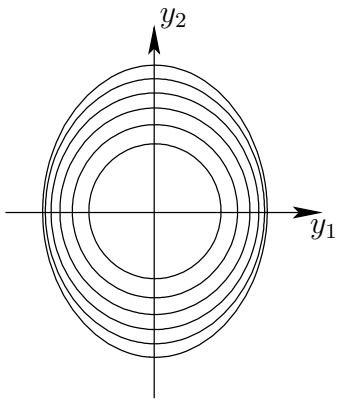


Figure 3.6: Family of possible lunar orbits in the ecliptic with $y_3 = 0$.

acts, which compensates the $1,2$-components of $-\boldsymbol{y}$ in (3.41). Finally, the *Coriolis acceleration* is given by

$$\boldsymbol{C} = 2(\dot{y}_2, -\dot{y}_1, 0)\,. \tag{3.44}$$

We thereby obtain the equations of motion of the Moon (with $r = |\boldsymbol{y}|$)

$$\begin{aligned} \ddot{y}_1 - 2\dot{y}_2 &= \Big(3 - \frac{1}{r^3}\Big)\, y_1\,, \\ \ddot{y}_2 + 2\dot{y}_1 &= -\frac{1}{r^3} y_2\,, \\ \ddot{y}_3 &= -\Big(\frac{1}{r^3} + 1\Big) y_3\,. \end{aligned} \tag{3.45}$$

Since the inclination of the lunar orbit to the ecliptic is small ($i \approx 5^\circ$) we first determine, following Hill (1878), periodic orbits in the -plane and then, as a perturbation thereof, the actual lunar orbit.

3.2.3 Periodic orbits

Let $y_3 \equiv 0$: we construct a one-parameter family of periodic solutions of (3.45) that circle counterclockwise around $\boldsymbol{y} = 0$, see Figure 3.6. Precisely one orbit (up to time translation) passes through each point of the (y_1, y_2)-plane near $\boldsymbol{y} = 0$. From (3.45) it is evident that with every solution $(y_1(t), y_2(t))$ also $(y_1(-t), -y_2(-t))$ and $(-y_1(-t), y_2(-t))$ are solutions. They arise by time reversal followed by reflection across the 1- or 2-axis, respectively.

For every initial point $(y_1(0) > 0, y_2(0) = 0)$ on the 1-axis one can always choose a vertical initial velocity $(\dot{y}_1(0) = 0, \dot{y}_2(0) > 0)$ so that the orbit intersects the 2-axis (at time $\pi\mu/2$) horizontally, i.e. $y_1(\pi\mu/2) = 0$, $\dot{y}_2(\pi\mu/2) = 0$ with $\mu > 0$. From the symmetries mentioned above one obtains the relations

$$\begin{aligned} y_1(t) &= y_1(-t)\,, & y_2(t) &= -y_2(-t)\,, \\ y_1(\pi\mu/2 + t) &= -y_1(\pi\mu/2 - t)\,, & y_2(\pi\mu/2 + t) &= y_2(\pi\mu/2 - t)\,. \end{aligned} \tag{3.46}$$

With these and the uniqueness of solutions for given initial conditions, one can extend every solution of $t \in [0, \pi\mu/2]$ to the whole $\mathbb{R}$. In particular, the orbits are periodic, $\boldsymbol{y}(t + 2\pi\mu) = \boldsymbol{y}(t)$, and oval (mirror-symmetric with respect to the 1- and 2-axes).

We choose as family parameter the orbital period $2\pi\mu$ (synodic month) and introduce the new time unit

$$\tau = \frac{t}{\mu} \tag{3.47}$$

so that the synodic month (in units of τ) is $= 2\pi$. Then (3.45) becomes, with $' = d/d\tau$,

$$\begin{aligned} y_1'' - 2\mu y_2' &= -\frac{\mu^2}{r^3} y_1 + \underline{3\mu^2 y_1}\,, \\ y_2'' + 2\mu y_1' &= -\frac{\mu^2}{r^3} y_2\,, \\ y_3'' &= -\mu^2 \Big(\frac{1}{r^3} + 1\Big) y_3\,. \end{aligned} \tag{3.48}$$

Omitting the underlined term (tidal forces), which we treat as a perturbation, the first two equations read

$$(y_1 + iy_2)'' + 2i\mu(y_1 + iy_2)' = -\frac{\mu^2}{r^3}(y_1 + iy_2).$$

The sought (now 2π-)periodic solution is a circular orbit

$$y_1(\tau) + iy_2(\tau) = r_0 e^{i\tau}\,, \qquad y_3(\tau) = 0 \tag{3.49}$$

with

$$r_0 = \frac{\mu^{\frac{2}{3}}}{(1 + 2\mu)^{\frac{1}{3}}}\,. \tag{3.50}$$

We can now justify a posteriori the neglect of the tidal perturbation in (3.48); if we substitute the solution (3.49) into the correction term, it is smaller by a factor of μ^2 than the leading term.

We now wish to take the influence of the tidal forces into account in perturbation theory at $\mu \ll 1$. Instead of $\boldsymbol{y}$ we introduce by

$$\begin{aligned} y_1 + iy_2 &= \mu^{2/3}(1 + 2\mu)^{-1/3} e^{i\tau} z\,, \\ y_3 &= \mu^{2/3}(1 + 2\mu)^{-1/3} \zeta \end{aligned}$$

new coordinates $(z, \zeta) \in \mathbb{C} \times \mathbb{R}$ ("variation of constants"). They are adapted to the above approximate solution (3.49) ($z(\tau) = 1, \zeta(\tau) = 0$) and correspond to a reference frame that rotates additionally by one revolution per month about the y_3-axis. Equations (3.48) now read

$$z'' + 2i(1 + \mu)z' = (1 + 2\mu)\Big(1 - \frac{1}{\rho^3}\Big) z + \underline{\frac{3}{2}\mu^2 \left(z + e^{-2i\tau} z^*\right)}\,, \tag{3.51}$$

$$\zeta'' = -\Big(\frac{1 + 2\mu}{\rho^3} + \mu^2\Big)\zeta\,, \tag{3.52}$$

with $\rho^2 = |z|^2 + \zeta^2$. Of course, without the perturbation term the previous approximate solution (i.e. $z(\tau) = 1$, $\zeta(\tau) = 0$) remains a solution.

We now set the sought 2π-periodic solution as a power series in μ [11]

$$z(\tau) = 1 + \sum_{k=2}^{\infty} \mu^k z_k(\tau) \;\equiv\; 1 + \tilde{z}(\tau) \tag{3.53}$$

with

$$\begin{aligned} z_k(\tau + 2\pi) &= z_k(\tau) \\ z_k(\tau)^* &= z_k(-\tau) \\ z_k\big(\frac{\pi}{2} + \tau\big)^* &= z_k\big(\frac{\pi}{2} - \tau\big)\,. \end{aligned} \tag{3.54}$$

To compute $z_2(\tau)$ we first expand $z\rho^{-3}$ in μ. To leading order $\zeta = 0$, and hence we obtain

$$\begin{aligned} \rho^2 = |z|^2 &= (1+\tilde{z})(1+\tilde{z}^*) = 1 + (z_2 + z_2^*)\mu^2 + O(\mu^4) \\ \rho^{-3} &= 1 - \frac{3}{2}(z_2 + z_2^*)\mu^2 + O(\mu^4) \\ z\rho^{-3} &= 1 + \tilde{z} - \frac{3}{2}(\tilde{z} + \tilde{z}^*) + O(\mu^4) = 1 - \Big(\frac{1}{2}z_2 + \frac{3}{2}z_2^*\Big)\mu^2 + O(\mu^4)\,. \end{aligned} \tag{3.55}$$

Comparing the terms $\propto \mu^2$ in (3.51) yields

$$z_2'' + 2iz_2' = \frac{3}{2}(z_2 + z_2^*) + \underline{\frac{3}{2}(1 + e^{-2i\tau})}\,. \tag{3.56}$$

The ansatz $z_2(\tau) = a_0 + a_- e^{-2i\tau} + a_+ e^{2i\tau}$ (with $a_{0,\pm} \in \mathbb{R}$ because of (3.54)) gives[12]

$$z_2(\tau) = -\frac{1}{2} - \frac{19}{16}e^{-2i\tau} + \frac{3}{16}e^{2i\tau}\,. \tag{3.57}$$

Equation (3.57) describes, to leading order, the oval shape of the periodic orbit. The difference of the "semi-axes" is given by

$$\Delta = \mu^2\Big(z_2\left(\frac{\pi}{2}\right) - z_2(0)\Big) \approx 2\mu^{2+2/3}\,. \tag{3.58}$$

For $\mu = 0.08085$ this is approximately $\Delta = 1.3 \times 10^{-2}$ (with $z = 1$ the Earth-Moon distance). In reality the change in distance is larger and the Moon's orbit is not perfectly periodic.

[11] The perturbation term is $O(\mu^2)$, so the expansion (3.53) begins with $k = 2$; (3.54) follows from (3.46).

[12] The solution $z_2^0(\tau) = ae^{i\tau} - 3a^*e^{-i\tau}$ of the homogeneous part of the equation (3.56) does not satisfy (3.54). It therefore does not contribute.

3.2.4 Variations of the orbit

The ansatz for orbits close to the periodic solution $(z(\tau), \zeta(\tau) \equiv 0)$ reads

$$\big(z(\tau) + \hat{z}(\tau), \hat{\zeta}(\tau)\big)\,, \tag{3.59}$$

with $\hat{z}$ and $\hat{\zeta}$ small. We substitute this ansatz into the equations of motion (3.51) and (3.52) and linearize in $\hat{z}, \hat{\zeta}$. One thereby obtains the *variational equations* of the periodic solution.

Since $\frac{\partial \rho}{\partial \zeta}\big|_{\zeta=0} = 0$, the variational equation of (3.51) contains no $\hat{\zeta}$, i.e. only $\hat{z}$ appears. In particular this equation is solved by $\hat{z} \equiv 0$, and we may consider a variation lying above the periodic orbit. The variation of (3.52) reads

$$\hat{\zeta}'' = -\Big(\frac{1+2\mu}{\rho^3} + \mu^2\Big)\hat{\zeta}\,, \tag{3.60}$$

where $\rho = \rho|_{\zeta=0}$ refers only to the periodic orbit. Inserting (3.57) into (3.55) leads to

$$\rho^{-3} = 1 + \frac{3}{2}\mu^2(1 + e^{2i\tau} + e^{-2i\tau}) + O(\mu^4)\,. \tag{3.61}$$

Hence (3.60) becomes

$$\begin{aligned}\hat{\zeta}'' &= -\Big(1 + 2\mu + \frac{3}{2}\mu^2(1 + 2\cos 2\tau) + \mu^2 + \ldots\Big)\hat{\zeta} \\ &= -\Big(1 + 2\mu + \frac{5}{2}\mu^2 + 3\mu^2\cos 2\tau + \ldots\Big)\hat{\zeta}\,.\end{aligned} \tag{3.62}$$

Neglecting first the term $\propto 3\mu^2 \cos 2\tau$, we obtain the differential equation of a harmonic oscillator in $\hat{\zeta}$ with angular frequency

$$r = \Big(1 + 2\mu + \frac{5}{2}\mu^2\Big)^{1/2} = 1 + \mu + \frac{3}{4}\mu^2 + \ldots\,. \tag{3.63}$$

in τ-units (with $\mu = 1$). In arbitrary units this is precisely the ratio (3.21) of synodic to draconic month. To first order $r = 1 + \mu$ in agreement with (3.23). For the Moon with $\mu = 0.08085$ (see (3.37)) one finds (including second-order correction)

$$r = 1.08575 \tag{3.64}$$

in very good agreement with (3.21).

The term $\propto 3\mu^2 \cos 2\tau$ has the consequence that $\hat{\zeta}(\tau)$ is no longer periodic but possesses a mean frequency r. Because of the oscillatory behavior r is affected by this only to order μ^4. By including further terms of the perturbation series in ζ, as well as in $\hat{z}(\tau) \neq 0$, the agreement with the observed value (3.21) improves still further; e.g. in the next order one finds

$$r = 1.08517\,. \tag{3.65}$$

Chapter 4

Lagrange Mechanics

We aim in this chapter to recast Newton's equations of motion into the Lagrangian formulation. This formulation offers several essential advantages. Conceptually, *all* coordinate systems are treated on an equal footing; the formalism remains valid even in accelerated and curved reference frames. As a consequence, fictitious forces and constraints are automatically accounted for. A further advantage is that the Lagrangian is a scalar, so one no longer needs to worry about the direction of forces. In a certain sense, this chapter is the core of the course. It is therefore worth studying the material carefully. In particular, we shall derive two of the most important results: the "*principle of Hamilton*" and "*Noether's theorem.*"

4.1 Configuration space and generalized coordinates

The reformulation of Newton's equations is especially useful for systems subject to (holonomic) *constraints*. In this viewpoint, part of the forces is described indirectly by restricting the possible motions of the mass points.

We introduce the relevant notions by means of three examples:

(A) The *simple pendulum:* The motion of a point mass swinging in a vertical plane is constrained by the thread's fixing the distance from the suspension point (*constraint*). The admissible positions of the pendulum are then given by

$$\boldsymbol{x} = l(t)\boldsymbol{e} \equiv \boldsymbol{x}(\boldsymbol{e}, t) \tag{4.1}$$

with $l(t)$ the (time-dependent) thread length. The *configuration space* of the pendulum is S^1 and its motion is determined by a function $\mathbb{R} \ni t \mapsto \boldsymbol{e}(t) \in S^1$. On the configuration space one may introduce the *generalized coordinate* $\theta \in K \subset \mathbb{R}$ with $\boldsymbol{e}(\theta) = (\cos\theta, \sin\theta)$. The open set K is called a *chart*. The map $\boldsymbol{e}(\theta)$ is not bijective, since $\boldsymbol{e}(2\pi) = \boldsymbol{e}(0)$. To obtain a bijective representation of the entire S^1 one needs at least two charts (an atlas). This subtlety is not important in this course.

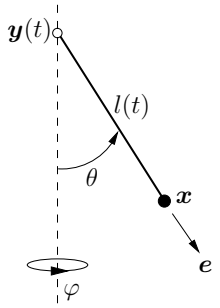


Figure 4.1: Spherical pendulum with variable suspension point $\boldsymbol{y}(t)$ and length $l(t)$.

(B) The *spherical pendulum:* The suspension point $\boldsymbol{y}(t)$ and the pendulum length $l(t)$ are prescribed functions of time t, cf. Figure 4.1. Under these constraints the admissible positions of the pendulum are

$$\boldsymbol{x} = \boldsymbol{y}(t) + l(t)\boldsymbol{e} \equiv \boldsymbol{x}(\boldsymbol{e}, t)\,, \tag{4.2}$$

where $\boldsymbol{e}$ is a point on the unit sphere S^2 (the configuration space of the pendulum). The motion of the pendulum is given by a function $t \mapsto \boldsymbol{e}(t) \in S^2$. On S^2 one may now introduce generalized coordinates; e.g. spherical coordinates θ, φ with

$$\boldsymbol{e}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,. \tag{4.3}$$

This maps the configuration space to a patch of the $\mathbb{R}^2$:

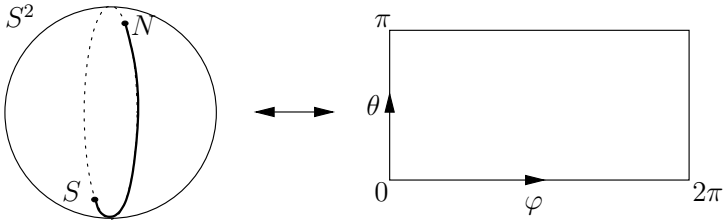


chart of S^2

The map is not bijective on the boundary of the rectangle (or on the zero meridian of S^2).

(C) A *bead on a rotating wire:* On a wire bent into a parabola and rotating about the 3-axis with angular speed ω sits a bead (point mass). The two constraints are met by the parametrization

$$\boldsymbol{x} = R(\omega t)\Big(r\boldsymbol{e}_1 + \frac{1}{2}\alpha r^2\boldsymbol{e}_3\Big) = R(\omega t)\Big(r, 0, \frac{1}{2}\alpha r^2\Big) \equiv \boldsymbol{x}(r, t) \tag{4.4}$$

with $R(\varphi)$ the rotation about the 3-axis by the angle φ and $r \in \mathbb{R}$ the generalized coordinate. The configuration space of the problem is $\{\boldsymbol{x}(r,t) \mid r \in \mathbb{R}\}$, i.e. isomorphic to $\mathbb{R}$.

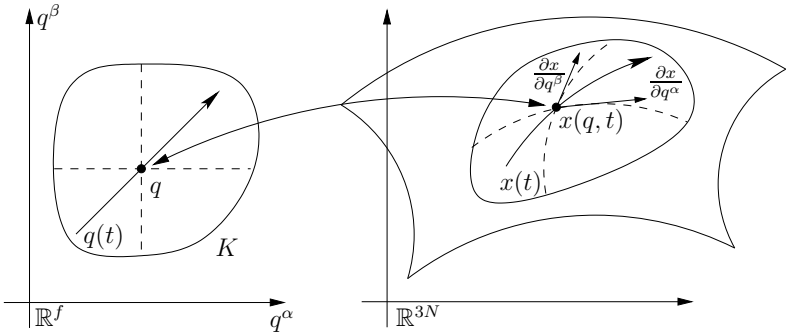


Figure 4.2: Mapping from the position coordinates (left) to the Cartesian coordinates (right). The tangent vectors $\partial x/\partial q^\alpha$ span in $\mathbb{R}^{3N}$ locally the vector space of virtual displacements.

In general we have a system of N particles whose configuration space in Cartesian coordinates $x = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \in \mathbb{R}^{3N}$ at time t forms a smooth f-dimensional manifold in $\mathbb{R}^{3N}$; constraints of this kind are called "*holonomic constraints*." The system then possesses f *degrees of freedom*.[1] We describe the system by generalized coordinates $q = (q^1, \dots, q^f)$; i.e. we have (locally) a chart $q \in K \subset \mathbb{R}^f$ and (for each $t \in \mathbb{R}$) a map

$$x \colon K \to \mathbb{R}^{3N}, \text{ with } q \mapsto x(q,t)\,, \tag{4.5}$$

which is differentiable in (q,t). We further require that the tangent map have rank f; i.e. the vectors

$$\frac{\partial x}{\partial q^\alpha} \in \mathbb{R}^{3N}\,, \qquad (\alpha = 1, \dots, f)\,, \tag{4.6}$$

must be linearly independent. Thus the tangent vectors (4.6) at every point (q,t) span a f-dimensional vector space, see Figure 4.2.

Every admissible (constraint-compatible) motion $x(t)$ of the system (within the chart K) is then induced by a function $t \mapsto q(t) \in K$ via

$$x(t) \equiv x(q(t), t) \tag{4.7}$$

To formulate the equations of motion of a system in the position coordinates $q = (q^1, \dots, q^f)$, we require the following notions.

Velocities, kinetic energy: With (4.7) the velocities $\dot{x} = (\dot{\boldsymbol{x}}_1, \dots, \dot{\boldsymbol{x}}_N)$ admissible at time t at position q are given by

$$\dot{x} = \sum_{\alpha=1}^{f} \frac{\partial x}{\partial q^\alpha}(q,t)\,\dot{q}^\alpha + \frac{\partial x}{\partial t}(q,t) \equiv \dot{x}(q,\dot{q},t) \tag{4.8}$$

[1] In Example (B) we have $N = 1$ and $f = 2$. Hence there are two degrees of freedom.

with arbitrary *generalized velocities* $\dot{q} = (\dot{q}^1, \ldots, \dot{q}^f) \in \mathbb{R}^f$. The velocities $\dot{x} \in \mathbb{R}^{3N}$ thus lie (necessarily) in the vector space spanned by $\partial x/\partial q^\alpha$ and $\partial x/\partial t$, but not necessarily in the tangent space.

With (4.8) $\dot{x}$ becomes a function of the *independent* variables[2]

$$(q, \dot{q}, t) \in K \times \mathbb{R}^f \times \mathbb{R}.$$

Hence the kinetic energy T can also be viewed as a new function

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i^2 \equiv T(q, \dot{q}, t) \tag{4.9}$$

By (4.8) one has

$$\frac{\partial \dot{x}}{\partial \dot{q}^\alpha} = \frac{\partial x}{\partial q^\alpha} \tag{4.10}$$

and thus

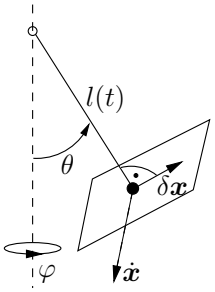
$$\frac{\partial T}{\partial \dot{q}^\alpha} = \sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \frac{\partial \dot{\boldsymbol{x}}_i}{\partial \dot{q}^\alpha} = \sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \frac{\partial \boldsymbol{x}_i}{\partial q^\alpha}. \tag{4.11}$$

Virtual displacements: At a point (q, t) the (directional) *derivative* of the function $x(q, t)$ at *fixed* t is a linear map $\mathbb{R}^f \ni \delta q \mapsto \delta x \in \mathbb{R}^{3N}$, defined by

$$\delta x = \sum_{\alpha=1}^{f} \frac{\partial x}{\partial q^\alpha} \, \delta q^\alpha. \tag{4.12}$$

The admissible vectors in the image δx are the tangent vectors at the point (q, t). One calls the admissible δx also the space of *virtual displacements* of the system from the configuration $x(q, t)$ at time t.

For Examples (A) and (C) the configuration space is one-dimensional. The space of virtual displacements is therefore simply the tangent to the circle (for (A)) or to the parabola (for (C)) at the point $x(q, t)$. For the spherical pendulum (Example (B)) $\delta \boldsymbol{x}$ corresponds to a variation of the position coordinates θ, φ at fixed time and hence is perpendicular to the string $\boldsymbol{e}$.



Comparing (4.12) with (4.8) shows that $\dot{x}$ is in general *not* a virtual displacement for time-dependent constraints.

[2] In the Lagrangian formalism it is essential to regard $(q, \dot{q}, t)$ as independent variables. Here $\dot{q}$ are, according to (4.8), the coordinates of the velocity vector in the tangent space while (q, t) fixes the point x at which the tangent space is formed.

Virtual work, generalized forces and impulses: Let $(\boldsymbol{F}_1, \ldots, \boldsymbol{F}_N)$ be the forces acting on the particles at the configuration $x(q,t)$. Their *virtual work* is defined as[3]

$$\delta A = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{x}_i. \tag{4.13}$$

Using (4.12) we can rewrite this relation as

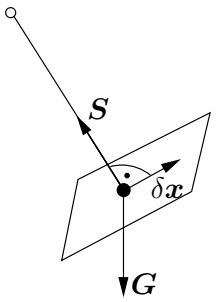
$$\delta A = \sum_{\alpha=1}^{f} K_\alpha \delta q^\alpha \equiv \langle K, \delta q \rangle \tag{4.14}$$

with the coefficients (*generalized forces*)

$$K_\alpha = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{x}_i}{\partial q^\alpha}(q,t). \tag{4.15}$$

By construction the expression (4.14) is independent of the choice of position coordinates.

The forces arising from the restriction (4.5) of the admissible configurations $x \in R^{3N}$ of the system are called *constraint forces*, the remaining ones *applied forces*. For Example (B) of the spherical pendulum the tension $\boldsymbol{S}$ is a constraint force and the gravitational force $\boldsymbol{G}$ an applied force. Observe that $\boldsymbol{S} \perp \delta\boldsymbol{x}$. *d'Alembert's principle* postulates that this statement holds in general: "Constraint forces are perpendicular to the virtual displacements and *do no virtual work*." Hence constraint forces do not contribute to the generalized forces K_α and need not be computed at all in the Lagrangian formalism. Although constraint forces do no virtual work, they can do *real work* under time-dependent constraints with $\boldsymbol{S} \cdot \dot{\boldsymbol{x}} \neq 0$ (even though $\boldsymbol{S} \cdot \delta\boldsymbol{x} = 0$).



Analog to the generalized forces K_α one also defines the *generalized momenta* p_α by

$$\sum_{i=1}^{N} \underbrace{m_i \dot{\boldsymbol{x}}_i}_{=\boldsymbol{p}_i} \cdot \delta \boldsymbol{x}_i = \sum_{\alpha=1}^{f} p_\alpha \, \delta q^\alpha \equiv \langle p, \delta q \rangle, \tag{4.16}$$

i.e.

$$p_\alpha = \sum_{i=1}^{N} \boldsymbol{p}_i \cdot \frac{\partial \boldsymbol{x}_i}{\partial q^\alpha} = \frac{\partial T}{\partial \dot{q}^\alpha}. \tag{4.17}$$

The last equation follows from (4.11). With (4.9) we thereby obtain the relation $p_\alpha = p_\alpha(q, \dot{q}, t)$.

[3]The virtual force δA is a map from the virtual displacement to the real numbers. Note that δA is in general not the derivative of a function $A(q,t)$; see also Chap. 4.3.

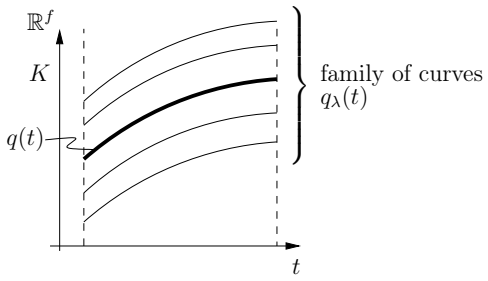


Figure 4.3: The path $q(t)$ is embedded in a family $q_\lambda(t)$ of curves so that $q_0(t) = q(t)$ (bold line).

Variation of a path: As a final preparation we introduce the concept of the variation of the path. The variation of a path generalizes the notion of the directional derivative (partial derivative with respect to a variable) to the "derivative" with respect to a function $\delta q^\alpha(t)$.

Given a path $q(t)$ we form an arbitrary 1-parameter family of curves $q_\lambda(t)$ such that $q_0(t) = q(t)$, see Figure 4.3. The *variation* of a function $F(q, \dot{q}, t)$ is then defined as

$$\delta F(t) = \frac{\partial}{\partial \lambda} F(q_\lambda(t), \dot{q}_\lambda(t), t)\Big|_{\lambda=0} \tag{4.18}$$

with

$$q^\alpha_\lambda(t) = q^\alpha(t) + \lambda\, \delta q^\alpha(t). \tag{4.19}$$

The f functions $\delta q^\alpha \colon \mathbb{R} \to K$ are here freely choosable. They correspond to the variations of $q^\alpha(t)$. We will later need the relation

$$\delta \dot{q}^\alpha(t) = \frac{\partial}{\partial \lambda} \dot{q}^\alpha_\lambda(t)\Big|_{\lambda=0} = \frac{d}{dt}\, \delta q^\alpha(t), \tag{4.20}$$

which links the variations $\delta \dot{q}^\alpha$ with the variations δq^α.

It is straightforward to verify that the variation of the position δx is precisely the virtual displacements from (4.12). Hence it follows that a variation δF of an arbitrary function $F(x, t)$ is a virtual displacement.

4.2 Equations of Motion

With these preliminaries we can formulate the equations of motion in a coordinate-free manner so that they remain valid in every reference frame.

Let $q(t)$ be a *mechanical path*, i.e. $x(q(t), t)$ is a solution of Newton's equations of motion

$$m_i \ddot{\boldsymbol{x}}_i = \boldsymbol{F}_i. \tag{4.21}$$

Then for *every* variation of a mechanical path

$$\frac{d}{dt}\langle p, \delta q\rangle \overset{(4.16)}{=} \frac{d}{dt}\sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \delta \boldsymbol{x}_i = \sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \delta \dot{\boldsymbol{x}}_i + \sum_{i=1}^{N} m_i \ddot{\boldsymbol{x}}_i \cdot \delta \boldsymbol{x}_i$$
$$= \delta\left(\frac{1}{2}\sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i^2\right) + \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{x}_i. \tag{4.22}$$

holds. Thus we have brought Newton's equations (4.21) for arbitrary position coordinates q into the form

$$\frac{d}{dt}\langle p, \delta q\rangle = \delta T + \delta A \tag{4.23}$$

The forces appear here only in the form of the virtual work. Since constraint forces do no virtual work, it suffices to determine the applied forces; henceforth we denote by $\boldsymbol{F}_i$ only the applied forces.

Starting from (4.23), we immediately obtain the explicit form

$$\sum_{\alpha=1}^{f}(\dot{p}_\alpha \delta q^\alpha + \underline{p_\alpha \delta \dot{q}^\alpha}) = \sum_{\alpha=1}^{f}\left(\frac{\partial T}{\partial q^\alpha}\delta q^\alpha + \underline{\frac{\partial T}{\partial \dot{q}^\alpha}\delta \dot{q}^\alpha} + K_\alpha \delta q^\alpha\right) \tag{4.24}$$

of the equations of motion. The underlined terms cancel according to (4.17). Because for fixed t the variations $\delta q^\alpha(t)$ are arbitrary, the prefactors must vanish, i.e.

$$\dot{p}_\alpha = \frac{\partial T}{\partial q^\alpha} + K_\alpha. \tag{4.25}$$

With the relation (4.17) we thereby obtain the equations of motion

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\alpha} - \frac{\partial T}{\partial q^\alpha} = K_\alpha, \qquad (\alpha = 1, \ldots, f). \tag{4.26}$$

for arbitrary position coordinates $q = (q^1, \ldots, q^f)$. These are f second-order differential equations for the functions $q^\alpha(t)$ given the generalized forces $K_\alpha = K_\alpha(q, \dot{q}, t)$. Note that here and in the following the total time derivative (d/dt) refers to differentiation along a path $q(t)$.

Remark: One may derive the equations of motion (4.26) also without invoking the concept of virtual displacements. With (4.9) and (4.11) we find

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\alpha} = \sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \frac{d}{dt}\frac{\partial \boldsymbol{x}_i}{\partial q^\alpha} + \sum_{i=1}^{N} m_i \ddot{\boldsymbol{x}}_i \cdot \frac{\partial \boldsymbol{x}_i}{\partial q^\alpha},$$
$$\frac{\partial T}{\partial q^\alpha} = \sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \frac{\partial \dot{\boldsymbol{x}}_i}{\partial q^\alpha} = \sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \frac{d}{dt}\frac{\partial \boldsymbol{x}_i}{\partial q^\alpha}. \tag{4.27}$$

Together with (4.15) and (4.21) this directly yields the final result (4.26).

4.3 Lagrangian Systems

The Lagrangian formalism becomes especially elegant when the applied forces $\boldsymbol{F}_i$ possess a potential $V(x,t)$ with $\boldsymbol{F}_i = -\partial_{\boldsymbol{x}_i} V(x,t)$; such a force field is called *conservative*. In this case one has

$$\delta A = -\sum_{i=1}^{N} \frac{\partial V}{\partial \boldsymbol{x}_i} \cdot \delta \boldsymbol{x}_i = -\sum_{\alpha=1}^{f} \frac{\partial V}{\partial q^\alpha} \delta q^\alpha, \tag{4.28}$$

so that $-\delta A$ is the variation of $V(q,t) \equiv V(x(q,t),t)$. Hence it follows immediately

$$K_\alpha = -\frac{\partial V}{\partial q^\alpha}, \tag{4.29}$$

so that $V(q,t)$ is the *potential* of the generalized forces. With the *Lagrangian*

$$L(q,\dot{q},t) = T(q,\dot{q},t) - V(q,t) \tag{4.30}$$

the *Euler-Lagrange equations*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0, \qquad (\alpha = 1,\ldots,f) \tag{4.31}$$

for a mechanical path follow from (4.26).

The equations of motion of a system with constraints can therefore generally be set up as follows:

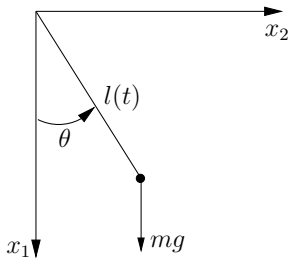
1. Write $T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i^2$, $V = V(x)$ (in an inertial frame) in Cartesian components without regard to the constraints.
2. Express the (admissible according to the constraints) configurations $x = (\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)$ by independent position coordinates $q = (q^1,\ldots,q^f)$. With $x = x(q,t)$ compute $\dot{x} = \dot{x}(q,\dot{q},t)$.
3. Substitute $x, \dot{x}$ into $L = T - V$.
4. Form the Euler-Lagrange equations (4.31).

These procedures will now be illustrated by a few examples.

Example 1: (see (A))

Planar pendulum with fixed suspension point and prescribed, time-dependent length $l(t)$. The position coordinate is θ. From

$$\boldsymbol{x} = l(\cos\theta, \sin\theta) = l\,\boldsymbol{e}(\theta)$$
$$\dot{\boldsymbol{x}} = \dot{l}(\cos\theta, \sin\theta) + l\dot{\theta}(-\sin\theta, \cos\theta)$$



one computes

$$T = \frac{m}{2}\left(\dot{l}^2 + \left(l\dot{\theta}\right)^2\right), \qquad V = -mgx_1 = -mgl\cos\theta\,,$$

and from this

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}\,, \qquad \frac{\partial L}{\partial \theta} = -mgl\sin\theta\,.$$

Thus the Euler-Lagrange equation (4.31)

$$\frac{d}{dt}(ml(t)^2\dot{\theta}) + mgl(t)\sin\theta = 0\,. \tag{4.32}$$

which is precisely the angular momentum theorem.

Remark: The string tension $\boldsymbol{S}$ as a constraint force is not taken into account in the formalism. However, it does work, since $\dot{\boldsymbol{x}}$ is not a virtual displacement in this example. Hence the energy $T + V$ is not conserved. One finds

$$\frac{d}{dt}(T+V) = m(\dot{l}\,\ddot{l} + l^2\dot{\theta}\ddot{\theta} + l\dot{l}\dot{\theta}^2) - mg\dot{l}\cos\theta + mgl\dot{\theta}\sin\theta\,. \tag{4.33}$$

Using the equation of motion (4.32) one obtains

$$\frac{d}{dt}(T+V) = m\dot{l}(\ddot{l} - l\dot{\theta}^2 - g\dot{l}\cos\theta)\,. \tag{4.34}$$

The right-hand side is precisely the power $\boldsymbol{S}\cdot\dot{\boldsymbol{x}}$ of the constraint force. Since we know that $\boldsymbol{S}$ points along the string, we can determine the string tension explicitly in this case and obtain

$$\boldsymbol{S} = m(\ddot{l} - l\dot{\theta}^2 - g\cos\theta)\boldsymbol{e}(\theta)\,. \tag{4.35}$$

The last term therefore corresponds to the counterforce of the gravitational force along the string. The first term is the counterforce of the guiding force and the second term the counterforce of the centrifugal force. All these forces must be balanced by the string tension so that the constraint $|\boldsymbol{x}| = l$ remains satisfied. These forces are not required in the Lagrangian formalism and hence are normally not even computed at all.

Example 2: (see (B))

We treat the spherical pendulum with $l(t) \equiv l$ and $\boldsymbol{y}(t) \equiv 0$. The position of the pendulum is given by

$$\boldsymbol{x} = l(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

Differentiating with respect to t yields the expression

$$\dot{\boldsymbol{x}} = l\dot{\theta}(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) + l\sin\theta\,\dot{\varphi}(-\sin\varphi, \cos\varphi, 0)$$

for the velocity vector. This gives the kinetic energy

$$T = \frac{1}{2}ml^2(\dot\theta^2 + \sin^2\theta\,\dot\varphi^2) \tag{4.36}$$

and the potential energy $V = -mgx_3 = -mgl\cos\theta$. The Lagrangian is

$$L = \frac{1}{2}ml^2(\dot\theta^2 + \sin^2\theta\,\dot\varphi^2) + mgl\cos\theta. \tag{4.37}$$

We compute

$$\frac{\partial L}{\partial\dot\theta} = ml^2\dot\theta, \qquad \frac{\partial L}{\partial\dot\varphi} = ml^2\sin^2\theta\,\dot\varphi, \qquad \frac{\partial T}{\partial\theta} = ml^2\sin\theta\cos\theta\,\dot\varphi^2.$$

The Euler-Lagrange equations then read

$$\frac{d}{dt}(ml^2\dot\theta) - ml^2\sin\theta\cos\theta\,\dot\varphi^2 + mgl\sin\theta = 0,$$
$$\frac{d}{dt}(ml^2\sin^2\theta\,\dot\varphi) = 0.$$

Example 3: (see (C)) For the bead on a rotating wire we obtain directly from (4.4) (note that $R^t\dot R\boldsymbol{x} = \boldsymbol{\omega}\times\boldsymbol{x}$ with $\boldsymbol{\omega} = \omega\boldsymbol{e}_3$)

$$\begin{aligned} R^t\dot{\boldsymbol{x}} &= \boldsymbol{\omega}\times\Big(r\boldsymbol{e}_1 + \frac{1}{2}\alpha r^2\boldsymbol{e}_3\Big) + (\boldsymbol{e}_1 + \alpha r\boldsymbol{e}_3)\dot r \\ &= \omega r\boldsymbol{e}_2 + (\boldsymbol{e}_1 + \alpha r\boldsymbol{e}_3)\dot r. \end{aligned}$$

Thus the kinetic energy is given by

$$T = \frac{1}{2}m\dot{\boldsymbol{x}}^2 = \frac{1}{2}m(R^t\dot{\boldsymbol{x}})^2 = \frac{1}{2}m\omega^2r^2 + \frac{1}{2}m(1+\alpha^2r^2)\dot r^2.$$

The potential energy due to gravity has the form $V(r) = mgx_3 = \frac{1}{2}mg\alpha r^2$. Hence we obtain

$$\frac{\partial L}{\partial\dot r} = m(1+\alpha^2r^2)\dot r, \qquad \frac{\partial L}{\partial r} = m(\omega^2 - \alpha g)r + m\alpha^2r\dot r^2.$$

The Euler-Lagrange equations take the form

$$\frac{d}{dt}\Big(m(1+\alpha^2r^2)\dot r\Big) - m(\omega^2 - \alpha g)r - m\alpha^2r\dot r^2 = 0.$$

Example 4: *Charged particle in an electromagnetic field*

From the homogeneous Maxwell equations[4]

$$\operatorname{div}\boldsymbol{B} = 0\,, \qquad \operatorname{curl}\boldsymbol{E} + \frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} = 0$$

[4]We use Gaussian units.

it follows directly that the magnetic $\boldsymbol{B}(\boldsymbol{x},t)$ and electric field $\boldsymbol{E}(\boldsymbol{x},t)$ can be expressed in terms of electromagnetic potentials $\varphi, \boldsymbol{A}$ with

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}\,, \qquad \boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\,. \tag{4.38}$$

The motion of a particle (in Cartesian coordinates $\boldsymbol{x} \in \mathbb{R}^3$) is determined by the Lagrangian

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) = \frac{1}{2} m \dot{\boldsymbol{x}}^2 - e\Big(\varphi(\boldsymbol{x},t) - \frac{\dot{\boldsymbol{x}}}{c} \cdot \boldsymbol{A}(\boldsymbol{x},t)\Big)\,, \tag{4.39}$$

where m is the mass and e the charge of the particle. Indeed one has

$$\frac{\partial L}{\partial x_k} = -e\frac{\partial \varphi}{\partial x_k} + \frac{e}{c}\sum_{i=1}^{3} \dot{x}_i \frac{\partial A_i}{\partial x_k}\,, \qquad \frac{\partial L}{\partial \dot{x}_k} = m\dot{x}_k + \frac{e}{c}A_k\,,$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} = m\ddot{x}_k + \frac{e}{c}\left(\frac{\partial A_k}{\partial t} + \sum_{i=1}^{3} \frac{\partial A_k}{\partial x_i}\,\dot{x}_i\right),$$

so that the Euler-Lagrange equations (4.31)

$$m\ddot{x}_k = e\underbrace{\Big(-\frac{\partial \varphi}{\partial x_k} - \frac{1}{c}\frac{\partial A_k}{\partial t}\Big)}_{E_k} + \frac{e}{c}\underbrace{\sum_{i=1}^{3} \dot{x}_i\Big(\frac{\partial A_i}{\partial x_k} - \frac{\partial A_k}{\partial x_i}\Big)}_{(\dot{\boldsymbol{x}} \times \boldsymbol{B})_k}\,,$$

agree with (1.16).

Example 5: *Double pendulum*

In the (planar) double pendulum two mass points move under the influence of a homogeneous gravitational field in a vertical plane, the first at a fixed distance l_1 from a fixed point (which we take as the origin), and the second at a fixed distance l_2 from the first mass point. We denote the deviations from the vertical from the respective suspension points by the position coordinates θ_1 and θ_2. Then the position coordinates (in the plane) are

$$\boldsymbol{x}_1 = l_1(\cos\theta_1, \sin\theta_1)\,, \qquad \boldsymbol{x}_2 = \boldsymbol{x}_1 + l_2(\cos\theta_2, \sin\theta_2)$$

and the configuration space is $S^1 \times S^1$.

For the kinetic energy one finds with $\dot{\boldsymbol{x}}_1 = l_1\dot{\theta}_1(-\sin\theta_1, \cos\theta_1)$

$$T = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + \frac{1}{2}m_2 l_2 \dot{\theta}_2^2\,,$$

while the potential energy is given by

$$V = -m_1 l_1 g\cos\theta_1 - m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2).$$

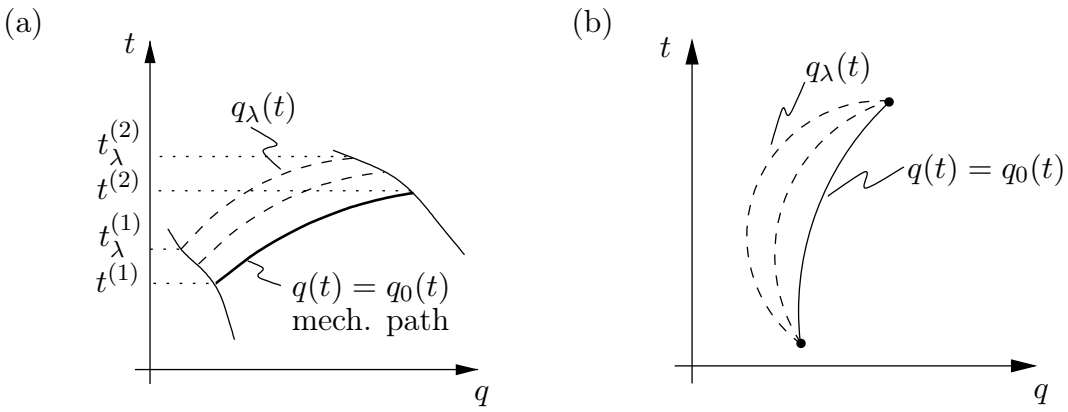


Figure 4.4: Variation of a mechanical trajectory $q(t)$ with (a) or without (b) change of the endpoints $q^{(0)}$, $q^{(1)}$.

The Lagrangian is simply $L = T - V$. From this one can easily derive the equations of motion. However, the double pendulum is *not integrable*: the equations of motion cannot be reduced to the evaluation of integrals. In fact, numerical computations (and experiments) indicate a "chaotic behavior." After a finite time the behavior of the double pendulum depends strongly on minimal changes in the initial conditions.

Remark: More generally than in (4.30), one calls a system a *Lagrangian system* if there exists a Lagrangian function $L(q, \dot{q}, t)$ such that the equations of motion are given by the Euler-Lagrange equations (4.31). We have already examined such a system in Example 4. Using (4.31) one obtains in this case

$$\begin{aligned} \frac{d}{dt}\langle p, \delta q\rangle &= \frac{d}{dt}\sum_{\alpha=1}^{f}\frac{\partial L}{\partial \dot{q}^\alpha}\,\delta q^\alpha \overset{(4.31)}{=} \sum_{\alpha=1}^{f}\Big(\frac{\partial L}{\partial q^\alpha}\,\delta q^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha}\,\delta \dot{q}^\alpha\Big) \\ &= \delta L \end{aligned} \tag{4.40}$$

for a mechanical trajectory with the *generalized momenta*

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}. \tag{4.41}$$

4.4 The Hamilton Principle

The laws of mechanics for Lagrangian systems can be derived from an extremal principle. This extremal principle is called the principle of "least" action or, equivalently, the Hamilton principle. To this end, we define the *action* of a trajectory $q(t)$ $(t^{(1)} \le t \le t^{(2)})$ as

$$S[q(t)] = \int_{t^{(1)}}^{t^{(2)}} dt\, L(q(t), \dot{q}(t), t). \tag{4.42}$$

We compute the variation δS of the action about a mechanical trajectory. In doing so, we allow for the possibility that the endpoints are varied as well; i.e., the endpoints $q_\lambda(t_\lambda^{(i)})$ of the comparison trajectories $q_\lambda(t)$ are in general also functions of the variation parameter λ, cf. Figure 4.4. We write for the *variation of the endpoints*,

$$\begin{aligned} \Delta t^{(i)} &\equiv \frac{dt_\lambda^{(i)}}{d\lambda}\Big|_{\lambda=0}, \\ \Delta q^{(i)} &\equiv \frac{d}{d\lambda} q_\lambda(t_\lambda^{(i)})\Big|_{\lambda=0} = \delta q^{(i)} + \dot{q}\,\Delta t^{(i)} \end{aligned} \tag{4.43}$$

with $\delta q^{(i)} = \delta q(t^{(i)})$, where δq (as always) refers to the variation at fixed time. With these preliminaries we obtain

$$\begin{aligned} \delta S &= \frac{d}{d\lambda} \int_{t_\lambda^{(1)}}^{t_\lambda^{(2)}} L(q_\lambda(t), \dot{q}_\lambda(t), t)\, dt \,\Big|_{\lambda=0} = \int_{(1)}^{(2)} \delta L\, dt + L\,\Delta t\Big|_{(1)}^{(2)} \\ &\overset{(4.40)}{=} \Big(\langle p, \delta q\rangle + L\,\Delta t \Big)\Big|_{(1)}^{(2)} \end{aligned}$$

Using (4.43) one deduces from this the general formula

$$\delta S = \Big(\sum_{\alpha=1}^{f} p_\alpha\, \Delta q^\alpha - \Big(\sum_{\alpha=1}^{f} p_\alpha\, \dot{q}^\alpha - L \Big) \Delta t \Big)\Big|_{(1)}^{(2)} \tag{4.44}$$

for the variation of a mechanical trajectory. In particular,

$$\delta S = \delta \int_{(1)}^{(2)} L\, dt = 0 \tag{4.45}$$

holds for an arbitrary variation of a mechanical trajectory with *fixed endpoints*.

This is the content of the *Hamilton principle*:

> *The action of a mechanical trajectory $q(t)$ is stationary (not necessarily minimal) with respect to any family $q_\lambda(t)$ of varied trajectories sharing fixed endpoints.*

Conversely, upon performing the variation, the Hamilton principle immediately yields the Euler-Lagrange equations. We have

$$\begin{aligned} 0 = \delta S &= \int_{(1)}^{(2)} \delta L\, dt = \int_{(1)}^{(2)} \sum_{\alpha=1}^{f} \Big(\frac{\partial L}{\partial q^\alpha} \delta q^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha} \delta \dot{q}^\alpha \Big) dt \\ &= \sum_{\alpha=1}^{f} p_\alpha\, \delta q^\alpha \Big|_{(1)}^{(2)} - \int_{(1)}^{(2)} \sum_{\alpha=1}^{f} \Big(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} \Big) \delta q^\alpha\, dt \end{aligned}$$

after partial integration. For fixed endpoints $\delta q^\alpha(t) = 0$ holds for $t = t^{(1)}, t^{(2)}$, and hence one directly obtains (4.31). We have thus shown that the Hamilton principle is equivalent to the Euler-Lagrange equations.

4.4.1 Equivalent Lagrangian Functions

Two Lagrangian functions L_1 and L_2 that differ only by the total time derivative of a function $F(q,t)$,

$$L_1 - L_2 = \frac{d}{dt}F(q,t) \equiv \sum_\alpha \frac{\partial F}{\partial q^\alpha}\dot{q}^\alpha + \frac{\partial F}{\partial t}\,, \tag{4.46}$$

describe the same dynamics, i.e., lead to the same Euler-Lagrange equations. This follows directly from the Hamilton principle. Indeed, for fixed endpoints and an otherwise arbitrary variation

$$\delta \int_{(1)}^{(2)} \frac{d}{dt}F(q,t)dt = \delta F(q,t)\Big|_{(1)}^{(2)} = 0 \tag{4.47}$$

and hence $\delta S_1 = \delta S_2$. Of course, one can also show directly that L_1 and L_2 lead to the same Euler-Lagrange equations.

Example 4 (continued): If one modifies the potentials $(\varphi, \boldsymbol{A})$ by a *gauge transformation*

$$\varphi \to \varphi - \frac{1}{c}\frac{\partial \chi}{\partial t}\,, \qquad \boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\chi\,, \tag{4.48}$$

where χ is an arbitrary scalar function, the electromagnetic field (4.38) does not change. Hence the equations of motion (1.16) should likewise remain invariant. In fact, L, given in Eq. (4.39), transforms under a gauge transformation (4.48) into the equivalent Lagrangian function

$$L + \frac{e}{c}\Big(\frac{\partial \chi}{\partial t} + \dot{\boldsymbol{x}}\cdot\boldsymbol{\nabla}\chi\Big) = L + \frac{e}{c}\frac{d\chi}{dt}$$

.

4.5 Conserved Quantities and Noether's Theorem

4.5.1 Conserved Quantities

A function $f(q,\dot{q},t)$ is called an *conserved quantity* or *constant of motion* if for *all* mechanical trajectories $q(t)$

$$\frac{d}{dt}f(q,\dot{q},t) = 0,$$

holds so that f remains constant under the motion. In Chapter 1.4 we have already encountered the conserved quantities momentum, angular momentum, and energy.

The Euler-Lagrange equation (4.31) is a second-order differential equation in the f degrees of freedom of the position coordinates. Hence one generally needs $2f$ integrations to solve the problem. Conserved quantities are extraordinarily important because each (independent) conserved quantity reduces by one the number of integrations required to solve the equations of motion.

Example: For the two-body problem from Chapter 2 we have $f = 3N = 6$ degrees of freedom (with $N = 2$). The Galilean relativity principle yields the 10 conservation laws from Chapter 1.4. Thus, in general, $2f - 10 = 2$ integrations of (2.12) and (2.14) remain. For the Kepler problem we found in addition the Laplace-Runge-Lenz vector, which provides an additional independent conserved quantity. Thereby we were able to determine the orbit without integration, and the only remaining integration is the determination of time $t(r)$ from (2.12).

4.5.2 Cyclic Variables

Let L be the Lagrangian function of a Lagrangian system. If L is independent of a position coordinate q^α, i.e., if $\partial L/\partial q^\alpha = 0$, then the coordinate q^α is called *cyclic*. For every cyclic variable q^α one then has, by the Euler-Lagrange equation

$$\frac{d}{dt}p_\alpha = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} = \frac{\partial L}{\partial q^\alpha} = 0, \tag{4.49}$$

i.e., the generalized momentum $p_\alpha = \partial L/\partial \dot{q}^\alpha$ is a conserved quantity.

Example 2 (continued): Let us reconsider the spherical pendulum. The Lagrangian function L does not depend on φ according to Eq. (4.37). The associated conserved quantity is the generalized momentum

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\, l^2 \sin^2\theta\, \dot{\varphi}\,,$$

, which is precisely the angular momentum about the vertical axis. This is conserved because no torque acts in the vertical direction.

The fact that the Lagrangian function L does not depend on a cyclic variable q^α is equivalent to the statement that the Lagrangian remains invariant under the shift $\phi_\lambda(q)$ of the position coordinates, with

$$[\phi_\lambda(q)]^\beta = q^\beta + \delta^{\beta\alpha}\lambda \tag{4.50}$$

i.e., with $q_\lambda(t) = \phi_\lambda(q(t))$ one has $L(q_\lambda, \dot{q}_\lambda, t) = L(q, \dot{q}, t)$ independently of λ. [5] This reformulation illustrates the idea underlying Noether's theorem.

4.5.3 Noether's Theorem

Many of the conserved quantities we have found so far can be obtained from Noether's theorem. We call a family $\phi_\lambda \colon \mathbb{R}^f \to \mathbb{R}^f$, $(\lambda \in \mathbb{R})$, of mappings of the position coordinates[6] onto itself a *flow* if it satisfies the group property[7]

$$\phi_0 = \mathrm{id}\,, \qquad \phi_\lambda \circ \phi_\mu = \phi_{\lambda+\mu} \tag{4.51}$$

[5] The Kronecker symbol $\delta^{\alpha\beta}$ is defined by $\delta^{\alpha\beta} = 1$ for $\alpha = \beta$ and $\delta^{\alpha\beta} = 0$ otherwise.
[6] We first treat, for simplicity, the case where the chart covers the entire $\mathbb{R}^f$.
[7] Here id denotes the identity map with $\mathrm{id}(x) = x$.

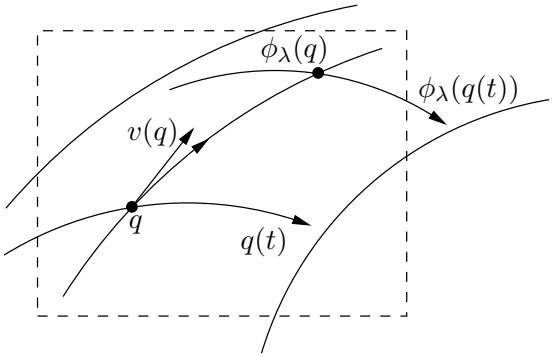


Figure 4.5: The vector field $v(q)$ at the point q generates the flow $\phi_\lambda(q)$, i.e., the point q is (locally) displaced along $v(q)$ by the flow. Starting from a trajectory $q(t)$ one thereby obtains the family $q_\lambda(t) = \phi_\lambda(q(t))$. If the chart K covers only a subspace of $\mathbb{R}^f$, one needs only a section of the flow (dashed rectangle).

Each flow possesses a *generating vector field* $v(q)$ on $\mathbb{R}^f$, defined by

$$v(q) = \frac{\partial}{\partial\lambda}\phi_\lambda(q)\Big|_{\lambda=0} \in \mathbb{R}^f \tag{4.52}$$

cf. Figure 4.5. Owing to the group property of the flow, (4.52) in fact holds for all $(\lambda, q) \in \mathbb{R} \times \mathbb{R}^f$, i.e.,

$$\frac{\partial}{\partial\lambda}\phi_\lambda(q) = \frac{\partial}{\partial\varepsilon}\phi_{\lambda+\varepsilon}(q)\Big|_{\varepsilon=0} = \frac{\partial}{\partial\varepsilon}\phi_\varepsilon(\phi_\lambda(q))\Big|_{\varepsilon=0} = v(\phi_\lambda(q))\,. \tag{4.53}$$

The function $f(\lambda) = \phi_\lambda(q)$ is thus the solution of the first-order differential equation

$$\frac{df(\lambda)}{d\lambda} = v(f(\lambda)), \qquad \text{with the initial condition } f(0) = q\,. \tag{4.54}$$

Hence, conversely, the vector field determines the flow.[8] The notion of a flow can be extended to a general configuration space: in a chart one can represent only a section of the flow ϕ_λ; in general one needs several charts (an atlas) to cover the entire configuration space.

Let $q(t)$ now be an arbitrary curve in the configuration space. The flow ϕ_λ maps the curve $q(t)$ to the family $q_\lambda(t) = \phi_\lambda(q(t))$. We call the flow ϕ_λ a *continuous symmetry* of the Lagrangian function $L(q, \dot{q}, t)$ if

$$L(q_\lambda(t), \dot{q}_\lambda(t), t) = L(q(t), \dot{q}(t), t) + \frac{d}{dt}F_\lambda(q(t), t) \tag{4.55}$$

holds for all $\lambda \in \mathbb{R}$ and for every curve $t \mapsto q(t) \in \mathbb{R}^f$; i.e., the Lagrangian is mapped under the flow to an equivalent Lagrangian function. If $F \equiv 0$ one says that the Lagrangian is *invariant* under the continuous symmetry.

Thus we can formulate the *Noether theorem*:

[8] Provided that (4.54) admits a global solution $f(q)$ for every initial value q.

Let ϕ_λ be a continuous symmetry of L; then

$$\langle p, v(q)\rangle - \delta F \equiv \sum_{\alpha=1}^{f} p_\alpha(q,\dot q,t)\, v^\alpha(q) - \delta F(q,t)\,, \tag{4.56}$$

with $p_\alpha = \partial L/\partial \dot q^\alpha$ is conserved, i.e. on every mechanical trajectory $q(t)$

$$\frac{d}{dt}\Big(\langle p, v(q)\rangle - \delta F\Big) = 0\,.$$

holds[9]

Remarks:

- To every continuous symmetry there corresponds a conservation law.
- The Noether theorem is useful because it is often easier to find continuous symmetries than to determine conserved quantities directly.
- The converse of Noether's theorem does not hold in general. Not every conserved quantity arises from a continuous symmetry; for example, the Laplace-Runge-Lenz vector $\boldsymbol{A}$ is not explained by any such symmetry. This deficiency of the Lagrangian formulation is remedied in the Hamiltonian framework, where every conserved quantity also induces a (canonical) flow, see Chap. (7.6).
- The Noether theorem also applies to flows $q_\lambda(t) = \phi_\lambda(q(t),t)$ that depend on time t explicitly.[10] The vector field is then given by $v\colon (q,t) \mapsto \partial_\lambda \phi_\lambda\big|_{\lambda=0} \in \mathbb{R}^f$. The proof remains unchanged and we obtain (4.56) with the time-dependent vector field $v(q,t)$ in place of $v(q)$.

Proof: For the variation $q_\lambda(t)$ about an arbitrary mechanical trajectory $q(t)$ one has by definition

$$\delta q(t) = \frac{\partial}{\partial\lambda} q_\lambda(t)\Big|_{\lambda=0} = v(q(t)).$$

By hypothesis (4.55) one also has

$$\delta L = \frac{d}{d\lambda} L(q_\lambda, \dot q_\lambda, t)\Big|_{\lambda=0} = \frac{d}{dt}\delta F(q(t),t).$$

With (4.40) this directly yields the claim.

[9] The conserved quantity depends only on δF in the expansion of $F_\lambda(q,t) = \delta F(q,t)\lambda + O(\lambda^2)$. In fact it suffices that in eq. (4.55) the terms linear in λ are a total time derivative, so that the function F_λ exists to every order λ. This result follows from the integral representation $F_\lambda(q,t) = \int_0^\lambda \delta F(q_\mu,t)\, d\mu$ of F_λ.

[10] To preserve the group property one formally defines the flow on the extended configuration space (q,t) by $\psi_\lambda(q,t) = (\phi_\lambda(q),t)$. The vector field is then given by $w(q,t) = \partial_\lambda \psi_\lambda(q,t)|_{\lambda=0} = (v,0) \in \mathbb{R}^f \times \mathbb{R}$.

Example 6: One can now easily obtain the conservation of the generalized momentum associated with a cyclic coordinate from Noether's theorem. If q^α is cyclic, then the flow

$$[\phi_\lambda(q)]^\beta = q^\beta + \delta^{\beta\alpha}\lambda$$

defines a continuous symmetry of the Lagrangian with $F \equiv 0$. The corresponding vector field

$$v^\beta = \delta^{\alpha\beta}$$

then has only one non-trivial component. The conserved quantity furnished by (4.56) is simply the conjugate momentum p_α.

Example 7: The Lagrangian

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{m}{2}\dot{\boldsymbol{x}}^2 - V(|\boldsymbol{x}|)$$

of a particle in two dimensions with position coordinates[11] $\boldsymbol{x} \in \mathbb{R}^2$ is invariant under all rotations $R \in \mathrm{SO}(2)$, i.e. $L(\boldsymbol{x}_\lambda, \dot{\boldsymbol{x}}_\lambda) = L(\boldsymbol{x}, \dot{\boldsymbol{x}})$ with $\boldsymbol{x}_\lambda = \phi_\lambda(\boldsymbol{x}) = R(\lambda)\boldsymbol{x}$ the coordinates rotated by the angle λ. We determine the associated vector field

$$\boldsymbol{v}(\boldsymbol{x}) = R'(\lambda)\boldsymbol{x}\Big|_{\lambda=0} = \boldsymbol{e}_3 \times \boldsymbol{x} = -x_2\boldsymbol{e}_1 + x_1\boldsymbol{e}_2$$

and the momenta

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = m\dot{\boldsymbol{x}}.$$

. Applying Noether's theorem then yields the conserved quantity

$$L_3 = \boldsymbol{p} \cdot \boldsymbol{v}(\boldsymbol{x}) = m(x_1\dot{x}_2 - x_2\dot{x}_1).$$

, which is precisely the angular momentum about the 3-axis. Had we taken the polar coordinates (r, φ) instead of $\boldsymbol{x}$ as the position coordinate, we would have found that φ is cyclic and hence

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}^2$$

is conserved. One may readily verify that $L_3 = p_\varphi$.

Example 8: A mass particle at position $x \in \mathbb{R}$ in a uniform gravitational field has the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - mgx.$$

. The flow (*pure Galilei transformation*)

$$x_\lambda(t) = \phi_\lambda(x(t), t) = x(t) + \lambda t$$

[11] Of course one may also use the Cartesian coordinates $\boldsymbol{x}$ as generalized coordinates.

is a symmetry of the problem because

$$L(x_\lambda, \dot{x}_\lambda, t) = \frac{1}{2}m(\dot{x}+\lambda)^2 - mg(x+\lambda t) = L(x, \dot{x}, t) + \frac{d}{dt}F_\lambda(x,t)$$

with

$$F_\lambda(x,t) = m\lambda\Big(x - \frac{1}{2}gt^2\Big) + \frac{1}{2}m\lambda^2 t.$$

. The corresponding vector field is $v(x,t) = t$ and

$$\delta F = m\Big(x - \frac{1}{2}gt^2\Big).$$

. Noether's theorem then gives the conserved quantity

$$\beta = m\dot{x}v(x,t) - \delta F = m\Big(\dot{x}t + \frac{1}{2}gt^2 - x\Big). \tag{4.57}$$

Energy conservation: The (generalized) energy conservation cannot be derived directly from our form of Noether's theorem. Energy conservation rests on the fact that the Lagrangian $L = L(q, \dot{q})$ does not depend explicitly on time (autonomous system). The symmetry underlying energy conservation is therefore time translation.

To prove energy conservation we consider the variation $q_\lambda(t) = q(t+\lambda)$ of a mechanical trajectory $q(t)$ with $\delta q(t) = \dot{q}(t)$. Equation (4.40) then demands that

$$\frac{d}{dt}\langle p, \dot{q}\rangle = \delta L = \frac{d}{dt}L - \frac{\partial L}{\partial t} \tag{4.58}$$

. Since L does not depend explicitly on t, the *generalized energy*

$$E = \langle p, \dot{q}\rangle - L = \sum_{\alpha=1}^{f} p_\alpha(q, \dot{q}, t)\dot{q}^\alpha - L(q, \dot{q}, t) \tag{4.59}$$

is conserved along every mechanical trajectory.

Example 9: If the Lagrange function is given by $L = T(q, \dot{q}) - V(q)$, with

$$T(q, \dot{q}) = \frac{1}{2}\sum_{\alpha,\beta=1}^{f} g_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta \qquad (g_{\alpha\beta} = g_{\beta\alpha}) \tag{4.60}$$

the relation

$$\sum_{\alpha=1}^{f}\dot{q}^\alpha\frac{\partial L}{\partial\dot{q}^\alpha} = \sum_{\alpha,\beta=1}^{f} g_{\alpha\beta}(q)\dot{q}^\alpha\,\dot{q}^\beta = 2T\,. \tag{4.61}$$

holds. Such systems arise, for example, when the constraints (4.5) do not depend on time t, cf. (4.8) and (4.9). The constraint forces then do no work, since $\dot{x}$ is a virtual displacement. In this case L does not depend explicitly on time, so energy is conserved in the form

$$E = T + V\,.$$

Example 2 (Continuation): Another example is the spherical pendulum. Since the Lagrange function (4.37) does not depend explicitly on time, the energy

$$E = \dot\theta \frac{\partial L}{\partial \dot\theta} + \dot\varphi \frac{\partial L}{\partial \dot\varphi} - L = \frac{l^2}{2}\dot\theta^2 + U(\theta)$$

is conserved with

$$U(\theta) = \frac{p_\varphi^2}{2l^2 \sin^2\theta} - g\, l\, \cos\theta.$$

The effective potential U is composed of the gravitational potential and the centrifugal potential, cf. (2.11).

Example 4 (Continuation): As an example, consider a charged particle in a *static* (time-independent) field. In this case the Maxwell equations are

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi(\boldsymbol{x})\,, \qquad \boldsymbol{B} = \operatorname{curl} \boldsymbol{A}(\boldsymbol{x})\,.$$

The Lagrange function is time-independent and hence energy is conserved. The generalized momentum is given by

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = m\dot{\boldsymbol{x}} + \frac{e}{c}\boldsymbol{A}.$$

Note that in this case the generalized momentum $\boldsymbol{p}$ does not coincide with the (kinematic) momentum $m\dot{\boldsymbol{x}}$ of Newtonian mechanics. Thus we obtain the energy

$$E = \boldsymbol{p}\cdot\dot{\boldsymbol{x}} - L = \frac{1}{2}m\dot{\boldsymbol{x}}^{\,2} + e\varphi(\boldsymbol{x}). \tag{4.62}$$

The first term is the kinetic energy and the second term the potential energy in the electric potential φ.

Example 8 (Continuation): For the mass particle in a homogeneous gravitational field energy is conserved. With Example 9 we obtain

$$E = T + V = \frac{1}{2}m\dot x^2 + mgx.$$

We have previously found the additional conserved quantity β. With these two conserved quantities the problem is completely integrable.

In fact, we can solve the equation for β for $\dot x$ with the result

$$\dot x = \frac{1}{t}\Big(x - \frac{1}{2}gt^2 + \frac{\beta}{m}\Big).$$

Substituting into the equation for E then yields, after a short calculation,

$$\gamma^2 \equiv \frac{2}{m}(E + g\beta) = \left(\frac{\frac{1}{2}gt^2 + x + \beta/m}{t}\right)^2.$$

This equation can be solved for x with the trajectory of a projectile

$$x = -\frac{1}{2}gt^2 + \gamma t - \frac{\beta}{m}$$

as the result.

4.6 The 10 classical conservation laws

For a (closed, Lagrangian) system $L = T - V$ whose potential V is invariant under Galilean transformations, we have already derived ten conserved quantities in Chapter 1.4. We now wish to trace these back via Noether's theorem to the 10 continuous parameters of the Galilei group (1.6).

Because of Galilean invariance the Lagrangian of such a system has the form

$$L(\underbrace{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N}_{=x}; \dot{\boldsymbol{x}}_1, \ldots, \dot{\boldsymbol{x}}_N) = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i^2 - V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \equiv T - V\,, \tag{4.63}$$

with

$$V(R\boldsymbol{x}_1 + \boldsymbol{b}, \ldots, R\boldsymbol{x}_N + \boldsymbol{b}) = V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)\,, \qquad (R \in \mathrm{SO}(3); \boldsymbol{b} \in \mathbb{R}^3),$$

see (1.36). The momenta are then given precisely by

$$\boldsymbol{p}_i = \frac{\partial L}{\partial \dot{\boldsymbol{x}}_i} = m_i \dot{\boldsymbol{x}}_i \tag{4.64}$$

We now discuss the parameters of the Galilei group one by one:

(i) **Time translations:** As we have already shown in Example 9, the conserved quantity is the energy $E = T + V$.

(ii) **Spatial translations:** L is invariant under the joint translation

$$\phi_\lambda(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = (\boldsymbol{x}_1 + \lambda\boldsymbol{b}, \ldots, \boldsymbol{x}_N + \lambda\boldsymbol{b})$$

by the vector $\lambda\boldsymbol{b}$. The generating vector field is $v(x,t) = (\boldsymbol{b}, \ldots, \boldsymbol{b})$. The corresponding conserved quantity is therefore

$$\sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \boldsymbol{b} \equiv \boldsymbol{P} \cdot \boldsymbol{b}\,,$$

i.e. the **total momentum** $\boldsymbol{P}$ is conserved, since $\boldsymbol{b}$ is arbitrary.

(iii) **Rotations:** L is invariant under the rotations

$$\phi_\lambda(\boldsymbol{x}_1, \ldots \boldsymbol{x}_N) = (R(\boldsymbol{e}, \lambda)\boldsymbol{x}_1, \ldots, R(\boldsymbol{e}, \lambda)\boldsymbol{x}_N)\,,$$

where $R(\boldsymbol{e}, \lambda)$ denotes the rotation about the axis $\boldsymbol{e}$ through the angle λ. The associated vector field $v(x) = (\boldsymbol{e} \times \boldsymbol{x}_1, \ldots, \boldsymbol{e} \times \boldsymbol{x}_N)$, cf. (1.39), yields the conserved quantity

$$\sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot (\boldsymbol{e} \times \boldsymbol{x}_i) = \boldsymbol{e} \cdot \sum_{i=1}^{N} \boldsymbol{x}_i \times m_i \dot{\boldsymbol{x}}_i \equiv \boldsymbol{e} \cdot \boldsymbol{L}\,,$$

i.e. the **total angular momentum** $\boldsymbol{L}$ is conserved.

(iv) **Special Galilean transformations:** a *special Galilean transformation* is a transformation to a frame moving uniformly with velocity $\boldsymbol{v}$. This corresponds to the flow

$$\phi_\lambda(\boldsymbol{x}_1, \dots \boldsymbol{x}_N, t) = (\boldsymbol{x}_1 + \lambda \boldsymbol{v} t, \dots, \boldsymbol{x}_N + \lambda \boldsymbol{v} t).$$

with generating vector field $v(x,t) = (\boldsymbol{v}t, \dots, \boldsymbol{v}t)$. The Lagrangian transforms as

$$\begin{aligned} L(x_\lambda, \dot{x}_\lambda) &= \frac{1}{2}\sum_{i=1}^N m_i(\dot{\boldsymbol{x}}_i + \lambda \boldsymbol{v})^2 - V(\boldsymbol{x}_1 + \lambda \boldsymbol{v}t, \dots, \boldsymbol{x}_N + \lambda \boldsymbol{v}t) \\ &= L(x, \dot{x}) + \sum_{i=1}^N m_i\Big(\lambda \dot{\boldsymbol{x}}_i \cdot \boldsymbol{v} + \frac{1}{2}\lambda^2 \boldsymbol{v}^2\Big). \end{aligned}$$

Hence the special Galilean transformations form a continuous symmetry (4.55) with

$$F_\lambda(x,t) = \sum_{i=1}^N m_i\Big(\lambda \boldsymbol{x}_i \cdot \boldsymbol{v} + \frac{1}{2}\lambda^2 \boldsymbol{v}^2 t\Big)\,, \qquad \delta F(x,t) = \sum_{i=1}^N m_i \boldsymbol{x}_i \cdot \boldsymbol{v}\,.$$

The associated conserved quantity (4.56) is

$$\sum_{i=1}^N m_i \dot{\boldsymbol{x}}_i \cdot \boldsymbol{v} t - \sum_{i=1}^N m_i \boldsymbol{x}_i \cdot \boldsymbol{v} = (\boldsymbol{P}t - M\boldsymbol{X}) \cdot \boldsymbol{v}\,;$$

i.e. the **center-of-mass integral** $M\boldsymbol{X} - \boldsymbol{P}t$ is conserved.

4.7 The Principle of Euler-Maupertuis

We restrict ourselves to autonomous systems so that the constraint (4.5) does not depend explicitly on time and the potential depends only on the coordinates, $V \equiv V(q)$, cf. Example 9. With (4.8) we thus obtain the kinetic energy in the form

$$T(q, \dot{q}) = \frac{1}{2}\sum_{\alpha,\beta=1}^f g_{\alpha\beta}(q)\, \dot{q}^\alpha \dot{q}^\beta\,, \qquad g_{\alpha\beta}(q) = \frac{1}{2}\sum_{i=1}^N m_i \frac{\partial \boldsymbol{x}_i}{\partial q^\alpha} \cdot \frac{\partial \boldsymbol{x}_i}{\partial q^\beta}. \tag{4.65}$$

The kinetic energy thereby defines a (riemannian) *metric* on the configuration space with the *mechanical line element*

$$ds^2 = \frac{1}{2}\sum_{i=1}^N m_i (d\boldsymbol{x}_i)^2 = \frac{1}{2}\sum_{\alpha,\beta=1}^f g_{\alpha\beta}(q)\, dq^\alpha dq^\beta. \tag{4.66}$$

This immediately yields that the kinetic energy

$$T(q, \dot{q}) = \frac{1}{2}\sum_{i=1}^N m_i \dot{\boldsymbol{x}}_i^2 = \left(\frac{ds}{dt}\right)^2 \tag{4.67}$$

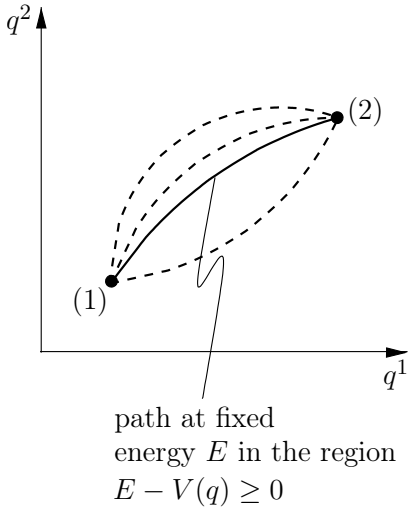


Figure 4.6: Variations of the trajectory in the principle of Euler-Maupertuis. The variations are at fixed energy and with the *position* of the initial and final points held fixed. Note that at fixed energy the *time* of the initial and final points will in general change naturally under a variation of the trajectory.

simply corresponds to the square of ds/dt.

We are interested in a variational principle for the *trajectory* (geometric shape of the curve) without specification of the temporal evolution. We use the energy conservation $E = T + V$ and wish to allow only variations at *fixed energy*. If we restrict the variations to fixed energy, we can hold only the endpoints of the trajectory fixed but no longer the initial and final times, see Figure 4.6. In fact the general result (4.44)

$$\delta S = -E\,\Delta t\Big|_{(1)}^{(2)} \tag{4.68}$$

holds for every variation of a mechanical trajectory at fixed energy $E = \langle p, \dot{q}\rangle - L$ and fixed endpoints ($\Delta q^{(i)} = 0$). This fact motivates the introduction of the *abbreviated action*[12]

$$S_0[q(t)] = S + E(t^{(2)} - t^{(1)}) = \int_{t^{(1)}}^{t^{(2)}} dt\,(L(q,\dot{q}) + E), \tag{4.69}$$

with which we obtain the new extremal principle $\delta S_0 = 0$, since

$$\Delta t\Big|_{(1)}^{(2)} = \delta \int_{(1)}^{(2)} dt.$$

It remains only to find a method for performing the variations of S_0 at fixed energy without reference to time (which will now change). The idea is to switch in (4.69) from integration over time to integration over the line element ds. Indeed, with $L + E = 2T$ and (4.67) we find that

$$(L+E)dt = 2T\frac{dt}{ds}ds = 2\sqrt{T}ds = 2\sqrt{E - V(q)}\,ds. \tag{4.70}$$

[12]The transition from S to S_0 corresponds to a Legendre transformation from t to E, see Chap. 8.

The last form is independent of time and we obtain the *principle of Euler-Maupertuis*[13]

$$0 = \delta S_0 = \delta \int_{(1)}^{(2)} \sqrt{E - V(q)}\, ds \tag{4.71}$$

for every variation of the trajectory at fixed endpoints $q^{(i)}$, $i = 1, 2$.

Remark: To compute the integral in (4.71), one must parametrize the trajectory $q(\tau)$ with τ an *arbitrary* parameter. Euler-Maupertuis principle then states explicitly that

$$0 = \delta \int_{\tau^{(1)}}^{\tau^{(2)}} \underbrace{\sqrt{E - V(q(\tau))}\sqrt{T(q(\tau), q'(\tau))}}_{\equiv \tilde{L}(q,q')}\, d\tau, \qquad \text{with} \quad q' = \frac{dq}{d\tau}\,.$$

The integrand then becomes a (new) Lagrangian $\tilde{L}(q, q')$ and the extremal principle leads to the Euler-Lagrange equations

$$\frac{d}{d\tau}\frac{\partial \tilde{L}}{\partial q'^{\alpha}} - \frac{\partial \tilde{L}}{\partial q^{\alpha}} = 0$$

cf. Eq. (4.31).

Example: As an example, consider the case of a free particle, $V(q) \equiv 0$. Then the variational principle (4.71)

$$\delta \int_{(1)}^{(2)} ds = 0\,,$$

i.e., the trajectories at fixed energy $E > 0$ are *geodesics* (with respect to the mechanical metric (4.66)), i.e. curves of extremal length.

An illustration is provided by a particle sliding without friction on a two-dimensional surface embedded in $\mathbb{R}^3$. In this case the kinetic energy is given by

$$T = \frac{m}{2}\dot{\boldsymbol{x}}^2 = \frac{m}{2}\left(\frac{ds}{dt}\right)^2,$$

where $ds^2 = (d\boldsymbol{x})^2$ refers here to the *Euclidean line element* of the $\mathbb{R}^3$. Comparing with (4.67) shows that the Euclidean line element differs from the mechanical one only by a constant factor. Since this amounts merely to a change of length unit, this difference is immaterial for the extremal principle (4.71). We may therefore replace the mechanical by the Euclidean length measure for a particle. Hence the particle's trajectories are simply geodesics in the Euclidean sense, e.g. great circles on a sphere.

[13] The constant factor 2 between S_0 and $\int \sqrt{E - V}\, ds$ is irrelevant for the extremal principle.

Relation to Fermat's Principle: Fermat's principle states that light (in geometrical optics) follows the path for which the travel time is an extremum. In a medium with a position-dependent *refractive index* $n(\boldsymbol{x})$, the effective speed of light $c/n(\boldsymbol{x})$ also becomes position dependent. The travel time along a path is thus given by

$$\frac{\mathcal{L}[\boldsymbol{x}(\tau)]}{c} = \int_{(1)}^{(2)} \frac{n(\boldsymbol{x})}{c}\, ds \tag{4.72}$$

with $\mathcal{L}[\boldsymbol{x}(\tau)]$ denoting the optical path length of the curve $\boldsymbol{x}(\tau)$.[14] Fermat's principle therefore asserts that

$$\delta\mathcal{L}[\boldsymbol{x}(\tau)] = \delta \int_{(1)}^{(2)} n(\boldsymbol{x})\, ds = 0 \tag{4.73}$$

for fixed endpoints. One now recognizes the close analogy to the Euler-Maupertuis principle (4.71) upon identifying

$$n(\boldsymbol{x}) \propto \sqrt{E - V(\boldsymbol{x})}. \tag{4.74}$$

To carry out the variation we parametrize the path by an arbitrary parameter τ. Then (4.73) reads

$$\delta \int_{\tau^{(1)}}^{\tau^{(2)}} \underbrace{n(\boldsymbol{x})|\boldsymbol{x}'|}_{L(\boldsymbol{x},\boldsymbol{x}')}\, d\tau = 0,$$

with $'$ the derivative with respect to τ.

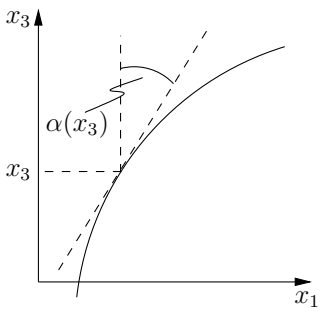
Example: In a layered medium with $n(\boldsymbol{x}) = n(x_3)$, x_1, x_2 are cyclic coordinates. This yields the conserved quantities

$$\frac{\partial L}{\partial x_i'} = n(x_3)\frac{x_i'}{|\boldsymbol{x}'|} = n(x_3)x_i', \qquad (i = 1, 2),$$

where in the last step we chose a parametrization by arc length with $d\tau = ds$ so that $|\boldsymbol{x}'| = 1$.

Hence we first find that the ray is planar ($x_2'/x_1' =$ constant , and thus without loss of generality $x_2' = 0$). Introducing the angle $\alpha(x_3)$ between the 3-axis and the path, with $\sin\alpha(x_3) = x_1'/|\boldsymbol{x}'| = x_1'$, we further obtain *Snellius's law of refraction*

$$n(x_3)\,\sin\big(\alpha(x_3)\big) = \text{ constant}$$



[14] Here the parameter τ denotes an arbitrary parametrization.

Chapter 5

Oscillation Problems

A further important class of mechanical motions are oscillations. In this chapter we investigate the concept of a normal mode of a conservative oscillating system and a selected example (parametric resonance) of a driven oscillation. Finally, we treat the problem of a vibrating string.

5.1 Conservative Oscillating Systems

We consider a system whose configuration is determined by f coordinates $(q^1, \ldots, q^f) \equiv q \in \mathbb{R}^f$. The system is assumed to conserve energy in the form

$$E = T(\dot{q}) + V(q) \tag{5.1}$$

with

$$T(q) = \frac{1}{2} \sum_{\alpha,\beta=1}^{f} g_{\alpha\beta}\, q^\alpha q^\beta \qquad \text{and} \qquad V(q) = \frac{1}{2} \sum_{\alpha,\beta=1}^{f} k_{\alpha\beta} q^\alpha q^\beta \tag{5.2}$$

quadratic forms on $\mathbb{R}^f$ ($g_{\alpha\beta} = g_{\beta\alpha}$, $k_{\alpha\beta} = k_{\beta\alpha}$), where T is taken to be positive definite. We may use the kinetic energy to introduce a *mechanical scalar product*

$$(x, y) = \frac{1}{2} \sum_{\alpha,\beta=1}^{f} g_{\alpha\beta}\, x^\alpha y^\beta = \frac{1}{2}\big(T(x+y) - T(x) - T(y)\big) \tag{5.3}$$

for two vectors $x, y \in \mathbb{R}^f$. It is straightforward to verify that (x, y) satisfies all properties of a scalar product. In particular $(q, q) = T(q)$ holds. Thus the space of configuration coordinates $\mathbb{R}^f$ becomes a Euclidean vector space.[1]

[1] A Euclidean vector space is a vector space equipped with a scalar product.

The mapping $\tilde{V} = g^{-1}k$ is then a (with respect to the mechanical scalar product) symmetric[2], linear mapping $\tilde{V}\colon \mathbb{R}^f \to \mathbb{R}^f$, so that the *potential energy* is given by

$$V(q) = (q, \tilde{V}q) = (\tilde{V}q, q). \tag{5.4}$$

Energy conservation now requires that

$$0 = \frac{d}{dt}E = \frac{d}{dt}[(\dot{q}, \dot{q}) + (q, \tilde{V}q)] = 2(\dot{q}, \ddot{q} + \tilde{V}q). \tag{5.5}$$

For these relations to hold for all $\dot{q}$, we obtain the *equations of motion*[3]

$$\ddot{q} = -\tilde{V}q. \tag{5.6}$$

As a symmetric mapping, $\tilde{V}$ possesses a complete, orthonormal system of eigenvectors $e_1, \ldots, e_f \in \mathbb{R}^f$

$$\tilde{V}e_k = \lambda_k e_k, \qquad (e_k, e_l) = \delta_{kl} \tag{5.7}$$

with $\lambda_k \in \mathbb{R}$. In this basis

$$q = \sum_{i=1}^{f} \xi_i e_i \tag{5.8}$$

with ξ_i the *normal coordinates*. In the normal coordinates the equation of motion (5.6) reduces to (*normal modes*)

$$\ddot{\xi}_i = -\omega_i^2 \xi_i, \qquad \omega_i^2 = \lambda_i. \tag{5.9}$$

The system is thereby equivalent to a system of f *uncoupled, harmonic oscillators* with frequencies $\omega_1, \ldots, \omega_f$. The general solution of (5.6) is thus a linear superposition

$$q(t) = \sum_{i=1}^{f} e_i [\underbrace{(e_i, q(0))}_{=\xi_i(0)} \cos(\omega_i t) + \underbrace{(e_i, \dot{q}(0))}_{=\dot{\xi}_i(0)} \frac{1}{\omega_i} \sin(\omega_i t)] \tag{5.10}$$

of the f normal modes. If the eigenfrequencies do not stand in a rational ratio to one another, however, the motion is in general not periodic.

A vibrating system is called *stable* if *no* solution $q(t)$ for $t \to \infty$ grows without bound. We immediately see that the system (5.1) is stable if $\omega_i^2 > 0$ for all i, i.e. if in addition to T also V is positive definite. If V is only positive semidefinite, then at least one $\omega_i = 0$. In this case we obtain

$$\frac{1}{\omega_i} \sin(\omega_i t) \xrightarrow{(\omega_i \to 0)} t, \tag{5.11}$$

i.e. in this case the system is unstable.

[2] A symmetric mapping $\tilde{V}$ satisfies $(\tilde{V}x, y) = (x, \tilde{V}y)$ for all $x, y, \in \mathbb{R}^f$.

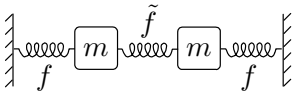
[3] One also obtains the equations of motion naturally as the Euler-Lagrange equations for $L = T(\dot{q}) - V(q)$.

Remark: The problem (5.1) is important because it describes a general autonomous mechanical system in the vicinity of an equilibrium position q^*; w. l. o. g. we set $q^* = 0$ to $V(q^*) = 0$. For the kinetic energy we have in this case found the general form (4.60). Since we are only interested in small displacements, we can identify $g_{\alpha\beta}$ from (5.2) simply with $g_{\alpha\beta}(q^*)$ from (4.61). Moreover, we can expand the potential $V(q)$ about the equilibrium point, yielding

$$V(q) \approx \sum_{\alpha=1}^{f} \overbrace{\frac{\partial V}{\partial q^\alpha}(q^*)}^{=0,\text{ since at equilibrium}} q^\alpha + \frac{1}{2} \sum_{\alpha,\beta=1}^{f} \frac{\partial^2 V}{\partial q^\alpha \partial q^\beta}(q^*)\, q^\alpha q^\beta, \tag{5.12}$$

i.e. $V(q)$ is, as required, a quadratic form.

Example: We consider as an example the simplest coupled oscillator from two identical harmonic oscillators with angular frequency $\Omega = \sqrt{f/m}$, which are connected by a spring with a spring constant $\tilde{f}$. The energy of the system is (with x_1, x_2 the displacement from the unstressed equilibrium position)



$$E = T(\dot{x}) + V(x) = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2}f(x_1^2 + x_2^2) + \frac{1}{2}\tilde{f}(x_2 - x_1)^2.$$

It is straightforward to verify that $V(x) = (x, \tilde{V}x)$ holds with the symmetric matrix

$$\tilde{V} = \begin{pmatrix} \Omega^2 + \tilde{\Omega}^2 & -\tilde{\Omega}^2 \\ -\tilde{\Omega}^2 & \Omega^2 + \tilde{\Omega}^2 \end{pmatrix}$$

and $\tilde{\Omega} = \left(\tilde{f}/m\right)^{1/2}$. The eigenvalue problem (5.7) has the solution

$$e_1 = \frac{1}{\sqrt{m}}(1,1), \qquad \omega_1 = \Omega,$$
$$e_2 = \frac{1}{\sqrt{m}}(1,-1), \qquad \omega_2 = \sqrt{\Omega^2 + 2\tilde{\Omega}^2}.$$

In the first normal mode the two oscillators simply oscillate in phase with constant separation, so that the coupling spring is never stretched. The frequency of this oscillation is naturally just the original frequency Ω. In the second normal mode the oscillators are in antiphase and oscillate with equal amplitude. The corresponding frequency ω_2 is therefore increased.

We see that the normal modes are intuitive, so that we could have guessed them intuitively. This procedure is facilitated by introducing the concept of symmetry.

Symmetries: A (discrete) symmetry is a linear map $S\colon \mathbb{R}^f \to \mathbb{R}^f$ that leaves T and V invariant; i.e. we require that $T(q) = T(Sq)$ and $V(q) = V(Sq)$ or (written with the mechanical scalar product)

$$(q, q) = (Sq, Sq) \qquad \text{and} \qquad (q, \tilde{V}q) = (Sq, \tilde{V}Sq). \tag{5.13}$$

The first relation means that S is orthogonal (with respect to the mechanical scalar product).[4] The second condition then requires,[5] that $S^{-1}\tilde{V}S = \tilde{V}$; i.e. S *commutes* with $\tilde{V}$

$$[S, \tilde{V}] \equiv S\tilde{V} - \tilde{V}S = 0. \tag{5.14}$$

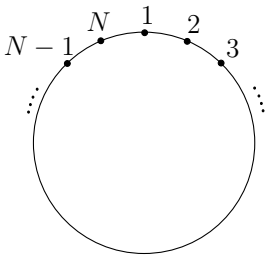
A symmetry is useful when one can determine its eigenspaces more easily than those of $\tilde{V}$. The reason is that, by (5.14), the eigenspaces of S are invariant under $\tilde{V}$; i.e.

$$\text{with} \quad Se_i = \lambda_i e_i \qquad \Longrightarrow \qquad S\tilde{V}e_i = \tilde{V}Se_i = \lambda_i \tilde{V}e_i. \tag{5.15}$$

It therefore suffices to solve the eigenvalue problem of $\tilde{V}$ in each eigenspace of S ("*symmetry sector*") separately. This leads to a reduction of the problem's dimension.[6]

Example: *Cyclic chain*

We consider a system of N identical mass points arranged on a circle and connected by identical springs (spring constant f, fundamental frequency $\Omega = \sqrt{f/m}$). The coordinates $(x_1, \ldots, x_N)$ describe the displacements from the equilibrium position.



The equations of motion read $m\ddot{x}_i = -f[(x_i - x_{i-1}) + (x_i - x_{i+1})]$, and hence from (5.6)

$$(\tilde{V}x)_i = \Omega^2(2x_i - x_{i-1} - x_{i+1}), \qquad (N + 1 \equiv 1).$$

Since all mass points are equivalent, the system has the symmetry

$$S\colon (x_1, \ldots, x_N) \mapsto (x_2, \ldots, x_{N+1} \equiv x_1).$$

The eigenvalue problem $Sx = \lambda x$ for S is easily solved. We find $x_{l+1} = \lambda x_l = \lambda^2 x_{l-1} = \cdots = \lambda^l x_1$. In particular one must have $x_1 = x_{N+1} = \lambda^N x_1$. The map S thus has N eigenvalues

$$\lambda_k = e^{ik\delta}, \qquad \left(k = 0, \ldots, N-1; \delta = \frac{2\pi}{N}\right)$$

[4] Because of (5.3) one can infer from $(q, q) = (Sq, Sq)$ that $(x, y) = (Sx, Sy)$.
[5] Note that $(q, \tilde{V}q) = (Sq, SS^{-1}\tilde{V}Sq) = (q, S^{-1}\tilde{V}Sq)$ holds for all q.
[6] It can even solve the problem completely if all eigenspaces of S are one-dimensional.

with the corresponding (non-normalized) eigenvectors

$$\tilde{e}_k = \left(1, e^{ik\delta}, e^{2ik\delta}, \ldots, e^{(N-1)ik\delta}\right).$$

Since the eigenspaces of S are one-dimensional, the problem reduced to a symmetry sector becomes trivial. We immediately obtain $\tilde{V}\tilde{e}_k = \omega_k^2 \tilde{e}_k$ with

$$\omega_k^2 = \Omega^2(2 - e^{-ik\delta} - e^{ik\delta}) = 2\Omega^2(1 - \cos k\delta) = 4\Omega^2 \sin^2 \frac{k\delta}{2},$$

i.e. the eigenfrequencies of the cyclic chain are

$$\omega_k = 2\Omega \sin \frac{k\pi}{N}.$$

We see that the modes k and $k' = N - k$ are degenerate. The general motion of the chain is given by (5.10).[7]

Special eigenmodes are:

• $k = 0$: with $\omega_0 = 0, \tilde{e}_0 = (1, \ldots, 1)$. This mode is called *null mode* and its solution $\xi_0(t) = \xi_0(0) + \dot{\xi}(0)t$ corresponds to a uniform rotation of the chain.

• $k = N/2$ (for N even): with $\omega_{N/2} = 2\Omega$, $\tilde{e}_{N/2} = (1, -1, \ldots, 1, -1)$. Thus adjacent mass points oscillate in opposite phase.

The remaining eigenvectors are not real. Here $\tilde{e}_k$ and $\tilde{e}_{-k} \equiv \tilde{e}_{N-k}$ are each degenerate. They correspond to counter-propagating waves $\mathrm{Re}(\tilde{e}_{\pm k} e^{-i\omega_k t})_j = \cos(kj\delta \mp \omega_k t)$ of equal frequency ω_k. The real eigenvectors of this frequency are the standing waves

$$\frac{1}{2}(\tilde{e}_k + \tilde{e}_{-k}) = (1, \cos k\delta, \ldots, \cos(N-1)k\delta),$$
$$\frac{1}{2i}(\tilde{e}_k - \tilde{e}_{-k}) = (0, \sin k\delta, \ldots, \sin(N-1)k\delta).$$

5.2 Parametric Resonance

Besides resonance in a driven oscillation, when the driver's frequency matches the natural frequency, there exists another kind of resonance phenomenon called *parametric resonance*, which we treat in this chapter. A generic example is the problem of a harmonic oscillator

$$\ddot{x} = -\omega^2(t)x, \qquad \omega(t+T) = \omega(t); \tag{5.16}$$

whose frequency is periodically modulated. Note that at no time is a force acting on the oscillator. In fact $x \equiv 0$ is always a solution of (5.16), and the oscillation is therefore not directly driven from the outside.

[7] Of course one should first form real and normalized eigenvectors e_k for $\tilde{V}$ from the degenerate $\tilde{e}_k$, $\tilde{e}_{N-k}$, see also below.

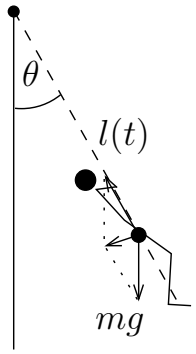


Figure 5.1: One person stands on a swing (a simple pendulum). They drive the motion by moving their center of mass up and down, thereby periodically changing the distance $l(t)$ from the suspension point to the center of mass.

Example: Such systems often arise through linearization about a periodic solution of a nonlinear problem (see, e.g., (3.62)). Equation (5.16) also describes standing on a swing, see Figure 5.1. The distance from the suspension point to the center of mass is thereby varied periodically along the rope according to $l(t) = l(t+T)$. The angular momentum theorem (4.32) requires that

$$\frac{d}{dt}(ml^2\dot{\theta}) = -mgl\sin\theta\,.$$

For small displacements ($\theta \ll 1$) it follows that

$$\frac{d}{dt}(l^2\dot{\theta}) = l\frac{d^2}{dt^2}(l\theta) - l\ddot{l}\theta = -gl\theta$$

and hence

$$\frac{d^2}{dt^2}(l\theta) = -\frac{g-\ddot{l}}{l}\,(l\theta)\,.$$

With $\omega^2(t) = (g-\ddot{l})/l$ and $x = l\theta$ this corresponds exactly to the equation of motion (5.16) of parametric resonance.

As a first step we rewrite equation (5.16) as the system

$$\dot{z} = A(t)z, \qquad z = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \qquad A(t) = \begin{pmatrix} 0 & 1 \\ -\omega^2(t) & 0 \end{pmatrix}. \tag{5.17}$$

of first-order differential equations; of course $A(t+T) = A(t)$ holds. Now let $P(t)$ be the propagator of (5.17) for the time interval from 0 to t, defined by

$$z(t) = P(t)z(0). \tag{5.18}$$

Example: We seek the propagator of (5.17) for $\omega(t) \equiv \omega$ constant. The general solution of the harmonic oscillator problem is then given (cf. (5.10)) by

$$x(t) = x(0)\cos(\omega t) + \dot{x}(0)\frac{1}{\omega}\sin(\omega t).$$

We obtain

$$z(t) = \begin{pmatrix} x(0)\cos(\omega t) + \dot{x}(0)\omega^{-1}\sin(\omega t) \\ -x(0)\omega\sin(\omega t) + \dot{x}(0)\cos(\omega t) \end{pmatrix}$$

and hence the propagator

$$P_\omega(t) = \begin{pmatrix} \cos(\omega t) & \omega^{-1}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix} \tag{5.19}$$

of an oscillator with constant angular frequency ω.

The periodicity of A requires that

$$P(nT) = P(T)^n\,, \qquad (n = 1, 2, 3, \ldots)\,. \tag{5.20}$$

Exponentially growing solutions therefore occur whenever $P(T)$ has an eigenvalue $\lambda \in \mathbb{C}$ with $|\lambda| > 1$; conversely, all solutions are bounded and the system is stable if all eigenvalues $|\lambda_k| \leq 1$.

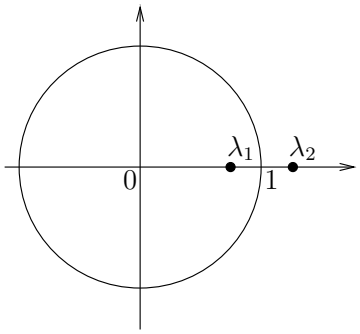
Since the trace of A vanishes ($\operatorname{Tr} A(t) = 0$) it follows[8]

$$\frac{d}{dt}\operatorname{Det} P(t) = \operatorname{Tr}\Big(\underbrace{\frac{dP}{dt}P(t)^{-1}}_{A(t)}\Big)\operatorname{Det} P(t) = 0 \tag{5.21}$$

and therefore $\operatorname{Det} P(t) = \operatorname{Det} P(0) = 1$.

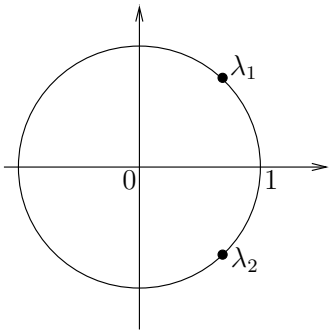
The two eigenvalues λ_1, λ_2 of $P(T)$ satisfy $\lambda_1\lambda_2 = 1$. Moreover $P(T)$ is real, and if λ_i is an eigenvalue then λ_i^* must also be an eigenvalue. One thereby finds the two possibilities:

1. $\lambda_i = \lambda_i^*$: λ_i real



$|\operatorname{Tr} P(T)| = |\lambda_1 + \lambda_2| > 2$

2. $\lambda_1 = \lambda_2^*$: $\lambda_i\lambda_i^* = 1$



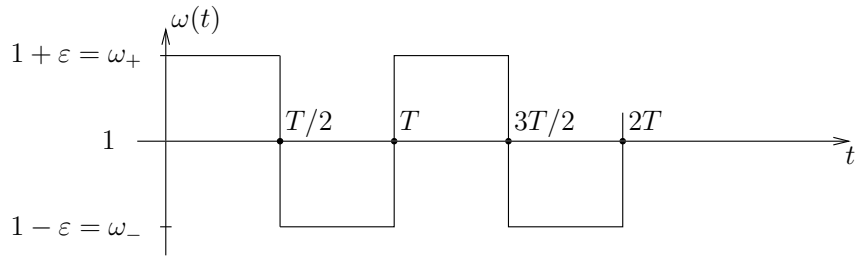
$|\operatorname{Tr} P(T)| = |\lambda_1 + \lambda_2| \leq 2$

In the first case the system is unstable ($|\lambda_2| > 1$), in the second case stable ($|\lambda_1| = |\lambda_2| = 1$).[9] The stability of the system is therefore determined by the trace of $P(T)$.

[8] We have $\operatorname{Det}(1 + \varepsilon A) = 1 + \varepsilon\operatorname{Tr}(A) + O(\varepsilon^2)$. Hence (to first order in ΔP) $\operatorname{Det}(P + \Delta P) = \operatorname{Det}(P)\operatorname{Det}(1 + \Delta P\,P^{-1}) = \operatorname{Det}(P) + \operatorname{Tr}(\Delta P\,P^{-1})\operatorname{Det}(P)$.

[9] The borderline case $\lambda_1 = \lambda_2 = 1$ is stable.

Example: We investigate the system for the simple case in which $\omega(t)$ is given simply by the step function



between the values $\omega_+ = 1+\varepsilon$ and $\omega_- = 1-\varepsilon$. The value $\omega = 1$ for the undisturbed oscillation with $\varepsilon = 0$ corresponds here to the choice of the time unit: i.e. we have set the period of the undisturbed oscillation to $T_0 = 2\pi$. The parameter ε characterizes the strength of the perturbation.

Because of the step form of $\omega(t)$ we obtain directly

$$P(T) = P_{\omega_-}(T/2)P_{\omega_+}(T/2)$$

for our problem with $P_\omega(t)$ from (5.19). For the stability analysis we determine[10]

$$\begin{aligned}
\operatorname{Tr} P(T) &= \cos\frac{\omega_+T}{2}\cos\frac{\omega_-T}{2} - \frac{\omega_+}{\omega_-}\sin\frac{\omega_+T}{2}\sin\frac{\omega_-T}{2} \\
&\quad + \cos\frac{\omega_+T}{2}\cos\frac{\omega_-T}{2} - \frac{\omega_-}{\omega_+}\sin\frac{\omega_+T}{2}\sin\frac{\omega_-T}{2} \\
&= (\cos T + \cos\varepsilon T) + \underbrace{\frac{1}{2}\left(\frac{\omega_+}{\omega_-}+\frac{\omega_-}{\omega_+}\right)}_{=(1+\varepsilon^2)(1-\varepsilon^2)}(\cos T - \cos\varepsilon T) \\
&= \frac{2}{1-\varepsilon^2}\left(\cos T - \varepsilon^2\cos\varepsilon T\right) \\
&= 2 - \frac{4}{1-\varepsilon^2}\left(\sin^2\frac{T}{2} - \varepsilon^2\sin^2\frac{\varepsilon T}{2}\right) \qquad (5.22) \\
\text{or:}\quad &= -2 + \frac{4}{1-\varepsilon^2}\left(\cos^2\frac{T}{2} - \varepsilon^2\cos^2\frac{\varepsilon T}{2}\right).
\end{aligned}$$

We now regard T (and hence the modulation frequency) as variable and seek, for given ε, the T-ranges in which the system is unstable (resonance ranges). There arise the two possibilities for instability:

$$\text{(a)} \qquad \operatorname{Tr} P(T) > 2: \qquad \sin^2\frac{T}{2} < \varepsilon^2\sin^2\frac{\varepsilon T}{2}, \qquad (5.23)$$

$$\text{(b)} \qquad \operatorname{Tr} P(T) < -2: \qquad \cos^2\frac{T}{2} < \varepsilon^2\cos^2\frac{\varepsilon T}{2}. \qquad (5.24)$$

[10] In the second equation we use the identities $\cos(a+b)+\cos(a-b) = 2\cos a\,\cos b$ and $\cos(a+b)-\cos(a-b) = -2\sin a\,\sin b$; in the fourth and fifth, respectively, the identities $\cos(2a) = 1-2\sin^2 a$ and $\cos(2a) = 2\cos^2 a - 1$.

We discuss the two cases for $\varepsilon \ll 1$:

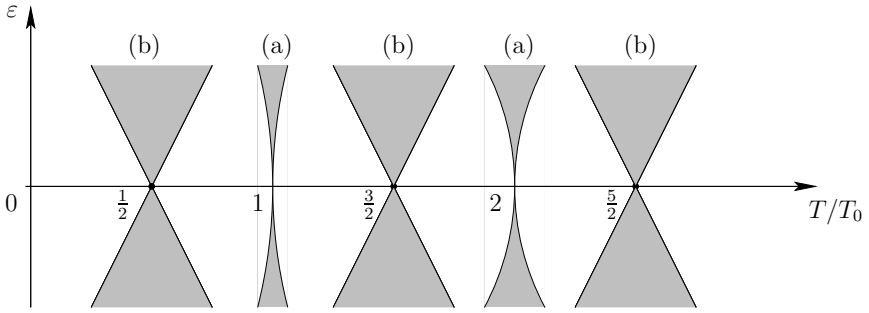
(a) For $\varepsilon = 0$ we have $\operatorname{Tr} P(T) = 2$ if $T = 2\pi n$, $(n = 1, 2, 3, \ldots)$. For small ε we seek solutions of (5.23) in the form $T = 2\pi n + x$ with $|x| \ll 1$. Approximately, for the resonance ranges,

$$|x| < 2\pi n \varepsilon^2 \,. \tag{5.25}$$

(b) For $\varepsilon = 0$ we have $\operatorname{Tr} P(T) = -2$ if $T = (2n+1)\,\pi$, $(n = 0, 1, 2, \ldots)$. For small ε we set $T = (2n+1)\pi + x$, and (5.24) then becomes

$$|x| < 2\varepsilon \,. \tag{5.26}$$

In case (b) the width of the resonance intervals is independent of n and vanishes like $O(\varepsilon)$ for $\varepsilon \to 0$. In case (a) this width grows, for fixed ε, in proportion to n and vanishes like $O(\varepsilon^2)$. In arbitrary units $T/2\pi = T/T_0$ is the ratio of the periods of the perturbation and of the free oscillation. Thus the stability diagram is approximately given by:



Note that the resonance region at $T = \frac{1}{2}T_0$ is larger than if one drives with the natural frequency $(T = T_0)$. That is, it is better to drive a parametric resonator at twice its eigenfrequency than at its natural frequency.

5.3 The Vibrating String

We model a string by a periodic array of identical point masses of mass m at the locations $x_i = ai$ along the 1-axis $(i \in \mathbb{Z})$, see Figure 5.2. Each pair of adjacent point masses interacts via a spring force with spring constant f. We assume that the string is tensioned by an external force F_0 so that in equilibrium the desired spacing $a = F_0/f$ is attained.

The potential energy of the springs has the form

$$V = \frac{1}{2} f \sum_{i \in \mathbb{Z}} (\boldsymbol{u}_{i+1} - \boldsymbol{u}_i + a\boldsymbol{e}_1)^2 \tag{5.27}$$

with the displacements $\boldsymbol{u}_i$ of the i-th point mass from its equilibrium position.

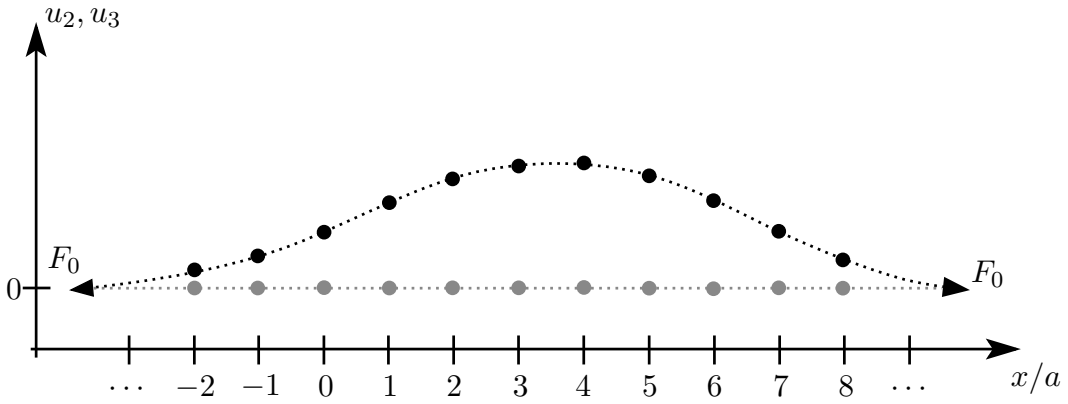


Figure 5.2: Modeling a string as an array of point masses at each position $x_i = ai$ along the 1-axis. Adjacent point masses are connected by a spring with spring constant f. The string is tensioned by an external force $F_0 = af$, which guarantees the correct equilibrium position (gray). The vector $\boldsymbol{u}_i$ then describes the displacement of the i-th point mass from its equilibrium position.

We are interested in describing small displacements. In particular, we consider transverse displacements with $\boldsymbol{u}_i \cdot \boldsymbol{e}_1 = 0$. Thereby we can rewrite the potential energy as

$$V = V_0 + \frac{1}{2} f \sum_{i \in \mathbb{Z}} (\boldsymbol{u}_{i+1} - \boldsymbol{u}_i)^2 \tag{5.28}$$

with V_0 a ($\boldsymbol{u}_i$-independent) constant. We now obtain directly the equations of motion

$$m \ddot{\boldsymbol{u}}_i = -\frac{\partial V}{\partial \boldsymbol{u}_i} = f(\boldsymbol{u}_{i+1} + \boldsymbol{u}_{i-1} - 2\boldsymbol{u}_i) \tag{5.29}$$

for the i-th point mass.

To obtain a realistic description of a string, we now let the density of point masses increase without bound; i.e. we take the *continuum limit*

$$a \to 0, \quad m \to 0, \qquad \text{with } \rho = m/a \text{ fixed.} \tag{5.30}$$

We replace $x_i = ai$ by the continuous coordinate $x \in \mathbb{R}$ and $\boldsymbol{u}_i$ becomes the differentiable function $\boldsymbol{u}(x)$. In particular (by Taylor expansion) we have

$$\boldsymbol{u}_{i\pm1} \approx \boldsymbol{u}(x) \pm a\boldsymbol{u}'(x) + \frac{1}{2} a^2 \boldsymbol{u}''(x). \tag{5.31}$$

Thereby we obtain the continuum limit

$$\frac{f}{m}(\boldsymbol{u}_{i+1} + \boldsymbol{u}_{i-1} - 2\boldsymbol{u}_i) \xrightarrow{(a\to 0)} \frac{F_0}{\rho} \boldsymbol{u}''(x). \tag{5.32}$$

The equations of motion (5.29) go over in the continuum limit to the *wave equation*

$$\frac{\partial^2}{\partial t^2}\boldsymbol{u}(x,t) = c^2\frac{\partial^2}{\partial x^2}\boldsymbol{u}(x,t) \tag{5.33}$$

with $\boldsymbol{u}(x,t)$ the displacement of the string at position x and time t and

$$c = \sqrt{\frac{F_0}{\rho}} \tag{5.34}$$

the propagation speed. The wave equation is a partial differential equation, since the parameter i, which initially labeled the particles, has now become an additional parameter $x = ai$ of the function $\boldsymbol{u}$.

We will restrict ourselves in the following to the case in which the string executes motion only in the 13-plane with $\boldsymbol{u}(x,t) = u(x,t)\boldsymbol{e}_3$. The general solution of the wave equation has the form (*d'Alembert*)

$$u(x,t) = f(x-ct) + g(x+ct) \tag{5.35}$$

with $f, g\colon \mathbb{R} \to \mathbb{R}$ two arbitrary (twice differentiable) functions. One can verify the solution by direct substitution. Here f describes a disturbance propagating to the right while keeping its shape invariant. Analogously g propagates shape-invariantly to the left. The general solution is a superposition of a right- and a left-moving wave.

Example: *Guitar*

For the case in which the string is clamped at the points $x = 0$ and $x = L$, i.e. $u(0,t) = 0$ and $u(L,t) = 0$, neither of the two functions provides a solution to the problem. The reason is that a right-moving wave is reflected at the point $x = L$ (and thereby converted into a left-moving wave). The solution to the problem is therefore necessarily a superposition of a right-moving and a left-moving wave, i.e. a *standing wave* of the form

$$\begin{aligned} u(x,t) &= A\sin(k(x-ct)) + A\sin(k(x+ct)) \\ &= 2A\sin(kx)\cos(kct). \end{aligned} \tag{5.36}$$

For the standing wave to satisfy the boundary condition at $x = L$ it must hold that

$$k = k_n = n\frac{\pi}{L}, \qquad (n = 1, 2, \dots) \tag{5.37}$$

With (5.36) the string then oscillates at frequency

$$\nu_n = \frac{k_n c}{2\pi} = n\frac{c}{2L}, \tag{5.38}$$

i.e. at frequency $\nu_1 = c/(2L)$ (and its overtones).

Chapter 6

Rigid Bodies

A rigid body is a system of mass points whose relative distances between *all* pairs of points remain constant during the motion. Although this is an idealization, this concept is very useful, and the mechanics of rigid-body motion deserves a detailed presentation. The example we will mainly deal with is the *gyroscope*, in which one point of the rigid body is held fixed.

6.1 Mass distribution and moment of inertia tensor

Mass distribution: A rigid body is described by a *mass distribution*

$$dm = \rho(\boldsymbol{y})\, d^3y \tag{6.1}$$

with mass density $\rho(\boldsymbol{y})$ relative to a *body-fixed* coordinate system $\boldsymbol{y} \in \mathbb{R}^3$. The general configuration in the *space-fixed* inertial system $\boldsymbol{x}$ is then given by

$$\boldsymbol{x}(t) = R(t)\boldsymbol{y} + \boldsymbol{b}(t), \qquad t \mapsto R(t) \in \mathrm{SO}(3) \text{ and } t \mapsto \boldsymbol{b}(t) \in \mathbb{R}^3, \tag{6.2}$$

cf. Figure 6.1. Unlike a mass point, the configuration space is six-dimensional: three coordinates describe the position $\boldsymbol{b}(t)$ of the reference point, while the additional three coordinates $R(t)$ describe the orientation of the rigid body relative to a standard configuration. Often we are interested in a situation in which one point of the rigid body (not necessarily the center of mass) is held fixed. In this case there is only rotational motion about this fixed point, which we set to $\boldsymbol{x} = \boldsymbol{y} = 0$, and no translation. A rigid body in which one point is held fixed is called a *gyroscope*.

With the mass distribution dm one can write various mechanical quantities compactly, e.g.

- the total mass: $M = \int dm$,
- the center of mass: $\boldsymbol{X} = M^{-1} \int dm\, \boldsymbol{x}$,
- the angular momentum: $\boldsymbol{L} = \int dm\, \boldsymbol{x} \times \dot{\boldsymbol{x}}$,

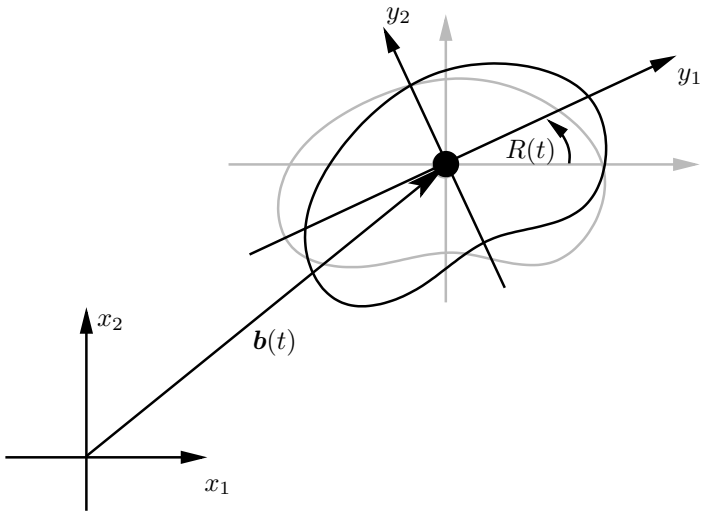


Figure 6.1: The general position of a rigid body is described by a translation $\boldsymbol{b}(t)$ and a rotation $R \in \mathrm{SO}(3)$. The configuration space is therefore six-dimensional.

- the kinetic energy: $T = \frac{1}{2}\int dm\, \dot{\boldsymbol{x}}^2$.

Example: (*concentrated mass points*) One may view the concept of a mass point as a special case of the general mass distribution dm. To do so, one introduces the notion of a concentrated mass point at location $\boldsymbol{y}_0$ of mass 1 as $dm = \delta(\boldsymbol{y} - \boldsymbol{y}_0)d^3y$ (Dirac delta function) with

$$\int dm\, f(\boldsymbol{y}) = \int d^3y\, f(\boldsymbol{y})\delta(\boldsymbol{y} - \boldsymbol{y}_0) = f(\boldsymbol{y}_0)$$

for any function $f\colon \mathbb{R}^3 \to \mathbb{R}$. With this one obtains for the mass distribution

$$dm = \sum_{i=1}^{N} m_i\, \delta(\boldsymbol{y} - \boldsymbol{y}_i)$$

the results from Ch. 1 for N mass points. As an example we consider the kinetic energy

$$T = \frac{1}{2}\int dm\, \dot{\boldsymbol{x}}^2 = \sum_{i=1}^{N} \frac{1}{2}\int d^3y\, \delta(\boldsymbol{y} - \boldsymbol{y}_i)\, \dot{\boldsymbol{x}}^2 = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\boldsymbol{x}}_i^2$$

with $\boldsymbol{x}(\boldsymbol{y}_i, t) = \boldsymbol{x}_i(t)$ the position of the i-th particle at time t. The expression for T agrees with (1.33).

Kinetic energy: We can use the relation (6.2) to express the kinetic energy in body-fixed coordinates. We obtain

$$\begin{aligned} T &= \frac{1}{2}\int dm\,\dot{\boldsymbol{x}}^2 = \frac{1}{2}\int dm(\dot{R}\boldsymbol{y}+\dot{\boldsymbol{b}})^2 = \frac{1}{2}\int dm(\dot{\boldsymbol{b}}^2 + \overbrace{\dot{R}\boldsymbol{y}\cdot\dot{R}\boldsymbol{y}}^{=R^t\dot{R}\boldsymbol{y}\cdot R^t\dot{R}\boldsymbol{y}} + 2\dot{R}\boldsymbol{y}\cdot\dot{\boldsymbol{b}}) \\ &= \frac{1}{2}M\dot{\boldsymbol{b}}^2 + \frac{1}{2}\int dm\,(\boldsymbol{\omega}\times\boldsymbol{y})^2 + M\dot{R}\boldsymbol{Y}\cdot\dot{\boldsymbol{b}} \end{aligned} \tag{6.3}$$

with the center of mass $\boldsymbol{Y} = R^t(\boldsymbol{X}-\boldsymbol{b})$ and the angular velocity $\boldsymbol{\omega}$ in y-coordinates.[1]

In the following we will place the body-fixed coordinate system either at the center of mass (so that $\boldsymbol{Y}=0$) or (in the case of a gyroscope) at the fixed point (so that $\boldsymbol{b}=0$). In both cases the last term of (6.3) vanishes. Thus the kinetic energy $T = \frac{1}{2}M\dot{\boldsymbol{b}}^2 + T_{\text{rot}}$ separates into the kinetic energy of the reference point (as if the entire mass were concentrated at that point) and the *rotational energy*

$$\begin{aligned} T_{\text{rot}} &= \frac{1}{2}\int dm\,(\boldsymbol{\omega}\times\boldsymbol{y})^2 = \frac{1}{2}\int dm\,\left(\boldsymbol{y}^2\boldsymbol{\omega}^2 - (\boldsymbol{y}\cdot\boldsymbol{\omega})^2\right) \\ &= \frac{1}{2}\sum_{i,k=1}^{3}\Theta_{ik}\omega_i\omega_k \end{aligned} \tag{6.4}$$

where we have defined the coefficients of the *moment of inertia tensor* Θ by

$$\Theta_{ik} = \int dm\,(\boldsymbol{y}^2\delta_{ik} - y_i y_k) \tag{6.5}$$

Angular momentum: For the calculation of the angular momentum we again consider the two cases with $\boldsymbol{Y}=0$ or $\boldsymbol{b}=0$. From (6.2) we thereby obtain the result

$$\boldsymbol{L} = \int dm\,\boldsymbol{x}\times\dot{\boldsymbol{x}} = M\boldsymbol{b}\times\dot{\boldsymbol{b}} + \underbrace{\int dm\,R\boldsymbol{y}\times\dot{R}\boldsymbol{y}}_{\equiv R\boldsymbol{S}}, \tag{6.6}$$

where the first term describes the angular momentum of the reference point. The second term is the angular momentum relative to this point (with $\boldsymbol{y}=0$). In y-coordinates it takes the form[2]

$$\begin{aligned} \boldsymbol{S} &= \int dm\,\boldsymbol{y}\times R^t\dot{R}\boldsymbol{y} = \int dm\,\overbrace{\boldsymbol{y}\times(\boldsymbol{\omega}\times\boldsymbol{y})}^{=\boldsymbol{y}^2\boldsymbol{\omega}-(\boldsymbol{y}\cdot\boldsymbol{\omega})\boldsymbol{y}} \\ &\equiv \Theta\,\boldsymbol{\omega}. \end{aligned} \tag{6.7}$$

With the angular momentum vector $\boldsymbol{S}$ we can rewrite the rotational energy as

$$T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{S}. \tag{6.8}$$

[1] As already in Chapter 1.5 we write $R^t\dot{R}\boldsymbol{y} = \boldsymbol{\omega}\times\boldsymbol{y}$ with $\boldsymbol{\omega}$ the y-components of the angular velocity.

[2] Note that $R(\boldsymbol{a}\times\boldsymbol{b}) = R\boldsymbol{a}\times R\boldsymbol{b}$ for arbitrary $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$.

Moment of inertia tensor: The mapping $\Theta\colon \boldsymbol{\omega} \mapsto \boldsymbol{S}$ given by (6.7) is the *moment of inertia tensor* of the rigid body relative to $\boldsymbol{y} = 0$. In components it has the form

$$S_i = \sum_{k=1}^{3} \Theta_{ik}\omega_k \tag{6.9}$$

with Θ_{ik} from (6.5). Since the mass distribution is fixed in the y-system, the components of Θ_{ik} do not depend on time; they rather characterize an intrinsic property of the rigid body. Furthermore, the Θ-matrix is symmetric, $\Theta_{ik} = \Theta_{ki}$. The moment of inertia tensor Θ therefore possesses three orthonormal eigenvectors $\boldsymbol{f}_i$ (*principal axes of inertia*) with

$$\Theta\, \boldsymbol{f}_i = \theta_i \boldsymbol{f}_i\,, \qquad (i = 1, 2, 3)\,. \tag{6.10}$$

As is evident from (6.4), the quadratic form associated with Θ, namely $2T_{\text{rot}} \geq 0$, is positive-semidefinite. In fact it is positive-definite provided the mass distribution is nondegenerate (i.e. not concentrated on a line through $\boldsymbol{y} = 0$). In the following we usually describe the system in the principal-axis system, in which

$$\Theta = \begin{pmatrix} \theta_1 & & \\ & \theta_2 & \\ & & \theta_3 \end{pmatrix}, \qquad S_i = \theta_i\omega_i, \qquad T = \frac{1}{2}\sum_{i=1}^{3} \theta_i\omega_i^2 \tag{6.11}$$

holds with the *principal moments of inertia* $\theta_i > 0$.

6.2 Equations of motion

The six equations of motion that determine the motion of a rigid body are given by

$$M\ddot{\boldsymbol{X}} = \boldsymbol{F} \qquad \text{(linear momentum theorem)}\,, \tag{6.12}$$

$$\dot{\boldsymbol{L}} = \boldsymbol{M} \qquad \text{(angular momentum theorem)}\,, \tag{6.13}$$

with $\boldsymbol{F}$ the total force and $\boldsymbol{M}$ the total torque (relative to $\boldsymbol{x} = 0$).

Because of the simple relation between $\boldsymbol{\omega}$ and $\boldsymbol{S}$ via the moment of inertia tensor in the body-fixed system, it is useful to transform the angular momentum theorem into the body-fixed system. After applying R^t to the angular momentum theorem (6.13) and using $\dot{\boldsymbol{L}} = M\boldsymbol{b} \times \ddot{\boldsymbol{b}} + \dot{R}\boldsymbol{S} + R\dot{\boldsymbol{S}}$ (from (6.6)), we obtain

$$\dot{\boldsymbol{S}} + \boldsymbol{\omega} \times \boldsymbol{S} + R^t(\boldsymbol{b} \times M\ddot{\boldsymbol{b}}) = R^t\boldsymbol{M}. \tag{6.14}$$

These equations of motion, together with the linear momentum theorem (6.12) and the relation $\boldsymbol{S} = \Theta\,\boldsymbol{\omega}$, determine the trajectory. We now simplify the equations of motion in the two cases a bit further:

Center of mass as reference point ($\boldsymbol{Y} = 0$): In this case $\boldsymbol{b} = \boldsymbol{X}$. Then from the linear momentum theorem we obtain the relation

$$\dot{\boldsymbol{S}} + \boldsymbol{\omega} \times \boldsymbol{S} = \boldsymbol{N} \tag{6.15}$$

with $\boldsymbol{N} = R^t(\boldsymbol{M} - \boldsymbol{X} \times \boldsymbol{F})$ the torque relative to the center of mass in y-components.

Gyroscope ($\boldsymbol{b} = 0$): In the case of a gyroscope held fixed at the suspension point $\boldsymbol{x} = \boldsymbol{y} = 0$, only the angular momentum theorem is needed. The relation between the space-fixed and the body-fixed coordinate systems is in this case given by $\boldsymbol{x} = R\boldsymbol{y}$. The equations of motion reduce again to (6.15), this time with $\boldsymbol{N} = R^t\boldsymbol{M}$, the torque relative to the suspension point.

In the principal-axis system (see (6.11)), the *Euler equations*

$$\begin{aligned} \theta_1\,\dot{\omega}_1 + (\theta_3 - \theta_2)\,\omega_2\,\omega_3 &= N_1, \\ \theta_2\,\dot{\omega}_2 + (\theta_1 - \theta_3)\,\omega_3\,\omega_1 &= N_2, \\ \theta_3\,\dot{\omega}_3 + (\theta_2 - \theta_1)\,\omega_1\,\omega_2 &= N_3. \end{aligned} \tag{6.16}$$

follow in components from (6.15). These equations define a system of nonlinear first-order differential equations. By $\boldsymbol{\omega}(t)$, $\Omega(t) = R^t\dot{R}$ is then also given,[3] and to determine $R(t)$ the linear differential equation

$$\dot{R} = R\,\Omega(t) \tag{6.17}$$

remains to be solved.

In general the explicit solution of the equations of motion of a rigid body is difficult. In the remainder of this chapter we restrict ourselves to the treatment of gyroscopes, since for these only $R(t)$ but not additionally $\boldsymbol{b}(t)$ must be determined. Depending on the principal moments of inertia we distinguish the following types of gyroscopes:

all θ_i distinct	(unsymmetric gyroscope),
$\theta_1 = \theta_2 \neq \theta_3$	(symmetric gyroscope),
$\theta_1 = \theta_2 = \theta_3$	(spherical gyroscope).

6.3 The free gyroscope

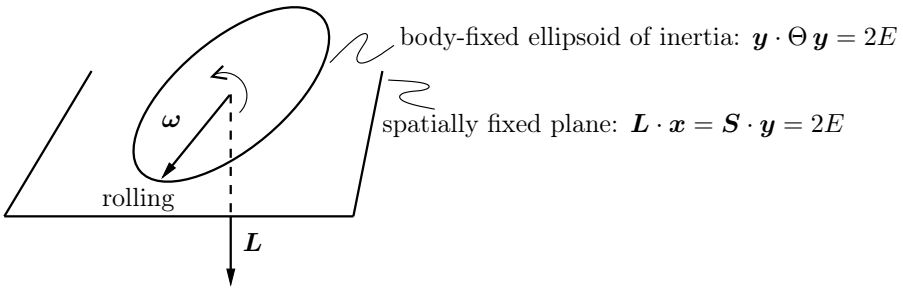
For the *free gyroscope* the Euler equations hold with $\boldsymbol{N} = 0$. A gyroscope is free if either no driving forces act or only the gravitational force in the case that the suspension point coincides with the center of mass. Since then $\boldsymbol{b} = 0$, it follows that $T = T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{S}$. With the equation of motion $\dot{\boldsymbol{S}} = -\boldsymbol{\omega} \times \boldsymbol{S}$ one obtains

$$\dot{T} = \boldsymbol{\omega} \cdot \Theta\,\dot{\boldsymbol{\omega}} = \boldsymbol{\omega} \cdot \dot{\boldsymbol{S}} = 0, \tag{6.18}$$

[3] We recall once more that $\boldsymbol{\omega} \times \boldsymbol{x} = \Omega\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^3$.

i.e. the (kinetic) energy is conserved with $T = E$. Moreover, $\boldsymbol{L}$ is of course also conserved.

The motions of the free gyroscope with energy E can thereby be represented geometrically by the *Poinsot construction*:



By construction the equations are satisfied for $\boldsymbol{y} = \boldsymbol{\omega}$. In fact they (for given R) $\boldsymbol{\omega}$ and hence $\dot{R}$ *uniquely* determine. For let $\boldsymbol{y}$ be any point of the space-fixed plane. Then we decompose $\boldsymbol{y} = \boldsymbol{\omega} + (\boldsymbol{y} - \boldsymbol{\omega})$ and find

$$\boldsymbol{y} \cdot \Theta\, \boldsymbol{y} = \underbrace{\boldsymbol{\omega} \cdot \Theta\, \boldsymbol{\omega}}_{=2E} + 2 \underbrace{(\boldsymbol{y} - \boldsymbol{\omega}) \cdot \overbrace{\Theta\, \boldsymbol{\omega}}^{=\boldsymbol{S}}}_{=2E-2E=0} + \underbrace{(\boldsymbol{y} - \boldsymbol{\omega}) \cdot \Theta (\boldsymbol{y} - \boldsymbol{\omega})}_{\geq 0} . \tag{6.19}$$

Thus $\boldsymbol{y} \cdot \Theta\, \boldsymbol{y} \geq 2E$ and '=' only for $\boldsymbol{y} = \boldsymbol{\omega}$.

The fixed plane (perpendicular to $\boldsymbol{L}$) is therefore the tangent plane of the ellipsoid of inertia at the point $\boldsymbol{\omega}$. Since $\boldsymbol{\omega}$ is the instantaneous axis of rotation, the point of contact of the ellipsoid with the tangent plane has zero velocity, i.e., the ellipsoid of inertia associated with the principal axes of inertia of a top *rolls* without slipping on the plane while its center is held fixed (torque $\boldsymbol{L}$ constant).

During this motion the angular velocity $\boldsymbol{\omega}$ (point of contact of the ellipsoid) moves around $\boldsymbol{L}$.

A simple analytical treatment of the free top is possible only in special cases. In the following we describe two of them.

Permanent rotations: A simple solution exists if the top rotates about one of its principal axes. As an example we choose

$$\boldsymbol{\omega} = (\omega_0, 0, 0)\,, \tag{6.20}$$

where ω_0 is constant. This is obviously a solution of the Euler equations. To investigate the stability of this solution, we consider a perturbation $\boldsymbol{\omega} = (\omega_0 + \omega_1, \omega_2, \omega_3)$ by keeping only linear terms in the small quantities $\omega_1, \omega_2, \omega_3$. In this approximation

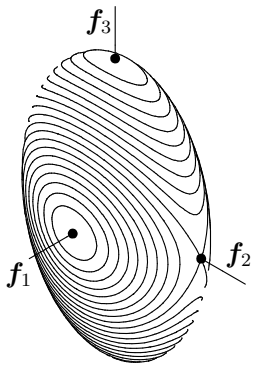


Figure 6.2: In the diagram the intersection curves are shown for the case $\theta_1 < \theta_2 < \theta_3$.

the Euler equations are given by

$$\begin{aligned} \theta_1\dot{\omega}_1 &= 0\,, \\ \theta_2\dot{\omega}_2 &= (\theta_3-\theta_1)\omega_0\omega_3\,, \\ \theta_3\dot{\omega}_3 &= (\theta_1-\theta_2)\omega_0\omega_2\,. \end{aligned} \tag{6.21}$$

The first equation states that ω_1 remains constant, and the remaining two can be written in matrix form as

$$\begin{pmatrix} \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\theta_3-\theta_1}{\theta_2}\omega_0 \\ \frac{\theta_1-\theta_2}{\theta_3}\omega_0 & 0 \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_3 \end{pmatrix}. \tag{6.22}$$

The eigenvalues $\pm\lambda$ of the 2×2 matrix are given by

$$\lambda^2 = \frac{(\theta_3-\theta_1)(\theta_1-\theta_2)}{\theta_2\theta_3}\omega_0^2 \tag{6.23}$$

and hence are either both real or both purely imaginary. The first case (with $\lambda \neq 0$) occurs if θ_1 lies *between* θ_2 and θ_3, i.e., if $\theta_2 < \theta_1 < \theta_3$, or if $\theta_3 < \theta_1 < \theta_2$. Then (6.22) has exponentially growing solutions, i.e., the rotation is *unstable*. In the remaining cases $\lambda^2 < 0$, the eigenvalues are purely imaginary and the solutions are bounded. This means that of the three principal axes only two exhibit stable permanent rotations, namely those with the smallest and the largest principal moment of inertia.

The same result also follows outside the linear approximation: as we have seen above, the kinetic energy T and the angular momentum $\boldsymbol{L}$ are conserved, and hence

$$T = \frac{1}{2}\sum_{i=1}^{3}\theta_i\omega_i^2 = \frac{1}{2}\sum_{i=1}^{3}\frac{S_i^2}{\theta_i} \equiv E\,, \tag{6.24}$$

$$\boldsymbol{L}\cdot\boldsymbol{L} = \boldsymbol{S}\cdot\boldsymbol{S} = \sum_{i=1}^{3}S_i^2 \equiv l^2\,. \tag{6.25}$$

Thus $\boldsymbol{S}$ always lies on the intersection of an ellipsoid (with semi-axes $\sqrt{2E\theta_i}$) and a sphere of radius l, see Figure 6.2. The intersection curves for $\theta_1 < \theta_2 < \theta_3$ near the

principal axes $\boldsymbol{f}_1$ and $\boldsymbol{f}_3$ are small closed curves, i.e., $\boldsymbol{S}$ (and hence $\boldsymbol{\omega}$) remains in their vicinity. Not so near $\boldsymbol{f}_2$.

The symmetric free top: In the special case of a symmetric body the ellipsoid of inertia is a spheroid, i.e., two of the three principal moments of inertia are equal. Without loss of generality let $\theta_1 = \theta_2$ and hence $\boldsymbol{f}_3$ is the *figure axis*. The Euler equations then simplify to

$$\begin{aligned} \dot{\omega}_1 &= -\alpha\,\omega_2\,, \\ \dot{\omega}_2 &= \alpha\,\omega_1\,, \\ \dot{\omega}_3 &= 0\,. \end{aligned} \qquad \left(\alpha = \frac{\theta_3 - \theta_1}{\theta_1}\omega_3\right), \tag{6.26}$$

The last equation requires that ω_3 is constant. Hence α is also constant. The solution of the differential equation for ω_1 and ω_2 is then given by

$$\omega_1(t) + i\omega_2(t) = (\omega_1(0) + i\omega_2(0))e^{i\alpha t}\,. \tag{6.27}$$

The vector $\boldsymbol{\omega}$ therefore rotates about the figure axis with constant angular velocity α. The motion $R(t)$ results as a special case of the next section or via the Poinsot construction: one simply sees that for the symmetric top the angular momentum $\boldsymbol{S}$, the axis of rotation $\boldsymbol{\omega}$, and the figure axis $\boldsymbol{f}_3$ always lie in one plane.[4]

If now the axis of rotation moves about the figure axis with angular velocity α, this implies in space-fixed coordinates that both the figure axis and the axis of rotation move about the space-fixed angular momentum $\boldsymbol{L}$ with angular velocity α. One says that the figure axis *precesses*.

Example (Eulerian theory of polar motion)

The kinematic North Pole (direction of the instantaneous axis of rotation $\boldsymbol{\omega}$) of the Earth rotates about the geographic North Pole (figure axis). The external torques acting on the Earth are so weak that the rotational motion can be regarded as torque-free. The Earth is symmetric about the polar axis and flattened at the poles, so that θ_1 is smaller than θ_3. Specifically, the numerical ratio of the moments of inertia is given by

$$\frac{\theta_1 - \theta_3}{\theta_1} = -0.0033\,. \tag{6.28}$$

The period of the precessional motion is therefore

$$T = \frac{2\pi}{\alpha} = \underbrace{\frac{2\pi}{\omega_3}}_{=1\text{Tag}}\,\underbrace{\frac{\theta_1}{\theta_3 - \theta_1}}_{\approx 300} \approx 300 \text{ days}\,. \tag{6.29}$$

[4] In body-fixed coordinates $\boldsymbol{\omega}\times\boldsymbol{f}_3 = (\omega_2, -\omega_1, 0)$, $\boldsymbol{S} = (\theta_1\omega_1, \theta_1\omega_2, \theta_3\omega_3)$ and hence $\boldsymbol{S}\cdot(\boldsymbol{\omega}\times\boldsymbol{f}_3) = 0$.

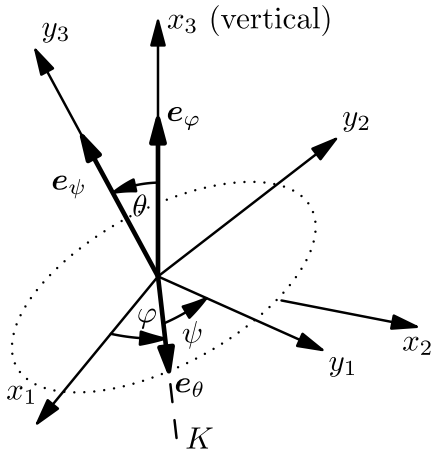


Figure 6.3: Coordinate system for describing the heavy symmetric top: the angle θ gives the inclination of the top. One also calls θ the "angle of nutation." The angle φ is called the "angle of precession." It gives the orientation of the projection of the top's axis (the nodal line K) onto the "ground." The angle ψ is the "angle of eigenrotation."

It follows that the rotation axis of the Earth describes a circle about the North Pole in the course of 10 months. An (albeit irregular) motion of this kind has been observed. The amplitude of the precession is very small, the rotation axis never deviating by more than about 4.5 m from the North Pole. The path is, however, completely irregular, and the fundamental period appears to be approximately 427 days (and not 300 days). The fluctuations are attributed to small displacements of the mass distribution on Earth, such as those caused by atmospheric motions, while the difference in the period from the value above arises from the fact that the Earth is not perfectly rigid but has the elastic properties of a material like steel.[5]

6.4 The heavy symmetric top

The other important example is the top in a uniform gravitational field. In the following we consider the case of the symmetric top, i.e., we again assume that $\theta_1 = \theta_2$. By symmetry the center of mass then lies on the figure axis. In the body-fixed coordinate system it therefore has the components

$$\boldsymbol{y}_S = (0, 0, l)\,, \qquad l > 0\,. \tag{6.30}$$

We choose the space-fixed x-system so that the x_3-axis points vertically upward. As configuration coordinates we use *the Euler angles* (φ, θ, ψ), cf. Figure 6.3. The body-fixed y-system (with basis vectors $\boldsymbol{f}_i$) is obtained from the space-fixed x-system

[5]The torque-free precession of the Earth's axis must not be confused with its slow precession about the normal to the ecliptic. That astronomical precession arises from gravitational forces of the Sun and the Moon, which were considered negligible in the above discussion. This assumption is justified because the period of the astronomical precession is very long (26,000 years).

(with basis vectors $\boldsymbol{e}_i$) by the following sequence of three rotations[6]

$$R = R_1(\varphi)R_2(\theta)R_3(\psi)$$

Rotation	Rotation axis	Rotation angle	
1.	x_3	φ	($\boldsymbol{e}_1$ goes over to the nodal line)
2.	K	θ	($\boldsymbol{e}_3$ goes onto $\boldsymbol{f}_3$)
3.	y_3	ψ	

where the "nodal line" K is the image of the x_1-axis under the first rotation. The corresponding chart on SO(3) is the domain

$$\{0 < \theta < \pi; 0 < \varphi, \psi < 2\pi\} \subset \mathbb{R}^3 \,. \tag{6.31}$$

One also calls the three Euler angles the angle of precession (φ), the angle of nutation (θ), and the angle of eigenrotation (ψ).

The rotation axes are given by the unit vectors $\boldsymbol{e}_\varphi$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\psi$. In the body-fixed coordinate system $\boldsymbol{e}_\psi = \boldsymbol{f}_3$ is just the unit vector in the y_3-direction. The other two rotation axes are somewhat more complicated. But one easily finds geometrically that

$$\begin{aligned} \boldsymbol{e}_\theta &= \cos\psi\, \boldsymbol{f}_1 - \sin\psi\, \boldsymbol{f}_2 \,, \\ \boldsymbol{e}_\varphi &= \sin\theta(\sin\psi\, \boldsymbol{f}_1 + \cos\psi\, \boldsymbol{f}_2) + \cos\theta\, \boldsymbol{f}_3 \end{aligned} \tag{6.32}$$

with $\boldsymbol{f}_i$ the unit vectors along the y_i-axes.

The angular velocity (in body-fixed components) is therefore given by

$$\begin{aligned} \boldsymbol{\omega} &= \dot\varphi\, \boldsymbol{e}_\varphi + \dot\theta\, \boldsymbol{e}_\theta + \dot\psi\, \boldsymbol{e}_\psi \,, \\ &= \left(\dot\varphi\, \sin\theta \sin\psi + \dot\theta\, \cos\psi, \dot\varphi\, \sin\theta \cos\psi - \dot\theta \sin\psi, \dot\varphi\, \cos\theta + \dot\psi\right) \,. \end{aligned} \tag{6.33}$$

The kinetic energy

$$T = \frac{\theta_1}{2}(\omega_1^2 + \omega_2^2) + \frac{\theta_3}{2}\omega_3^2 \tag{6.34}$$

then evaluates to

$$T = \frac{\theta_1}{2}\left(\dot\varphi^2 \sin^2\theta + \dot\theta^2\right) + \frac{\theta_3}{2}\left(\dot\psi + \dot\varphi\cos\theta\right)^2 \,. \tag{6.35}$$

The center of mass has in the space-fixed coordinate system the form

$$\boldsymbol{x}_S = (*, *, l\cos\theta) \,. \tag{6.36}$$

This leads to the Lagrangian

$$L = T - V \,, \tag{6.37}$$

with

$$V = mgl\cos\theta \,. \tag{6.38}$$

[6]We describe the rotations as transformations from the x- to the y-system. The order of the rotations is thus the reverse of that in the definition $\boldsymbol{x} = R\boldsymbol{y}$. The order of the Euler angles is chosen so that typically $\dot\psi > \dot\theta > \dot\varphi$.

Conservation laws: Since L does not depend on t, φ, and ψ, the system possesses the three conserved quantities:

$$T + V = E\,, \tag{6.39}$$

$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = \dot\varphi\,\theta_1 \sin^2\theta + (\dot\psi + \dot\varphi\cos\theta)\,\theta_3\cos\theta \equiv L_3\,, \tag{6.40}$$

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = (\dot\psi + \dot\varphi\cos\theta)\,\theta_3 \equiv S_3\,. \tag{6.41}$$

We denote the corresponding constant values by E, L_3, and S_3. Here, L_3 and S_3 are the projections of the angular momentum onto $\boldsymbol{e}_\varphi = \boldsymbol{e}_3$ (the vertical) and onto $\boldsymbol{e}_\psi = \boldsymbol{f}_3$ (the figure axis), respectively. From $\boldsymbol{S} = \Theta\,\boldsymbol{\omega}$ it follows directly that

$$p_\varphi = \boldsymbol{S}\cdot\boldsymbol{e}_\varphi\,, \qquad p_\psi = \boldsymbol{S}\cdot\boldsymbol{e}_\psi\,. \tag{6.42}$$

The conservation of $T+V$ and p_φ holds more generally even if the top is not symmetric; p_ψ, however, is conserved only for symmetric tops.

Integration of the equations of motion: From (6.40) and (6.41) it follows that

$$\dot\psi + \dot\varphi\cos\theta = \frac{S_3}{\theta_3}\,, \qquad \dot\varphi = \frac{L_3 - S_3\cos\theta}{\theta_1\sin^2\theta}\,. \tag{6.43}$$

Substituting into (6.39) yields

$$E' \equiv E - \frac{S_3^2}{2\theta_3} = \frac{\theta_1}{2}\dot\theta^2 + \frac{(L_3 - S_3\cos\theta)^2}{2\theta_1\sin^2\theta} + mgl\cos\theta\,. \tag{6.44}$$

This is a first-order differential equation for the *nutation motion* $\theta(t)$. To solve it, we introduce the variable $u = \cos\theta$. With $\dot u = -\sin\theta\,\dot\theta$ one then finds

$$\dot u^2 = (\alpha - \beta u)(1-u^2) - (a - bu)^2 \equiv f(u)\,, \tag{6.45}$$

where the constants α, β, a, and b are given by

$$\alpha = \frac{2E'}{\theta_1}\,, \qquad \beta = \frac{2mgl}{\theta_1} > 0\,, \qquad a = \frac{L_3}{\theta_1}\,, \qquad b = \frac{S_3}{\theta_1} \tag{6.46}$$

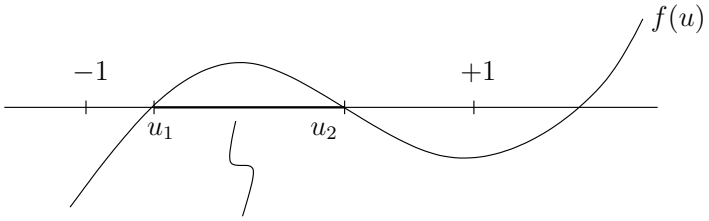
Note that u is defined only for $-1 \le u \le 1$. Since the left-hand side is nonnegative, $\dot u^2 \ge 0$, the general solution is restricted to the region where $f(u) \ge 0$; there it reads

$$t(u) - t(v) = \int_v^u \frac{dx}{\sqrt{f(x)}}\,. \tag{6.47}$$

The function $f(u)$ is a cubic polynomial. The zeros of this cubic correspond to the angles at which $\dot\theta$ changes sign, i.e., the turning angles of θ. For large u, βu^3 is

the dominant term; since $\beta > 0$, $f(u)$ is therefore positive for large positive u and negative for large negative u.

At the points $u = \pm 1$, $f(u) = -(a \mp b)^2$, and is therefore always negative (we consider the general case $b \neq \pm a$)[7]. Hence at least one of the zeros must lie in the region $u > 1$, which does not correspond to any physical angle. By (6.45), $f(u) \geq 0$ somewhere in the physical range ($|u| \leq 1$). For a top, $f(u)$ must therefore have two zeros $u_1 \leq u_2$ lying in the interval $[-1, 1]$, with the general behavior



of $f(u)$. The points u_1 and u_2 are the turning points of the nutation motion. This motion is periodic with period

$$T = 2 \int_{u_1}^{u_2} \frac{dx}{\sqrt{f(x)}} \,. \tag{6.48}$$

Using (6.43), from a solution for $u(t) = \cos\theta(t)$ one can also determine $\varphi(t)$ and $\psi(t)$. In particular,

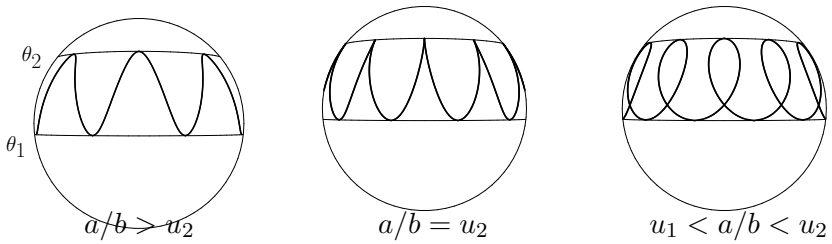
$$\dot{\varphi} = \frac{a - bu}{1 - u^2} \,, \tag{6.49}$$

holds, and hence $\dot{\varphi}(t)$ has *the same* period T. On the other hand, $\varphi(t)$ describes the precession of the figure axis about the vertical. By integrating (6.49), we find that this precession is the sum of a linear function in t (mean precession) and a periodic function of period T.

It is customary to describe the motion of the top by plotting the curve traced by the figure axis on a unit sphere centered at the suspension point. This curve is called the *locus* of the figure axis. The spherical coordinates of a point on the locus in the space-fixed system are identical with the Euler angles θ and φ.[8] The third angle, ψ, describes only the rotation of the top about its own symmetry axis. As we have seen above, the locus lies between the two limiting circles $\theta_1 = \arccos u_1$ and $\theta_2 = \arccos u_2$, where $\dot{\theta}$ vanishes on both circles. The shape of the locus curve is largely determined by the value of the zero $u_0 \equiv a/b$ of $a - bu$. One can show that $u_0 > u_1$. From (6.49) we obtain the result that the precession reverses direction for $u > u_0 = a/b$. Thus we obtain the following locus curves:

[7] The limiting cases correspond to the sleeping top ($b = a$, $S_3 = L_3$) and the hanging top ($b = -a$, $S_3 = -L_3$).

[8] The angle φ is measured unconventionally from the $(-x_2)$ axis.



In the first case $\dot{\varphi} > 0$ for all time. The figure axis thus performs a *precession* about the vertical while θ executes a *nutation* between the limiting circles. The second case with $a/b = u_2$ is not as unusual as one might initially believe. It corresponds to the initial conditions $\dot{\theta}(0) = \dot{\varphi}(0) = 0$ (releasing the top without any initial precession of the figure axis). Then $u(0) = u_0 = a/b$ (because $\dot{\varphi}(0) = 0$) and $f(u_0) = 0$ (because $\dot{\theta}(0) = 0$). In this case the top "falls" from θ_2 down to θ_1 and then rises again. In the last case $\dot{\varphi}$ changes sign at the limiting circle $\theta_0 = \arccos u_0$. For $\theta_1 \leq \theta < \theta_0$, normal precession occurs, whereas for a more upright top (with $\theta_0 < \theta \leq \theta_2$) the precession proceeds backward.

6.4.1 Examples

Pure precession: We obtain a *pure precession* as a special case if $u_1 = u_2 \equiv \bar{u}$. Then $\dot{\varphi}$ is constant and $f(u)$ has a double zero at $\bar{u}$, i.e., $f(\bar{u}) = f'(\bar{u}) = 0$.

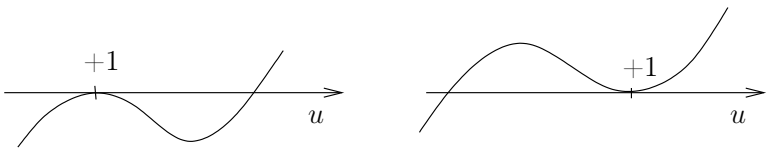
Sleeping top: We consider the stability of the *vertically upright top* with angular velocity ω. Initially $\theta = \dot{\theta} = 0$ and hence $L_3 = S_3 = \theta_3\omega$, $E' = mgl$. We obtain

$$\alpha = \beta = \frac{2mgl}{\theta_1}, \qquad a = b = \frac{\theta_3\omega}{\theta_1}.$$

Thus

$$f(u) = (1-u)^2[\alpha(1+u) - a^2]$$

has a double zero at $u = 1$ and hence $\theta = 0$ is an equilibrium point. To analyze the stability of this motion we consider the behavior of f. Depending on the sign of $[\cdots]_{u=1} = 2\alpha - a^2$ we obtain



$2\alpha < a^2$
d.h. $\omega^2 > 4\theta_1 mgl/\theta_3^2$
(stable)

$2\alpha > a^2$
d.h. $\omega^2 < 4\theta_1 mgl/\theta_3^2$
(unstable)

as possible behaviors.

The first case corresponds to a *fast top* with $|\omega| > \omega_c \equiv 2\sqrt{\theta_1 mgl}/\theta_3$ and $u = 1$, $\theta = 0$ as the only physical solution. The upright position is thus stable and

the top *sleeps*. In the second case ($|\omega| < \omega_c$) the equilibrium solution $\theta = 0$ is unstable, since to the left of $u = 1$ there is an energetically allowed interval. The vertically upright spinning top begins to wobble as soon as $|\omega|$ falls below the stability threshold ω_c due to frictional losses. Thus the *sleeping* top *awakens* after some time and begins to tumble.

Chapter 7

Hamiltonian Systems

For various aspects of mechanics (and especially also for quantum mechanics and statistical mechanics) the Hamiltonian description of mechanics is important. The basic idea is that one does not use the position coordinates and their velocities to describe the system, but rather replaces the velocities by the conjugate momenta. This procedure will now be explained in detail.

7.1 Hamiltonian Function

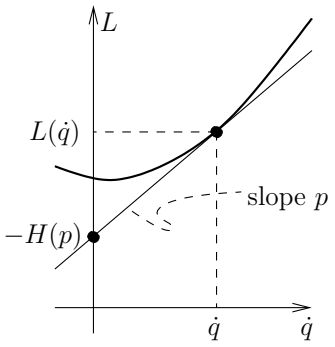
Consider a Lagrangian system for which the Lagrangian function L depends on the position coordinates q^α, their velocities $\dot{q}^\alpha$, and possibly explicitly on time t, $L \equiv L(q, \dot{q}, t)$. The (canonically) conjugate momenta are defined by

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} \equiv p_\alpha(q, \dot{q}, t) \ , \qquad (\alpha = 1, \ldots, f) \tag{7.1}$$

As we will see later, the motion of a mechanical system can often be described more naturally in the *phase space*, whose coordinates are the position coordinates q^α and the corresponding momenta p_α. To arrive at this description, we must therefore express the velocities $\dot{q}^\alpha$ in terms of p_α.

The *Hamiltonian function* $H(q, p, t)$ is obtained from the Lagrangian function $L(q, \dot{q}, t)$ by a Legendre transformation at fixed (q, t). In this process the variable $\dot{q}$ is replaced by p via:

Geometrically:



Analytically:

One solves (7.1) for

$$\dot{q} = \dot{q}(q, p, t) \tag{7.2}$$

and substitutes this into

$$\langle p, \dot{q} \rangle - L = \sum_{\alpha=1}^{f} p_\alpha \, \dot{q}^\alpha - L(q, \dot{q}, t) \equiv H(q, p, t) \tag{7.3}$$

(cf. (4.59)).

We wish to determine the relation between the partial derivatives of H and those of L. We obtain directly

$$\begin{aligned}
\frac{\partial H}{\partial p_\alpha} &= \dot{q}^\alpha + \sum_{\beta=1}^{f} \left(p_\beta - \frac{\partial L}{\partial \dot{q}^\beta} \right) \frac{\partial \dot{q}^\beta}{\partial p_\alpha} = \dot{q}^\alpha \\
\frac{\partial H}{\partial q^\alpha} &= \sum_{\beta=1}^{f} \left(p_\beta - \frac{\partial L}{\partial \dot{q}^\beta} \right) \frac{\partial \dot{q}^\beta}{\partial q^\alpha} - \frac{\partial L}{\partial q^\alpha} = -\frac{\partial L}{\partial q^\alpha} \\
\frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t}
\end{aligned} \tag{7.4}$$

since $\partial L / \partial \dot{q}^\beta = p_\beta$.

7.2 The Canonical Equations of Motion

The equations of motion for the phase space coordinates (q, p) can now be easily derived from the Euler-Lagrange equations $\dot{p}_\alpha = \partial L / \partial q^\alpha$. The resulting Hamiltonian (*canonical*) equations of motion are

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha} \,, \qquad \dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha} \,, \qquad (\alpha = 1, \ldots f) \,. \tag{7.5}$$

These are $2f$ first-order differential equations for the $2f$ unknown functions $q^1(t), \ldots q^f(t)$, $p_1(t), \ldots p_f(t)$.

Examples:

(i) A point mass in an external potential $V(\boldsymbol{x})$:

$$\begin{aligned}
L &= \frac{m}{2} \dot{\boldsymbol{x}}^2 - V(\boldsymbol{x}) \,, \qquad \boldsymbol{p} = m \dot{\boldsymbol{x}} \,, \\
H &= \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{x}) \,.
\end{aligned}$$

(ii) The same system in polar coordinates

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r)\,, \qquad p_r = m\dot{r}\,, \qquad p_\varphi = mr^2\dot{\varphi}\,,$$
$$H = \frac{1}{2m}(p_r^2 + r^{-2}p_\varphi^2) + V(r)\,.$$

(iii) The general autonomous case (cf. Example 9 from Chapter 4.5.3)

$$L = \frac{1}{2}\sum_{\alpha,\beta=1}^{f} g_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta - V(q)\,, \qquad p_\alpha = \sum_{\beta=1}^{f} g_{\alpha\beta}(q)\dot{q}^\beta\,,$$
$$H = \frac{1}{2}\sum_{\alpha,\beta=1}^{f} g^{\alpha\beta}(q)p_\alpha p_\beta + V(q)\,,$$

where $(g^{\alpha\beta}(q))_{\alpha,\beta=1,\dots,f}$ is the inverse of the matrix $(g_{\alpha\beta}(q))_{\alpha,\beta=1,\dots,f}$.

(iv) A charged particle in an external electromagnetic field:

$$L = \frac{m}{2}\dot{\boldsymbol{x}}^2 - e\Big(\varphi(\boldsymbol{x},t) - \frac{\dot{\boldsymbol{x}}}{c}\cdot\boldsymbol{A}(\boldsymbol{x},t)\Big)\,, \qquad \boldsymbol{p} = m\dot{\boldsymbol{x}} + \frac{e}{c}\boldsymbol{A}\,,$$
$$H = \frac{1}{2m}\Big(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{x},t)\Big)^2 + e\varphi(\boldsymbol{x},t)\,.$$

Since the Lagrangian does not depend explicitly on time in these cases, the Hamiltonian function is simply the generalized energy, cf. (4.59). One must however also note that H is a function of (q,p,t), i.e. $\dot{q}$ must be expressed in terms of p.

7.3 Phase Space and Poisson Brackets

We denote the $2f$ variables

$$x \equiv (x_1,\dots,x_{2f}) \equiv (q^1,p_1,\dots,q^f,p_f) \tag{7.6}$$

as the *phase coordinates* of the system, and the associated space as the *phase space*. If the configuration space is $\mathbb{R}^f$, then the phase space is simply $\mathbb{R}^{2f}$. The canonical equations (7.5) take the form

$$\dot{x}_i = \sum_{k=1}^{2f}\varepsilon_{ik}\frac{\partial H}{\partial x_k}\,, \qquad \varepsilon = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}. \tag{7.7}$$

The matrix ε has the important properties $\varepsilon^t = \varepsilon^{-1} = -\varepsilon$.

Let $F(q,p) = F(x)$ be an arbitrary function on the phase space. Along a mechanical trajectory $x(t) = (q(t), p(t))$, $F(x(t))$ changes according to

$$\begin{aligned}\frac{d}{dt}F(q(t),p(t)) &= \sum_{k=1}^{2f} \frac{\partial F}{\partial x_k}\dot{x}_k = \sum_{i,k=1}^{2f} \frac{\partial F}{\partial x_k}\varepsilon_{ki}\frac{\partial H}{\partial x_i} \\ &= \sum_{\alpha=1}^{f}\left(\frac{\partial F}{\partial q^\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial H}{\partial q^\alpha}\right). \end{aligned} \tag{7.8}$$

For any two functions on the phase space $F_1(q,p)$ and $F_2(q,p)$ one defines the *Poisson bracket*

$$\{F_1, F_2\} \equiv \sum_{i,k=1}^{2f} \frac{\partial F_1}{\partial x_i}\varepsilon_{ik}\frac{\partial F_2}{\partial x_k} = \sum_{\alpha=1}^{f}\left(\frac{\partial F_1}{\partial q^\alpha}\frac{\partial F_2}{\partial p_\alpha} - \frac{\partial F_1}{\partial p_\alpha}\frac{\partial F_2}{\partial q^\alpha}\right) \tag{7.9}$$

of F_1 and F_2. With this notation the time derivative of F (along a mechanical trajectory) can be rewritten as

$$\frac{d}{dt}F(q(t),p(t)) = \{F,H\}(q(t),p(t))\,, \tag{7.10}$$

where we have made explicit that the Poisson bracket is evaluated at the point $x(t)$. For $F(x) = x_i$ one thus obtains the canonical equations of motion in the form

$$\dot{x}_i = \{x_i, H\}. \tag{7.11}$$

Example: Particularly simple Poisson brackets arise between momentum and position coordinates. In fact one easily verifies that the "canonical Poisson brackets" hold

$$\{q^\alpha, q^\beta\} = 0\,, \qquad \{q^\alpha, p_\beta\} = \delta^\alpha{}_\beta\,, \qquad \{p_\alpha, p_\beta\} = 0\,,$$

where $\delta^\alpha{}_\beta$ is the Kronecker delta.

The Poisson bracket is linear

$$\{aF_1 + bF_2, F_3\} = a\{F_1,F_3\} + b\{F_2,F_3\}, \qquad (a,b \in \mathbb{R}) \tag{7.12}$$

and antisymmetric (since $\varepsilon^t = -\varepsilon$)

$$\{F_1, F_2\} = -\{F_2, F_1\}. \tag{7.13}$$

It also satisfies the *Jacobi identity*

$$\{\{F_1,F_2\},F_3\} + \{\{F_2,F_3\},F_1\} + \{\{F_3,F_1\},F_2\} = 0\,. \tag{7.14}$$

We will provide the proof of this at the end of Chapter 7.5.

7.4 Canonical Transformations

As we saw at the beginning of Chapter 7.3, the Hamiltonian equations of motion in phase space have the simple form $\dot{x}_i = \{x_i, H\}$ or also (with $\varepsilon^{-1} = \varepsilon^t$)

$$\sum_{i=1}^{2f} \varepsilon_{ik}\, \dot{x}_i = \frac{\partial H}{\partial x_k}\,. \tag{7.15}$$

We now wish to study the structure of bijective coordinate transformations

$$\bar{x}_i = \phi_i(x_1, \ldots, x_{2f}) \tag{7.16}$$

that leave the canonical equations (7.15) form-invariant. That is, for *every* Hamiltonian function H it should hold that the equations of motion

$$\sum_{i=1}^{2f} \varepsilon_{ik} \dot{\bar{x}}_i = \frac{\partial \bar{H}}{\partial \bar{x}_k}\,, \tag{7.17}$$

with $\bar{H}(\bar{x}) = (\bar{H} \circ \phi)(x) = H(x)$ are equivalent to (7.15). Transformations with this property are called *canonical transformations*.

From (7.16) and (7.17) one directly obtains

$$\sum_{i,j=1}^{2f} \varepsilon_{ik} \frac{\partial \bar{x}_i}{\partial x_j} \dot{x}_j = \sum_{l=1}^{2f} \frac{\partial H}{\partial x_l} \frac{\partial x_l}{\partial \bar{x}_k}\,. \tag{7.18}$$

We now define the *Jacobi matrix*

$$A_{ij}(x) \equiv \frac{\partial \phi_i}{\partial x_j} = \frac{\partial \bar{x}_i}{\partial x_j}, \tag{7.19}$$

and thereby obtain the condition[1]

$$\sum_{j=1}^{2f} \underbrace{\left(\sum_{i,k=1}^{2f} A_{ij} \varepsilon_{ik} A_{kl} \right)}_{=\varepsilon_{jl}\text{, since invariant to (7.15)}} \dot{x}_j = \frac{\partial H}{\partial x_l}\,.$$

We conclude from this that canonical transformations are characterized by the fact that

$$A^t(x) \varepsilon A(x) = \varepsilon \qquad (\text{for all } x)\,. \tag{7.20}$$

Linear maps $A\colon \mathbb{R}^{2f} \to \mathbb{R}^{2f}$ with the property $A^t \varepsilon A = \varepsilon$ are called *symplectic* and form the group $\mathrm{Sp}(2f)$. In particular one has

$$(\mathrm{Det}\, A)^2 = 1 \tag{7.21}$$

and even $\mathrm{Det}\, A = +1$, which we will not prove here. This implies that the inverse of a canonical transformation exists. If $A \in \mathrm{Sp}(2f)$ then the following properties hold:

[1] Note that $\partial x_j / \partial \bar{x}_i = (A^{-1})_{ji}$.

- $A^{-1} \in \mathrm{Sp}(2f)$, since $(A^{-1})^t \varepsilon A^{-1} = (A^{-1})^t (A^t \varepsilon A) A^{-1} = \varepsilon$,
- $A^t \in \mathrm{Sp}(2f)$, since $\varepsilon = -\varepsilon^{-1} = -((A^{-1})^t \varepsilon A^{-1})^{-1} = -A\varepsilon^{-1}A^t = A\varepsilon A^t$.

A canonical transformation preserves *all* Poisson brackets, i.e.

$$\{F_1, F_2\}(x) = \{\bar{F}_1, \bar{F}_2\}(\bar{x})\,, \quad \text{where } F_1(x) = \bar{F}_1(\bar{x}),\ F_2(x) = \bar{F}_2(\bar{x}) \tag{7.22}$$

This follows simply from the fact that[2]

$$\begin{aligned} \{F_1, F_2\} &= \sum_{j,k=1}^{2f} \frac{\partial F_1}{\partial x_j}\, \varepsilon_{jk}\, \frac{\partial F_2}{\partial x_k} = \sum_{i,l=1}^{2f} \frac{\partial \bar{F}_1}{\partial \bar{x}_i} \underbrace{\left(\sum_{j,k=1}^{2f} A_{ij}\, \varepsilon_{jk}\, A_{lk} \right)}_{=(A\varepsilon A^t)_{il} = \varepsilon_{il},\ \text{since } A^t \in \mathrm{Sp}(2f)} \frac{\partial \bar{F}_2}{\partial \bar{x}_l} \\ &= \{\bar{F}_1, \bar{F}_2\}\,. \end{aligned} \tag{7.23}$$

With the form (7.11) of the canonical equations of motion it is of course also clear in reverse that every transformation which preserves all Poisson brackets (7.22) defines a canonical transformation.

Example 1: (*Point transformations*) In the Lagrangian formalism arbitrary new position coordinates $\bar{q}^\alpha = \bar{q}^\alpha(q^1, \ldots, q^f)$ can be introduced. The Lagrangian function remains defined by $T - V$ invariant with $\bar{L}(\bar{q}, \dot{\bar{q}}) = L(q, \dot{q})$. The conjugate momenta are therefore given by

$$\bar{p}_\alpha = \frac{\partial \bar{L}}{\partial \dot{\bar{q}}^\alpha} = \sum_{\beta=1}^{f} \frac{\partial L}{\partial \dot{q}^\beta} \frac{\partial \dot{q}^\beta}{\partial \dot{\bar{q}}^\alpha} \overset{(4.10)}{=} \sum_{\beta=1}^{f} p_\beta \frac{\partial q^\beta}{\partial \bar{q}^\alpha}\,. \tag{7.24}$$

As we will see in Chapter 7.7.1, this transformation is naturally also canonical. One can also obtain this result by directly computing the Jacobi matrix.

Example 2: The group of canonical transformations is however substantially larger than the group of point transformations. This circumstance is one of the advantages of the Hamiltonian over the Lagrangian formalism. For example, the transformation ($f = 1$)

$$\bar{q} = \sqrt{pq^3}\,, \qquad \bar{p} = \sqrt{\frac{p}{q}}\,.$$

is also a canonical transformation. To prove this we compute the Jacobi matrix

$$A = \frac{1}{2} \begin{pmatrix} 3\sqrt{pq} & \sqrt{p^{-1}q^3} \\ -\sqrt{pq^{-3}} & \sqrt{p^{-1}q^{-1}} \end{pmatrix}.$$

One can easily verify that this is symplectic.

[2] In the literature this relation is often also written as $\{F_1, F_2\}_{\bar{x}} = \{F_1, F_2\}_x$, where the index of the Poisson bracket refers to the "variables" of the functions F_1, F_2 in question.

7.5 Canonical Flows

Let $x = (x_1, \ldots, x_{2f})$ be phase coordinates on the phase space $\Gamma = \mathbb{R}^{2f}$. Let $\phi_\lambda \colon x \mapsto \bar{x} \equiv x(\lambda)$ be a *canonical flow*, i.e. a family of canonical transformations with $x(0) = \phi_0(x) = x$. By differentiating with respect to λ we obtain the generating vector field $v(x)$ (cf. (4.53))

$$\frac{dx(\lambda)}{d\lambda} = \frac{\partial}{\partial\lambda}\phi_\lambda(x) = v(\phi_\lambda(x)). \tag{7.25}$$

The fact that the transformation ϕ_λ is to be canonical for every λ means that the Jacobi matrix

$$(A_\lambda)_{ik}(x) = \frac{\partial(\phi_\lambda)_i}{\partial x_k} = \frac{\partial \bar{x}_i}{\partial x_k} \tag{7.26}$$

is symplectic for all x and λ.

We now wish to characterize the vector fields $v(x)$ that generate *canonical flows*. From (7.25) it follows (at every point x)

$$\frac{\partial}{\partial\lambda}(A_\lambda)_{ik} = \frac{\partial}{\partial x_k} v_i(\phi_\lambda(x)) = \sum_{l=1}^{2f} \underbrace{\frac{\partial v_i}{\partial \bar{x}_l}(\bar{x})}_{\equiv V_{il}(\bar{x})} (A_\lambda)_{lk}\,, \tag{7.27}$$

or in matrix notation

$$\frac{\partial}{\partial\lambda} A_\lambda(x) = V(\bar{x})\, A_\lambda(x)\,, \qquad \text{with } \bar{x} = \phi_\lambda(x)\,. \tag{7.28}$$

For $\lambda = 0$ we have trivially a symplectic transformation with $A_0(x) = 1$. In order for the transformation to remain symplectic for all λ, we must therefore require only that

$$0 = \frac{\partial}{\partial\lambda}(A_\lambda^t \varepsilon A_\lambda) = A_\lambda^t (V^t\varepsilon + \varepsilon V) A_\lambda\,. \tag{7.29}$$

Because of $\operatorname{Det} A_\lambda \neq 0$ and $\varepsilon^t = -\varepsilon$ this is equivalent to $(\varepsilon^t V)^t = \varepsilon^t V$ or (the kl-component) written out

$$\frac{\partial}{\partial \bar{x}_k} \underbrace{\sum_{i=1}^{2f} \varepsilon_{il} v_i(\bar{x})}_{\equiv g_l(\bar{x})} = \frac{\partial}{\partial \bar{x}_l} \underbrace{\sum_{i=1}^{2f} \varepsilon_{ik} v_i(\bar{x})}_{= g_k(\bar{x})}\,. \tag{7.30}$$

These conditions mean that $g(x)$ is the gradient of a function $F(x)$, i.e.

$$g_k(x) = \sum_{i=1}^{2f} \varepsilon_{ik} v_i(x) = \frac{\partial F}{\partial x_k}(x) \quad \text{or} \quad v_i(x) = \sum_{k=1}^{2f} \varepsilon_{ik} \frac{\partial F}{\partial x_k}(x)\,. \tag{7.31}$$

The vector fields of canonical flows are characterized by (7.31).

We thus obtain the important result that the differential equation (7.25), which characterizes canonical flows, takes the form of canonical equations (cf. (7.11))

$$\frac{dx_i(\lambda)}{d\lambda} = \sum_{k=1}^{2f} \varepsilon_{ik} \frac{\partial F}{\partial x_k}(x(\lambda)) = \{x_i, F\}\,, \tag{7.32}$$

Here the parameter λ plays the role of time and F that of the Hamiltonian.

From the derivation it also follows directly the converse: canonical equations generate canonical flows on the phase space. The function $F(x)$ is called *generating function* of the canonical flow.

Example: The dynamics (time evolution) of an *autonomous* Hamiltonian system $H = H(x)$ generates a canonical flow on the phase space with $\phi_t(q,p) = (q(t), p(t))$. Here $q(t), p(t)$ satisfy the canonical equations (7.15) with initial conditions $q(0) = q$ and $p(0) = p$. The flow is canonical because the associated vector field $v = (\dot{q}, \dot{p})$ satisfies the condition (7.31) with $F = H$.

An important consequence is the fact that the Poisson bracket is compatible with time evolution. Indeed, for a fixed time t with $\bar{q} = q(t)$, $\bar{p} = p(t)$ and $F_i = \bar{F}_i \circ \phi_t$ from (7.23) we immediately obtain that

$$\{\bar{F}_1, \bar{F}_2\}(\bar{q}, \bar{p}) = \{F_1, F_2\}(q, p)$$

and thus

$$\{\bar{F}_1, \bar{F}_2\} \circ \phi_t = \{\bar{F}_1 \circ \phi_t, \bar{F}_2 \circ \phi_t\}; \tag{7.33}$$

i.e. the time evolution ϕ_t "commutes" with the Poisson bracket.

Proof of the Jacobi identity: Let $F_i(x)$, $i = 1, 2, 3$ be functions on the phase space. We consider the flow $x(\lambda) = \phi_\lambda(x)$ generated by F_3. With (7.10) we obtain for the changes of the quantities F_1, F_2 and $\{F_1, F_2\}$ under the flow the conditions

$$\frac{d}{d\lambda} F_i(x(\lambda))\Big|_{\lambda=0} = \{F_i, F_3\}, \quad \frac{d}{d\lambda}\{F_1, F_2\}(x(\lambda))\Big|_{\lambda=0} = \{\{F_1, F_2\}, F_3\}, \tag{7.34}$$

where the Poisson brackets are evaluated at $x = x(0)$. From (7.33) we know that the Poisson bracket commutes with the flow, i.e.

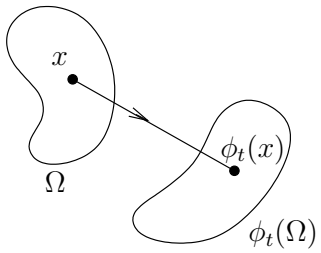
$$\{F_1, F_2\}(x(\lambda)) = \{F_1 \circ \phi_\lambda, F_2 \circ \phi_\lambda\}(x).$$

Differentiating this relation with respect to λ, one obtains (using the product rule)

$$\frac{d}{d\lambda}\{F_1, F_2\}(x(\lambda))\Big|_{\lambda=0} = \left\{\frac{dF_1(x(\lambda))}{d\lambda}\Big|_{\lambda=0}, F_2\right\} + \left\{F_1, \frac{dF_2(x(\lambda))}{d\lambda}\Big|_{\lambda=0}\right\}. \tag{7.35}$$

Substituting (7.34) then yields (using antisymmetry) the Jacobi identity (7.14).

7.5.1 Liouville's Theorem



The phase-space volume

$$\mu(\Omega) = \int_\Omega dx_1 \dots dx_{2f} \tag{7.36}$$

of every subset $\Omega \subset \Gamma$ of phase space is invariant under time evolution, i.e.

$$\mu(\phi_t(\Omega)) = \mu(\Omega)\,. \tag{7.37}$$

To prove this, we need only the fact that ϕ_t is a canonical flow, so that at every time t the map $\phi_t \colon x \mapsto \bar{x} = x(t)$ is a canonical transformation. Thus we obtain

$$\mu(\phi_t(\Omega)) = \int_{\phi_t(\Omega)} d\bar{x}_1 \dots d\bar{x}_{2f} = \int_\Omega \underbrace{\left|\mathrm{Det}\, \frac{\partial \bar{x}_i}{\partial x_j}\right|}_{=1,\text{ because of (7.21)}} dx_1 \dots dx_{2f} = \mu(\Omega)\,. \tag{7.38}$$

Equation (7.37) states that every canonical flow ϕ_t is volume-preserving.

7.6 Conserved Quantities

As we saw in Chapter 7.3, the time derivative of a quantity $F(x)$ along a mechanical trajectory is given by

$$\frac{d}{dt} F(x(t)) = \{F, H\}(x(t)) \tag{7.39}$$

Hence F is conserved precisely when its Poisson bracket with F and H vanishes.

Now let F_1 and F_2 be two conserved quantities, i.e.

$$\{F_1, H\} = \{F_2, H\} = 0\,. \tag{7.40}$$

From the Jacobi identity it follows that $\{F_1, F_2\}$ also has vanishing Poisson bracket with H. Thus the Poisson bracket of two conserved quantities is again a conserved quantity (*Poisson's theorem*).

Example: The angular momentum $\boldsymbol{L}$ is defined by $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$. If two components of the angular momentum $\boldsymbol{e}_1 \cdot \boldsymbol{L}$ and $\boldsymbol{e}_2 \cdot \boldsymbol{L}$ are conserved, then so is the third component, since

$$\{\boldsymbol{e}_1 \cdot \boldsymbol{L}, \boldsymbol{e}_2 \cdot \boldsymbol{L}\} = (\boldsymbol{e}_1 \times \boldsymbol{e}_2) \cdot \boldsymbol{L}\,. \tag{7.41}$$

Moreover,

$$\{\boldsymbol{e}_1 \cdot \boldsymbol{L}, \boldsymbol{e}_2 \cdot \boldsymbol{p}\} = (\boldsymbol{e}_1 \times \boldsymbol{e}_2) \cdot \boldsymbol{p}\,; \tag{7.42}$$

i.e. if one component of momentum and the angular momentum are conserved, then in fact all components of momentum are conserved.

We denote the canonical flows generated by F and H in the sense of Chapter 7.5 by ψ_λ and ϕ_t. It holds that

$$\frac{d}{dt}F(\phi_t(x))\Big|_{t=0} = \{F,H\} = -\{H,F\} = -\frac{d}{d\lambda}H(\psi_\lambda(x))\Big|_{\lambda=0}\,. \tag{7.43}$$

Thus F is conserved if and only if H is invariant under the canonical flow generated by F, with

$$H(\psi_\lambda(x)) = H(x). \tag{7.44}$$

Hence to every conserved quantity there corresponds a continuous symmetry ψ_λ of H, and conversely. In a certain sense, this statement is therefore the converse of Noether's theorem.

Example: Let $\mathbb{R}^6$ be the phase space with coordinates $(\boldsymbol{x},\boldsymbol{p})$ and

$$F(\boldsymbol{x},\boldsymbol{p}) = \boldsymbol{e}\cdot\boldsymbol{L} = \boldsymbol{e}\cdot(\boldsymbol{x}\times\boldsymbol{p})\,, \tag{7.45}$$

, where $\boldsymbol{e}$ is a fixed unit vector. The differential equations (7.32) of the canonical flow generated by F are (in the form (7.5))

$$\frac{d\boldsymbol{x}}{d\lambda} = \frac{\partial F}{\partial \boldsymbol{p}} = \boldsymbol{e}\times\boldsymbol{x}\,, \qquad \frac{d\boldsymbol{p}}{d\lambda} = -\frac{\partial F}{\partial \boldsymbol{x}} = \boldsymbol{e}\times\boldsymbol{p}\,. \tag{7.46}$$

After integration one obtains the corresponding canonical flow

$$\psi_\lambda\colon (\boldsymbol{x},\boldsymbol{p}) \mapsto (R(\lambda)\boldsymbol{x}, R(\lambda)\boldsymbol{p})\,, \tag{7.47}$$

where $R(\lambda)$ denotes the rotation in $\mathbb{R}^3$ about the axis $\boldsymbol{e}$ through an angle λ. Hence the components (7.45) of the angular momentum are the generating functions of these rotations. With (7.43) one therefore obtains the most general form of the angular momentum theorem

$$\frac{d}{dt}\boldsymbol{e}\cdot(\boldsymbol{x}\times\boldsymbol{p})\Big|_{t=0} = -\frac{d}{d\lambda}H(R(\lambda)\boldsymbol{x}, R(\lambda)\boldsymbol{p})\Big|_{\lambda=0}\,. \tag{7.48}$$

Thus $\boldsymbol{e}\cdot(\boldsymbol{x}\times\boldsymbol{p})$ is conserved if and only if H is invariant under rotations about the axis $\boldsymbol{e}$. Analogously, the momentum component $\boldsymbol{e}\cdot\boldsymbol{p}$ is the generator of the translations $\psi_\lambda\colon (\boldsymbol{x},\boldsymbol{p}) \mapsto (\boldsymbol{x}+\lambda\boldsymbol{e},\boldsymbol{p})$. From this follows the momentum theorem

$$\frac{d}{dt}\boldsymbol{e}\cdot\boldsymbol{p}\Big|_{t=0} = -\frac{d}{d\lambda}H(\boldsymbol{x}+\lambda\boldsymbol{e},\boldsymbol{p})\Big|_{\lambda=0}\,. \tag{7.49}$$

Occasionally one is interested in the time rate of change of $F(x,t)$, i.e., a quantity F that depends explicitly on time. We obtain (instead of (7.39))

$$\frac{d}{dt}F(x(t),t) = \sum_{i=1}^{2f}\frac{\partial F}{\partial x_i}\dot{x}_i + \frac{\partial F}{\partial t} = \{F,H\} + \frac{\partial F}{\partial t}\,. \tag{7.50}$$

In particular, for $H = H(x,t)$ the *energy theorem*

$$\frac{dH}{dt} = \underbrace{\{H,H\}}_{=0} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \tag{7.51}$$

holds because of (7.13). For autonomous systems with $H = H(x)$ the energy H is therefore conserved.

7.7 The Hamiltonian principle in phase space

By the definition of the Hamiltonian function (7.3) we have

$$L(q,\dot{q},t) = \langle p,\dot{q}\rangle - H = \sum_{\alpha=1}^{f} p_\alpha \dot{q}^\alpha - H(q,p,t)\,. \tag{7.52}$$

We therefore expect that the equations of motion can be derived from the variational principle

$$\delta \int_{(1)}^{(2)} \Big(\langle p,\dot{q}\rangle - H\Big) dt = 0 \tag{7.53}$$

where the position $q^\alpha(t^{(i)})$ is held fixed at the endpoints $t^{(i)}$.

In the phase space, however, q^α and p_β are treated as independent variables, whereas in the original formulation of the variational principle the variation of $\dot{q}$ was prescribed by that of q. It is therefore *a priori* not obvious that (7.53) yields the correct equations of motion. In fact, the variational principle (7.53) is equivalent to the canonical equations of motion. Under a variation of the independent variables q^α, p_β (with fixed endpoints $q^{(i)}$) we obtain

$$\begin{aligned}
&\delta \int_{(1)}^{(2)} \Big(\sum_{\alpha=1}^{f} p_\alpha \dot{q}^\alpha - H\Big) dt \\
&= \int_{(1)}^{(2)} \sum_{\alpha=1}^{f} \left(p_\alpha \delta\dot{q}^\alpha + \dot{q}^\alpha \delta p_\alpha - \frac{\partial H}{\partial q^\alpha}\delta q^\alpha - \frac{\partial H}{\partial p_\alpha}\delta p_\alpha \right) dt \\
&= \sum_{\alpha=1}^{f} p_\alpha \underbrace{\delta q^\alpha}_{=0}\Big|_{(1)}^{(2)} + \int_{(1)}^{(2)} \sum_{\alpha=1}^{f} \left[\left(\dot{q}^\alpha - \frac{\partial H}{\partial p_\alpha} \right) \delta p_\alpha - \left(\dot{p}_\alpha + \frac{\partial H}{\partial q^\alpha} \right) \delta q^\alpha \right] dt\,,
\end{aligned}$$

where we have performed a partial integration on the first term in the second line. The boundary terms vanish and we recover precisely the Hamiltonian equations of motion.

7.7.1 Generating canonical transformations

As preparation for the discussion of the Hamilton-Jacobi equation in Chapter 8 we now discuss how one may systematically construct canonical transformations. Consider a Hamiltonian system with phase-space coordinates $x = (q^1, p_1, \ldots, q^f, p_f)$ and Hamiltonian function $H(x,t)$. We subject the system to a (possibly) time-dependent canonical transformation

$$\bar{x}_i = \phi_i(x_1, \ldots, x_{2f}, t). \tag{7.54}$$

Since the transformation is canonical, the equations of motion in the new coordinates $\bar{x} = (Q^1, P_1, \ldots, Q^f, P_f)$ are again the Hamiltonian equations. Because the canonical transformation depends on time, however, we must in general introduce a new Hamiltonian function $K(\bar{x}, t)$. Equivalently, one may assert that the two variational problems

$$\delta \int_{(1)}^{(2)} \Big(\langle p, \dot{q} \rangle - H \Big) dt = 0 \quad \text{and} \quad \delta \int_{(1)}^{(2)} \Big(\langle P, \dot{Q} \rangle - K \Big) dt = 0 \tag{7.55}$$

possess solutions $x(t)$ and $\bar{x}(t)$, respectively, that correspond under (7.54). A sufficient condition for this is (cf. (4.46)) that the two integrands in (7.55) differ by a total time derivative of a function $F(q, Q, t)$.[3]

This means that for *all* curves $x(t)$ one has[4]

$$\langle p(t), \dot{q}(t) \rangle - H(x(t), t) = \langle P(t), \dot{Q}(t) \rangle - K(\bar{x}(t), t) + \frac{dF}{dt}. \tag{7.56}$$

Rearranging (7.56) yields the expression

$$\frac{dF}{dt} = \Big(\langle p, \dot{q} \rangle - \langle P, \dot{Q} \rangle \Big) + (K - H). \tag{7.57}$$

It is useful to express the function F not by the variables q and Q but rather by the old coordinates q and the new momenta P. The change from Q to P as independent variables is achieved by a Legendre transformation. We define

$$S(q, P, t) = F(q, Q, t) + \langle P, Q \rangle \qquad \text{with} \quad P_\alpha = -\frac{\partial F}{\partial Q^\alpha}(q, Q, t). \tag{7.58}$$

Analogous to (7.4) we obtain for the partial derivatives[5]

$$\frac{\partial S}{\partial P_\alpha} = Q^\alpha, \qquad \frac{\partial S}{\partial q^\alpha} = \frac{\partial F}{\partial q^\alpha}, \qquad \frac{\partial S}{\partial t} = \frac{\partial F}{\partial t}. \tag{7.59}$$

[3] Since in the calculus of variations the endpoints δq^α and δQ^α are held fixed, F does not alter the Euler-Lagrange equations.

[4] Often one writes this relation also in compact form

$$\langle p, dq \rangle - H dt = \langle P, dQ \rangle - K dt + dF$$

as an identity of differentials. Hence $dF = \langle p, dq \rangle - \langle P, dQ \rangle + (K - H) dt$. The Legendre transformation to S is then obtained by the product rule $d\langle P, Q \rangle = \langle dP, Q \rangle + \langle P, dQ \rangle$ and $dS = dF + d\langle P, Q \rangle$.

[5] Replace $(L, H, \dot{q}, p)$ in (7.4) by $(-F, S, Q, P)$.

Thus we obtain directly

$$\begin{aligned}\frac{dF}{dt} &\equiv \sum_{\alpha=1}^{f}\left(\frac{\partial F}{\partial q^\alpha}\dot{q}^\alpha + \frac{\partial F}{\partial Q^\alpha}\dot{Q}^\alpha\right) + \frac{\partial F}{\partial t} \\ &= \sum_{\alpha=1}^{f}\left(\frac{\partial S}{\partial q^\alpha}\dot{q}^\alpha - P_\alpha\dot{Q}^\alpha\right) + \frac{\partial S}{\partial t}\end{aligned} \tag{7.60}$$

for the total time derivative of the function $F(q, Q, t)$.

For a given function $F(q, Q, t)$ one obtains by comparison of (7.57) with (7.60) the relations[6]

$$p_\alpha = \frac{\partial S}{\partial q^\alpha}(q, P, t)\,, \qquad Q^\alpha = \frac{\partial S}{\partial P_\alpha}(q, P, t)\,, \qquad K = H + \frac{\partial S}{\partial t}\,, \tag{7.61}$$

between the old (q, p) and the new (Q, P) canonical coordinates (and the new Hamiltonian K). To obtain the canonical transformation (7.54) explicitly, one must solve the first equation for P_α and substitute the result into the second equation. This yields

$$Q^\alpha = Q^\alpha(q, p, t), \qquad P_\alpha = P_\alpha(q, p, t). \tag{7.62}$$

The condition for solvability at a point (q_0, P_0, t_0) is that there

$$\mathrm{Det}\left(\frac{\partial^2 S}{\partial q^\alpha \partial P_\beta}\right) \neq 0\,. \tag{7.63}$$

We call $S(q, P, t)$ the *generating function* of the (at any time t) canonical transformation 7.62. The transformation is canonical for every *fixed* time $t = t_0$, since it is generated there by $S_0(q, P) = S(q, P, t_0)$. For the generating function S_0 the time derivative vanishes, and hence the canonical equations of motion with the Hamiltonian $\bar{H}(\bar{x}) \equiv K_0(\bar{x}) = H(x)$ hold for the new variables $\bar{x} = \phi(x, t_0)$. This precisely corresponds to the definition (7.17) of a canonical transformation.

By a free choice of the function S one can construct canonical transformations in a very elegant way. To demonstrate this, we consider two examples (analogous to the examples in Sec. 7.4).

Example 1: (*Point transformations*) Let $S = \sum_{\beta=1}^{f} P_\beta Q^\beta(q^1, \ldots, q^f)$, where the functions $Q^\alpha(q)$ are given. Then (7.61)

$$Q^\alpha = Q^\alpha(q^1, \ldots, q^f)\,, \qquad p_\alpha = \sum_{\beta=1}^{f} P_\beta \frac{\partial Q^\beta}{\partial q^\alpha}\,. \tag{7.64}$$

After solving the second equations for P_α with $P_\alpha = \sum_{\beta=1}^{f} p_\beta(\partial q^\beta/\partial Q^\alpha)$, these are the point transformations discussed in connection with (7.24); we have now proved that the transformation is canonical.

[6] The first and last relation follow from the comparison. The middle relation follows from (7.59).

Example 2: For $f = 1$ we consider the generating function $S(q,P) = \frac{1}{2}q^2\,P^2$. Equation (7.61) yields

$$Q = q^2 P\,, \qquad p = qP^2. \tag{7.65}$$

Thus we obtain the canonical transformation

$$Q = \sqrt{pq^3}\,, \qquad P = \sqrt{\frac{p}{q}}\,. \tag{7.66}$$

Remark: The generating function $S(q,P,t)$ of the canonical transformation is not the same as the generating function $F(q,p)$ of a canonical flow $(Q,P) = x(\lambda) = \phi_\lambda(q,p)$ from (7.32). The concepts are however closely related. For small λ it is true that

$$S(q,P,\lambda) = \sum_{\alpha=1}^{f} P_\alpha q^\alpha + \lambda F(q,P) + O(\lambda^2)\,. \tag{7.67}$$

Proof: From (7.61) we know

$$p_\alpha = P_\alpha + \lambda \frac{\partial F}{\partial q^\alpha}\bigg|_{(q,P)} + O(\lambda^2)\,, \quad Q^\alpha = q^\alpha + \lambda \frac{\partial F}{\partial p_\alpha}\bigg|_{(q,P)} + O(\lambda^2)\,. \tag{7.68}$$

Since $Q^\alpha - q^\alpha = O(\lambda)$ and $P_\alpha - p_\alpha = O(\lambda)$, we can rewrite the result as

$$Q^\alpha = q^\alpha + \lambda \frac{\partial F}{\partial p_\alpha}\bigg|_{(Q,P)} + O(\lambda^2)\,, \quad P_\alpha = p_\alpha - \lambda \frac{\partial F}{\partial q^\alpha}\bigg|_{(Q,P)} + O(\lambda^2)\,. \tag{7.69}$$

Hence $(Q,P) = (q(\lambda),p(\lambda))$ is locally the solution of the canonical equations $dq^\alpha(\lambda)/d\lambda = \partial F/\partial p_\alpha$, $dp_\alpha(\lambda)/d\lambda = -\partial F/\partial q^\alpha$ with initial condition $(q(0),p(0)) = (q,p)$ with $F(q,p)$ the generating function, cf. (7.32).

Chapter 8

The Hamilton-Jacobi Theory

The Hamilton-Jacobi theory is a further reformulation of classical mechanics. It encompasses all results of the Lagrangian and Hamiltonian formulations, but also provides a deeper understanding of integrability. Moreover, as shown in Chapter 8.5, it allows a direct transition from classical mechanics to quantum mechanics.

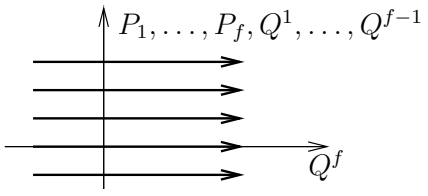
8.1 The Time-Independent Case

We now consider an autonomous system, i.e. a system whose Hamiltonian does not depend explicitly on t, $H = H(q^1, \ldots, q^f, p_1, \ldots, p_f)$. We attempt to find a time-independent canonical transformation such that K is precisely one of the new momentum coordinates, say $K(Q, P) = P_f$, where P_f naturally corresponds to the energy E. If this can be achieved, the equations of motion in the new coordinates are given by

$$\begin{aligned} \dot{P}_\alpha &= -\frac{\partial K}{\partial Q^\alpha} = 0\,, && (\alpha = 1, \ldots, f)\,, && \\ \dot{Q}^\alpha &= \frac{\partial K}{\partial P_\alpha} = 0\,, && (\alpha = 1, \ldots, f-1)\,, && \dot{Q}^f = \frac{\partial K}{\partial P_f} = 1\,. \end{aligned} \tag{8.1}$$

The equations of motion are then trivially solvable with

$$\begin{aligned} P_\alpha(t) &= P_\alpha(0)\,, && (\alpha = 1, \ldots, f)\,, \\ Q^\alpha(t) &= Q^\alpha(0)\,, && (\alpha = 1, \ldots, f-1)\,, \\ Q^f(t) &= Q^f(0) + t\,. \end{aligned}$$



of the Hamiltonian $K = P_f$.

As we saw in Chapter 7.7.1, to this end one must determine a generating function

$S_0 = S_0(q, P)$ so that (see (7.61))

$$H\Big(q^1, \ldots, q^f, \frac{\partial S_0}{\partial q^1}, \ldots, \frac{\partial S_0}{\partial q^f}\Big) = P_f \equiv E\,. \tag{8.2}$$

This equation is called the *time-independent Hamilton-Jacobi equation*. It is a first-order partial differential equation (parametrized by P_f) for S_0. In order that S_0 can actually be solved for q^α, one needs a family of solutions[1]

$$S_0(q^1, \ldots, q^f, P_1, \ldots, P_f) \qquad \text{with} \quad \mathrm{Det}\left(\frac{\partial^2 S_0}{\partial q^\alpha \partial P_\beta}\right) \neq 0\,, \tag{8.3}$$

parametrized by $(f-1)$-parameters $P_1, \ldots, P_{f-1}$ and, of course, P_f. Such a solution is called a *complete integral*.[2]

Here it is important that $Q(q, P, t)$ can be solved for q subject to the condition (cf. (7.63))

$$\mathrm{Det}\left(\frac{\partial^2 S_0}{\partial q^\alpha \partial P_\beta}\right) \neq 0. \tag{8.4}$$

A necessary condition for solvability is therefore at least that

$$\mathrm{Rang}\left(\frac{\partial^2 S_0}{\partial q^\alpha \partial P_\beta}\right)_{\substack{\alpha=1,\ldots,f \\ \beta=1,\ldots,f-1}} = f - 1\,. \tag{8.5}$$

This condition is in fact also sufficient. From (8.2) one obtains by differentiation with respect to $P_1, \ldots, P_f$

$$\sum_\alpha \frac{\partial H}{\partial p_\alpha}\frac{\partial^2 S_0}{\partial q^\alpha \partial P_\beta} = 0\,, \quad (\beta = 1, \ldots, f-1)\,, \qquad \sum_\alpha \frac{\partial H}{\partial p_\alpha}\frac{\partial^2 S_0}{\partial q^\alpha \partial P_f} = 1\,.$$

The last column ($\beta = f$) of the matrix (8.4) can therefore not be a linear combination of the first $f-1$. Moreover, as a necessary condition for the solvability of (8.2), it follows that in the region of phase space under consideration nowhere

$$\frac{\partial H}{\partial p_1} = \ldots = \frac{\partial H}{\partial p_f} = 0 \tag{8.6}$$

holds.

From the complete solution of the Hamilton-Jacobi equation one obtains the motion in the original coordinates as follows. For given values of the conserved quantities $P_1, \ldots, P_f$, each of the equations

$$\frac{\partial S_0}{\partial P_\beta}(q, P) = Q^\beta\,, \qquad (\beta = 1, \ldots, f-1) \tag{8.7}$$

[1] In general $S_0(q, P) + A$ with $A \in \mathbb{R}$ is a solution. The additional additive constant A is, however, irrelevant in mechanics.

[2] As we saw in Chapter 5.3, a *general solution* of a partial differential equation depends on arbitrary functions (and not merely parameters).

for $q = (q^1, \ldots, q^f)$ defines a surface in configuration space that, by (8.5), has linearly independent normals. Their intersection is the (one-dimensional) trajectory. The traversal of this trajectory in time is given by

$$\frac{\partial S_0}{\partial E}(q, P) = Q^f(0) + t \equiv t_0 + t. \tag{8.8}$$

The $2f$ constants $Q^1, P_1, \ldots, t_0, E$ are determined by the initial conditions.

Example 1: The harmonic oscillator has the Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{1}{2}fq^2.$$

with angular frequency $\omega = \sqrt{f/m}$. The Hamilton-Jacobi equation for $S_0(q)$ is the ordinary differential equation

$$\frac{1}{2m}\left(\frac{dS_0}{dq}\right)^2 + \frac{1}{2}fq^2 = E$$

with solution

$$S_0(q) = \int_0^q dx\, \sqrt{m(2E - fx^2)}. \tag{8.9}$$

The motion is given by (8.8). With

$$\frac{\partial S_0}{\partial E} = \int_0^q dx \sqrt{\frac{m}{2E - fx^2}} = \frac{1}{\omega}\arcsin\Big(q/\sqrt{2E/f}\Big) = t_0 + t$$

we obtain the general solution

$$q(t) = \sqrt{2E/f}\, \sin\big(\omega(t_0 + t)\big).$$

8.2 Separable Problems

If one can write the Hamilton-Jacobi equation (8.2) in the form

$$f\Big(q^1, \frac{\partial S_0}{\partial q^1}\Big) = F\Big(q^2, \ldots, q^f, \frac{\partial S_0}{\partial q^2}, \ldots, \frac{\partial S_0}{\partial q^f}\Big), \tag{8.10}$$

then the coordinate q^1 is said to be *separable*. The separation ansatz

$$S_0(q^1, \ldots, q^f) = S_1(q^1) + \tilde{S}_0(q^2, \ldots, q^f) \tag{8.11}$$

leads to the two equations

$$f\Big(q^1, \frac{dS_1}{dq^1}\Big) = P_1\,, \qquad F\Big(q^2, \ldots, q^f, \frac{\partial \tilde{S}_0}{\partial q^2}, \ldots, \frac{\partial \tilde{S}_0}{\partial q^f}\Big) = P_1\,, \tag{8.12}$$

where P_1 is constant, since the left and right sides of (8.10) do not depend on $q^2, \ldots, q^f$ and q^1, respectively. From (8.12) one finds, by solving the first equation, the function $S_1(q^1, P_1)$.

The problem is *completely separable* if one can proceed similarly with the second equation, and so on. In this case the result is a complete solution

$$S_0(q^1, \ldots, q^f, P_1, \ldots, P_{f-1}, P_f) = S_1(q^1, P_1) + S_2(q^2, P_1, P_2) + \ldots + S_f(q^f, P_1 \ldots P_{f-1}) \quad (8.13)$$

of the Hamilton-Jacobi equation (8.2), since each term may still depend on $P_f = E$. The (complete) separability of a problem always refers to particular coordinates $q = (q^1, \ldots, q^f)$, whose existence is an exception rather than the rule. For a list of problems that are completely separable, see Landau & Lifshitz.

Example 2: (*Cyclic coordinate*) If the coordinate q^1 is cyclic, it does not enter explicitly into the Hamilton-Jacobi equation. One can then trivially separate the system with $f(q^1, dS_1/dq^1) = dS_1/dq^1$. This immediately yields $S_1 = P_1 q^1$ and

$$S_0 = P_1 q^1 + \tilde{S}_0.$$

The constant P_1 is, of course, nothing other than the conjugate momentum $\partial S_0/\partial q^1$.

Example 3: (*The planar central-force problem*) We consider the planar central-force problem in polar coordinates (see Example (ii) in Chapter 7.2), for which the Hamiltonian is given by

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r)$$

The Hamilton-Jacobi equation is

$$\frac{1}{2m}\left[\left(\frac{\partial S_0}{\partial r}\right)^2 + r^{-2}\left(\frac{\partial S_0}{\partial \varphi}\right)^2\right] + V(r) = E\,,$$

with $E \equiv P_2$. This equation is separable, since it can be rewritten as

$$\left(\frac{\partial S_0}{\partial \varphi}\right)^2 = 2mr^2(E - V(r)) - r^2\left(\frac{\partial S_0}{\partial r}\right)^2. \quad (8.14)$$

With the separation ansatz $S_0(r, \varphi) = S_r(r) + S_\varphi(\varphi)$ both sides of (8.14) must equal a separation constant $l^2 \equiv P_1^2$. It follows that

$$S_\varphi(\varphi) = l\varphi\,, \qquad S_r(r) = \int^r dx\, \sqrt{2m(E - V(x)) - l^2 x^{-2}}\,, \quad (8.15)$$

up to irrelevant additive constants. The orbit $\varphi(r)$ and the time dependence $t(r)$ are now obtained from (8.7) and (8.8)

$$\frac{\partial S_0}{\partial l} = \varphi - \int^r \frac{l}{x^2} \frac{dx}{\sqrt{2m(E - V(x)) - l^2 x^{-2}}} = \varphi_0 \,,$$
$$\frac{\partial S_0}{\partial E} = \int^r \frac{m\, dx}{\sqrt{2m(E - V(x)) - l^2 x^{-2}}} = t_0 + t \,,$$

where the constants t_0, φ_0 are determined by the initial conditions. This result, of course, agrees with (2.14) and (2.12).

Remark: A complete solution of the Hamilton-Jacobi equation directly provides the f conserved quantities P_β. That P_β are conserved follows from

$$\{H(x), P_\beta(x)\} = \{P_f, P_\beta\} = 0, \tag{8.16}$$

where we have used (7.22) and (8.2). Moreover, they are in *involution*, i.e.

$$\{P_\alpha(x), P_\beta(x)\} = \{P_\alpha, P_\beta\} = 0, \quad \text{(for all } \alpha, \beta). \tag{8.17}$$

The conserved quantities are also *independent*[3] of one another, since (from (8.4))

$$\mathrm{Det}\left(\frac{\partial P_\alpha(q,p)}{\partial p_\beta}\right) = \mathrm{Det}\left(\frac{\partial^2 S_0}{\partial q^\beta \partial P_\alpha}\right)^{-1} \neq 0 \,. \tag{8.18}$$

A mechanical system with f degrees of freedom is called *completely integrable* (in the Liouville sense) if it possesses f independent conserved quantities in involution. Thus every system for which a complete solution of the Hamilton-Jacobi equation exists is completely integrable.[4]

Note that in Lagrangian mechanics $2f$ integrals of the motion are required to integrate a problem completely. It turns out that with the Hamiltonian method only f conserved quantities are needed, which, however, must be in involution. Each additional conserved quantity reduces the number of degrees of freedom by two.

8.3 Angle-Action Variables

We wish to examine more closely the structure of the phase space for a completely separable problem with

$$S_0(q, P) = \sum_{\alpha=1}^{f} S_\alpha(q^\alpha, P), \tag{8.19}$$

[3]The quantities P_α are independent if the matrix $\partial P_\alpha / \partial x_k$ has full rank f.

[4]This result also holds conversely: a completely integrable system has a (global) solution of the Hamilton-Jacobi equation.

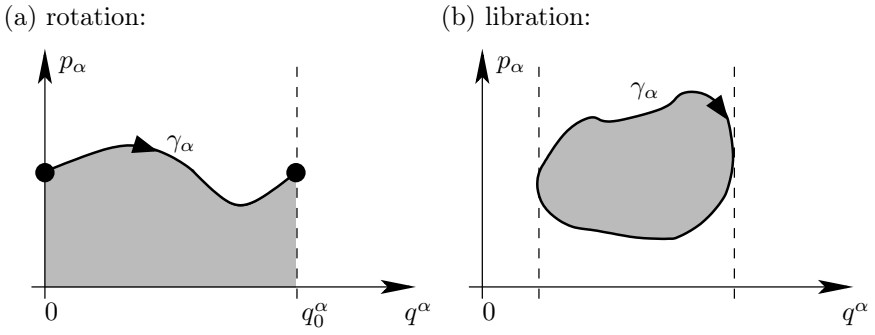


Figure 8.1: The action variable J_α is the (oriented) area (divided by 2π) enclosed by the curve $p_\alpha(q^\alpha)$. There are two classes of trajectories. (a) Rotation (possible only for periodic variables): the curve connects two equivalent points with $q^\alpha = 0$ and $q^\alpha = q_0^\alpha$; q_0^α denotes the period. (b) Libration: the curve encloses an area in phase space. The motion in q is bounded by two turning points (dashed lines) at P fixed.

that exhibits bounded motion in all coordinates.[5] One can then make a special choice of canonical coordinates, the so-called *angle-action variables*, in which the configuration space becomes a torus and the motion becomes conditionally periodic. These variables are useful for perturbation theory, problems with slowly (adiabatically) varying parameters, and for the semiclassical description of quantum-mechanical problems.

The canonical momentum $p_\alpha(q^\alpha, P) = \partial S_\alpha/\partial q^\alpha$ is (for fixed P) a function only of the coordinate q^α. Since the motion is bounded, the trajectory γ_α in the subspace (q^α, p_α) is closed. We define the *action variables* J_α by the line integral along γ_α

$$J_\alpha(P) = \frac{1}{2\pi}\oint_{\gamma_\alpha} p_\alpha(q^\alpha)\,dq^\alpha = \frac{1}{2\pi}\oint_{\gamma_\alpha}\langle p, dq\rangle, \qquad (P \text{ fixed}). \tag{8.20}$$

There are two classes of trajectories, see Figure 8.1:

(a) *Rotation* (e.g. angle φ in polar coordinates): q^α is a periodic variable with period q_0^α and the curve γ_α connects $q^\alpha = 0$ with $q^\alpha = q_0^\alpha$,

(b) *Libration* (e.g. position q for the harmonic oscillator): the curve γ_α is a closed curve in phase space (q^α, p_α).

From the definition it follows that $J_\alpha = J_\alpha(P)$ depends only on the constants of motion P and hence J_α themselves are constants of motion. If

$$\mathrm{Det}\left(\frac{\partial J_\alpha}{\partial P_\beta}\right) \neq 0$$

the relation $J(P)$ is invertible with $P = P(J)$ and one may take J as new momenta. The change from (q, p) to (ϕ, J) is then effected by the generating function $\tilde{S}_0(q, J) =$

[5] Although technically more difficult, most results remain valid for a general completely integrable system with the necessary adjustments.

$S_0(q, P(J))$. The new coordinates

$$\phi^\alpha(q, J) = \frac{\partial \tilde{S}_0}{\partial J_\alpha}(q, J) \tag{8.21}$$

are called *angle variables*. In general ϕ^α is a function of (all) $q^1, \ldots, q^f, J_1, \ldots, J_f$.

The new Hamiltonian is given by $K(J) = P_f(J)$ and does not depend on the angle variables. The angle variables therefore satisfy the canonical equations of motion

$$\dot{\phi}^\alpha = \frac{\partial K(J)}{\partial J_\alpha} = \omega^\alpha(J). \tag{8.22}$$

Since J are constants of motion, all angle variables depend linearly on time with

$$\phi^\alpha = \omega^\alpha(J)t + \phi_0^\alpha; \tag{8.23}$$

ϕ_0^α are integration constants determined by the initial conditions.

We now show that ϕ^α are in fact periodic variables, so that ϕ^α and $\phi^\alpha + 2\pi$ correspond to the same state (q, p). Thus the configuration space T with $\phi \in T$ is a f-dimensional torus ($T = S^1 \times \cdots \times S^1$) and the motion (8.23) corresponds to *conditionally periodic* motion, each angle variable ϕ^α executing periodic motion with angular frequency ω^α.

To prove that ϕ^α is a periodic variable (with period 2π) we compute the change $\Delta\phi^\alpha$ when the system returns to the same state along the path γ_β.[6] This question concerns only the configuration space and hence the relation $\phi(q, P)$ and not whether the mechanical trajectory is periodic. We therefore consider the (virtual) displacement (at fixed P or J)

$$\delta\phi^\alpha = \sum_{\beta=1}^{f} \frac{\partial \phi^\alpha}{\partial q^\beta}\, \delta q^\beta = \sum_{\beta=1}^{f} \frac{\partial^2 \tilde{S}_0}{\partial J_\alpha \partial q^\beta}\, \delta q^\beta = \frac{\partial}{\partial J_\alpha} \langle p(q, J), \delta q \rangle. \tag{8.24}$$

After integration over the curve γ_β we obtain

$$\Delta\phi^\alpha = \frac{\partial}{\partial J_\alpha} \underbrace{\oint_{\gamma_\beta} \langle p, dq \rangle}_{=2\pi J_\beta} = 2\pi\delta_{\alpha\beta}, \tag{8.25}$$

so that after traversing the curve γ_β only the angle variable ϕ^β has changed by 2π. Since the system returns to the same state along every curve γ_β ($\beta = 1, \ldots, f$), we obtain the desired result that all ϕ^α are periodic variables.

With this general result we can make statements about the motion of an integrable system without solving the equations of motion explicitly. We assume for simplicity that all motions are of the libration type.[7] The angle-action variables specify the

[6] Note that γ_β is a *closed* curve.

[7] For rotations the result (8.26) applies for $\tilde{q}^\alpha = q^\alpha - q_0^\alpha \phi^\alpha / 2\pi$.

position $q(\phi, J)$ uniquely and it must hold that $q(\phi + 2\pi m, J) = q(\phi, J)$ with $m \in \mathbb{Z}^f$. We may therefore expand the (periodic) function $q(\phi)$ in a Fourier series. This yields the time dependence of $q^\alpha(t)$ as

$$q^\alpha(t) = \sum_{m\in\mathbb{Z}^f} A^\alpha_m e^{i\langle m,\phi\rangle} \equiv \sum_{m\in\mathbb{Z}^f} A^\alpha_{m_1,\ldots,m_f} \exp\left(i\sum_{\beta=1}^{f} m_\beta(\omega^\beta t + \phi^\beta_0)\right). \tag{8.26}$$

A similar expansion is obtained for $p(t)$ and hence for any function $F(q,p)$.

Since the frequencies are in general not commensurate, the motion $q(t)$ of the system is not periodic but only *conditionally periodic*. In some cases two (or several) of the angular frequencies ω^α may be commensurate. If, for example, $\omega^1/\omega^2 \in \mathbb{Q}$ holds, one speaks of a *degeneracy* of the motion of ϕ^1 and ϕ^2. In the extreme case in which all f angular frequencies are commensurate the motion of the system is called *completely degenerate*. In this case the motion is of course strictly periodic and all trajectories are closed.

Example 1 (continued): The trajectory of the harmonic oscillator (at fixed E) is a libration (between the turning points $\pm q^*$ with $q^* = \sqrt{2E/f}$). We obtain the action variable

$$J = \frac{1}{2\pi}\oint p(q)\,dq = \frac{2}{2\pi}\int_{-q^*}^{q^*} \sqrt{m(2E - fq^2)}\,dq = \frac{E}{\omega}.$$

The action $\tilde{S}_0(J)$ is given by $S_0(E)$ from (8.9) with the relation $E = \omega J$. We obtain the corresponding angle variable

$$\phi = \frac{\partial \tilde{S}_0}{\partial J} = \omega\frac{\partial S_0}{\partial E} = \arcsin\left(\frac{q}{q^*}\right).$$

The motion is completely degenerate (as always with a single degree of freedom) with period $2\pi/\omega$.

Remark: We have seen that the configuration space of a completely integrable system is a f-dimensional torus. The KAM theorem (Kolmogorov (1956), Arnol'd (1965), Moser (1965)) describes the stability of this result under (small) perturbations with

$$H = \underbrace{H_0}_{\text{integrable}} + \varepsilon H_1.$$

The result is: if H_0 angular frequencies ω^α are 'sufficiently incommensurate,' then the perturbed system H for $\varepsilon \ll 1$ has predominantly solutions that are also conditionally periodic and differ little from those of H_0. Sufficiently incommensurate means in this case that $\beta > 0$ and $\tau > f - 1$ exist such that

$$|\langle k, \omega\rangle| = \left|\sum_{\alpha=1}^{f} k_\alpha \omega^\alpha\right| \geq \frac{\beta}{\left(\sum_{\alpha=1}^{f} |k_\alpha|\right)^\tau}, \quad \text{(for all } k \in \mathbb{Z}^n, k \neq 0).$$

Quasiclassical quantization: With the angle-action variables one can quantize integrable systems (quasiclassically) (*Einstein-Keller quantization*). For this one requires that the action variables are not continuous quantities but integer multiples of the *Planck quantum of action*

$$\hbar = 1.055 \times 10^{-34}\,\mathrm{J\,s} \tag{8.27}$$

Specifically one demands that

$$J_\alpha = \hbar(n_\alpha + \nu_\alpha), \qquad (n_\alpha \in \mathbb{Z}), \tag{8.28}$$

where $\nu_\alpha = 0$ for rotation and $\nu_\alpha = \frac{1}{2}$ for libration.[8] For a libration the enclosed area in Figure 8.1 is always positive and hence only $n_\alpha = 0, 1, 2, \ldots$ make sense.

Example 1 (continued): For the harmonic oscillator we have found $J = E/\omega$. Since the motion is of libration type, quasiclassical quantization demands that

$$E_n = \hbar\omega\Big(n + \frac{1}{2}\Big), \qquad (n = 0, 1, \ldots),$$

which exactly matches the spectrum of the harmonic oscillator in quantum mechanics.

8.3.1 Adiabatic invariants

We consider an (autonomous) integrable system governed by the Hamiltonian $H \equiv H(q, p; \lambda)$, which depends on an external parameter λ. The motion of the system is characterized by the time scale

$$T = \frac{2\pi}{\omega_{\min}}$$

with the smallest angular frequency $\omega_{\min} = \min_\alpha \omega^\alpha$. We wish to understand the mechanical system when the parameter $\lambda = \lambda(t)$ in the Hamiltonian is varied *adiabatically* (i.e. slowly) with[9]

$$T\left|\frac{d\lambda}{dt}\right| \ll |\lambda|\,. \tag{8.29}$$

Since the (autonomous) system is integrable, there exists a generating function $S_0(q, J; \lambda)$ with $E = K(J; \lambda)$, so that J are the action variables (at fixed λ). The corresponding angle variables we denote (as usual) by ϕ^α.

We now transform the (full) time-dependent problem $H(q, p; \lambda(t))$ with the time-dependent generating function $S(q, J, t) = S_0(q, J; \lambda(t))$. This yields the new (time-dependent) Hamiltonian

$$\tilde{K} = K(J; \lambda) + \frac{\partial S}{\partial t} = K(J; \lambda) + \Lambda(\phi, J; \lambda)\dot{\lambda} \tag{8.30}$$

[8] This rule is somewhat simplified; in general $4\nu_\alpha$ is given by the Keller-Maslov index.

[9] It is difficult to give a precise criterion for the validity of the adiabatic result. In the derivation we will implicitly take the limit $T|\dot{\lambda}| \to 0$.

with

$$\Lambda = \frac{\partial S_0(q, J; \lambda)}{\partial \lambda}. \tag{8.31}$$

In order to interpret $\tilde{K}$ as a new Hamiltonian in the canonical variables (ϕ, J) we must still express q in S_0 via (8.21) in terms of ϕ and J.

The canonical equations for the angle-action variables then read

$$\dot{J}_\alpha = -\frac{\partial \tilde{K}}{\partial \phi^\alpha} = -\frac{\partial \Lambda}{\partial \phi^\alpha}\dot{\lambda}, \tag{8.32}$$

$$\dot{\phi}^\alpha = \frac{\partial \tilde{K}}{\partial J} = \omega^\alpha(J; \lambda) + \frac{\partial \Lambda}{\partial J_\alpha}\dot{\lambda} \tag{8.33}$$

with angular frequencies $\omega^\alpha = \partial K(J; \lambda)/\partial J_\alpha$. It is thus immediately evident that the variables J are no longer constants of motion. Equations (8.32) and (8.33) are still exact. We now average the first equation (8.32) over a time τ, with $\tau\left|\dot{\lambda}\right| \ll |\lambda|$ and $\tau \gg T$.

Because of the first condition we have $\dot{\overline{\lambda}} = \overline{\dot{\lambda}} \approx \dot{\lambda}$ and we may pull $\dot{\lambda}$ out of the average on the right-hand side of (8.32). This yields

$$\overline{\dot{J}_\alpha} = -\overline{\frac{\partial \Lambda}{\partial \phi^\alpha}}\dot{\lambda} \approx -\overline{\frac{\partial \Lambda}{\partial \phi^\alpha}}\dot{\lambda} \tag{8.34}$$

with $\overline{f}(t) = \tau^{-1}\int_t^{t+\tau} dt'\, f(t')$.

The second condition yields

$$\overline{f(\phi, J)} = \frac{1}{\tau}\int_t^{t+\tau} dt'\, f(\omega t' + \phi_0, J) \approx \frac{1}{(2\pi)^f}\int_{[0,2\pi]^f} d^f\phi\, f(\phi, J) \tag{8.35}$$

for an arbitrary function $f(\phi, J)$ on the phase space, since the motion proceeds over many periods of ϕ_α and thus covers the entire phase space (at fixed J); here we have assumed that the motion is non-degenerate.[10] Hence we immediately obtain that

$$\overline{\frac{\partial \Lambda}{\partial \phi^\alpha}} \approx \frac{1}{(2\pi)^f}\int_{[0,2\pi]^f} d^f\phi\, \frac{\partial \Lambda}{\partial \phi^\alpha} = 0 \tag{8.36}$$

since $\Lambda(\phi, J; \lambda)$ (like any physical quantity) is a 2π-periodic function in ϕ^α. The equation (8.34) gives the central result

$$\overline{\dot{J}_\alpha} = \dot{\overline{J}}_\alpha \overset{(8.35)}{\approx} \dot{J}_\alpha \approx 0, \tag{8.37}$$

i.e. the action variables J_α remain constant under a slow change of an external parameter λ and are therefore *adiabatic invariants*.

[10] The adiabatic theorem remains valid also for a degenerate motion. One then simply replaces the averaging over the torus in (8.35) by the corresponding subset of accessible coordinates.

Example 1 (continued): We now consider the situation in which the angular frequency of the oscillator is varied slowly. According to the considerations in this chapter, the angle variable $J = E/\omega$ remains approximately constant. We thereby obtain the result that

$$E(t) = \omega(t)J,$$

i.e. the energy of the oscillator changes in proportion to the angular frequency.

8.4 Time-dependent Hamilton-Jacobi equation

We now consider a (non-autonomous) system whose Hamilton function $H \equiv H(q^1, \ldots, q^f, p_1, \ldots, p_f, t)$ now also depends on time. We seek a *time-dependent* canonical transformation such that in the new coordinates the Hamiltonian $K(Q^1, \ldots, Q^f, P_1, \ldots, P_f, t) \equiv 0$ is. Then $Q^1, \ldots, P_f$ are constant and the motion is at rest in the transformed coordinates. By (7.61), the corresponding generating function $S(q, P, t)$ is a solution of the *time-dependent Hamilton-Jacobi equation*

$$\frac{\partial S}{\partial t} + H\Big(q, \frac{\partial S}{\partial q}, t\Big) = 0\,, \tag{8.38}$$

with the condition

$$\mathrm{Det}\left(\frac{\partial^2 S}{\partial q^\alpha \partial P_\beta}\right) \neq 0\,. \tag{8.39}$$

The motion in the original coordinates follows from the f equations

$$\frac{\partial S}{\partial P_\alpha}(q, P, t) = Q^\alpha\,, \qquad \text{hence} \quad q^\alpha = q^\alpha(Q, P, t)\,, \tag{8.40}$$

where the constants $(Q, P) = (Q^1, \ldots, Q^f, P_1, \ldots, P_f)$ are fixed by the initial conditions. Insertion into $p_\alpha = (\partial S/\partial q^\alpha)(q, P, t)$ then yields $p_\alpha = p_\alpha(Q, P, t)$.

If the system is autonomous, then (8.38) is equivalent to (8.2). In this case we may choose the separation ansatz

$$S(q, t) = S_0(q) + S_t(t) \tag{8.41}$$

and obtain

$$H\Big(q, \frac{\partial S_0}{\partial q}\Big) = -\dot{S}_t \equiv P_f\,, \tag{8.42}$$

where $P_f \equiv E$ is the separation constant with $S_t(t) = -P_f t$. Since

$$\frac{\partial S}{\partial P_\beta} = \frac{\partial S_0}{\partial P_\beta} - \delta_{\beta f} t\,, \tag{8.43}$$

the equations (8.40) reduce precisely to the time-independent Hamilton-Jacobi equations (8.7) and (8.8).

Remark: The generating function $S(q,P,t)$ of Hamilton-Jacobi theory is related to the action $S[q(t)]$ from the Hamiltonian principle. With (7.61) we obtain for the derivative along the mechanical path

$$\frac{d}{dt}S(q,P,t) = \sum_{\alpha=1}^{f}(p_\alpha \dot{q}^\alpha + Q^\alpha \dot{P}^\alpha) + (K-H) = \sum_{\alpha=1}^{f} p_\alpha \dot{q}^\alpha - H = L, \tag{8.44}$$

where we have used $K=0$ and $\dot{P}_\alpha = 0$. Integration over time leads to the result

$$S(q^{(1)},P,t^{(1)}) - S(q^{(0)},P,t^{(0)}) = \int_{(0)}^{(1)} dt\, L(q,\dot{q},t) = S[q(t)]. \tag{8.45}$$

The content of the equation is that $S(q,P,t)$ is given by the action $q\colon t \mapsto q(t)$ computed along the mechanical path $S[q(t)]$ of the problem: the difference of the generating *function* $S(q,P,t)$ is the action *functional* evaluated on the mechanical path connecting $q^{(0)}$ with $q^{(1)}$. For this reason the generating function of the canonical transformation is also called the action. An analogous relation holds between the reduced action $S_0[q(t)]$ of the Euler-Maupertuis principle and the generating function $S_0(q,P)$ in the time-independent case.

One sees very clearly from the above construction how, in the Hamilton-Jacobi method, the difficulty of the problem shifts from solving the equations of motion (trivial in the new coordinates) to finding the coordinate transformation (non-trivial). The generating function that mediates the coordinate transformation is given by an integration over the solution of the problem. Hence knowledge of the generating function is equivalent to solving the problem.

8.5 Outlook on Wave Mechanics

The starting point is the analogy between the mechanics of a particle and ray optics, particularly evident in the variational principles

$$\delta \int_{(1)}^{(2)} ds\, \sqrt{E - V(\boldsymbol{x})} = 0 \qquad \text{(Euler-Maupertuis)}, \tag{8.46}$$

$$\delta \int_{(1)}^{(2)} ds\, n(\boldsymbol{x}) = 0 \qquad \text{(Fermat)}. \tag{8.47}$$

Schrödinger (1926) built upon this a wave mechanics that relates to classical mechanics in the same way that wave optics relates to ray optics; i.e., mechanics is underpinned by a wave mechanics according to the scheme

$$\begin{array}{ccc} \text{geometrical optics} & \longleftrightarrow & \text{mechanics} \\ \uparrow & & \uparrow \\ \text{wave optics} & \longleftrightarrow & \text{wave mechanics}\,. \end{array}$$

. As a wavelength, de Broglie (1923) postulated the quantity

$$\lambda = \frac{2\pi\hbar}{p},$$

(or also $p = \hbar k$) depending on the momentum p of the particle.

Mechanics

For the moment we restrict ourselves to a particle in $\mathbb{R}^3$ with the Hamiltonian

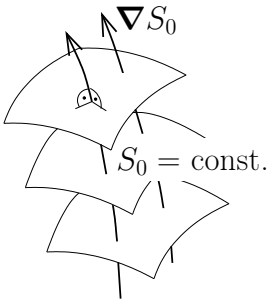
$$H = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{x}). \tag{8.48}$$

Then the Hamilton-Jacobi equation reads

$$(\boldsymbol{\nabla} S_0)^2 = 2m\big(E - V(\boldsymbol{x})\big). \tag{8.49}$$

Each solution of this equation describes a bundle of mechanical trajectories of energy E. Here the momentum is determined by

$$\boldsymbol{p} = m\dot{\boldsymbol{x}} = \boldsymbol{\nabla} S_0. \tag{8.50}$$



The trajectory curves are thus orthogonal trajectories of the surfaces $S_0 = \text{const}$. On each of these trajectory curves $d\boldsymbol{x}/ds = \dot{\boldsymbol{x}}/(ds/dt) = \boldsymbol{\nabla} S_0/|\boldsymbol{\nabla} S_0|$ holds, i.e.,

$$\sqrt{2m(E - V(\boldsymbol{x}))}\,\frac{d\boldsymbol{x}}{ds} = \boldsymbol{\nabla} S_0, \qquad (s = \text{arc length}). \tag{8.51}$$

They thereby satisfy (8.46). The vector $d\boldsymbol{x}/ds$ is here the unit vector along the trajectory.

Ray Optics

We write Fermat's principle (8.47) in the form

$$\delta \int_{(1)}^{(2)} ds\, k(\boldsymbol{x}) = 0, \quad k(\boldsymbol{x}) = \frac{\omega}{c} n(\boldsymbol{x}) \tag{8.52}$$

with k the wavenumber and ω the angular frequency of the light. This corresponds to the Euler-Maupertuis principle for the trajectory curves of energy E in the potential $V(\boldsymbol{x})$, provided that

$$2m(E - V(\boldsymbol{x})) = C^2 k^2(\boldsymbol{x}) \tag{8.53}$$

with C an arbitrary constant. From the discussion in mechanics we can directly adopt the following result: each solution $S_0(\boldsymbol{x})$ of the Hamilton-Jacobi equation (cf. (8.49))

$$(\boldsymbol{\nabla} S_0)^2 = C^2 k^2(\boldsymbol{x}) \tag{8.54}$$

describes a bundle of light rays; the orthogonal trajectories of the surfaces $S_0(\boldsymbol{x}) =$ const. This bundle is determined by (cf. (8.51))

$$Ck(\boldsymbol{x})\frac{d\boldsymbol{x}}{ds} = C\boldsymbol{k}(\boldsymbol{x}) = \boldsymbol{\nabla} S_0\,. \tag{8.55}$$

The choice $C = 1$ is expedient in view of wave optics, as we shall see immediately. In optics $S_0(\boldsymbol{x})$ is then called the *eikonal* and (8.54) the *eikonal equation*.

Wave Optics → Ray Optics

In scalar wave optics monochromatic light is described by a complex field $\psi(\boldsymbol{x}, t) = u(\boldsymbol{x})e^{-i\omega t}$ that satisfies the wave equation (with $C = 1$)

$$(C^2\Delta + k^2)u = 0 \tag{8.56}$$

We decompose

$$u(\boldsymbol{x}) = A(\boldsymbol{x})e^{i\tilde{S}_0(\boldsymbol{x})} \tag{8.57}$$

into the amplitude A and the phase $\tilde{S}_0 = S_0/C$. With

$$\begin{aligned}
&\boldsymbol{\nabla}(Ae^{i\tilde{S}_0}) = (\boldsymbol{\nabla}A + iA\boldsymbol{\nabla}\tilde{S}_0)e^{i\tilde{S}_0},\\
&\Delta(Ae^{i\tilde{S}_0}) = \operatorname{div}\boldsymbol{\nabla}(Ae^{i\tilde{S}_0}) = \Big(\Delta A + iA\Delta\tilde{S}_0 + 2i\boldsymbol{\nabla}A\cdot\boldsymbol{\nabla}\tilde{S}_0 - A(\boldsymbol{\nabla}\tilde{S}_0)^2\Big)e^{i\tilde{S}_0}
\end{aligned}$$

it follows from (8.56) the two equations

$$C^2\Delta A - A(\boldsymbol{\nabla}S_0)^2 + Ak^2 = 0, \qquad A\Delta S_0 + 2\boldsymbol{\nabla}A\cdot\boldsymbol{\nabla}S_0 = 0. \tag{8.58}$$

Ray optics, in the form of the eikonal equation (8.54) with $C = 1$, is thus a good approximation in regions where

$$\left|\frac{\Delta A}{A}\right| \ll k^2. \tag{8.59}$$

Roughly speaking, this means that the eikonal approximation is good in regions where the amplitude $A(\boldsymbol{x})$ changes little over a wavelength $\approx k^{-1} = |\boldsymbol{\nabla}S_0|^{-1}$. The eikonal $S_0(\boldsymbol{x})$ is then the phase of the light wave, and the light rays are orthogonal trajectories of the surfaces of constant phase. The vector field $\boldsymbol{\nabla}S_0$ describes (cf. (8.55)) the local wave vector $\boldsymbol{k}(\boldsymbol{x}) = k(\boldsymbol{x})\,d\boldsymbol{x}/ds$.

Mechanics ← Wave Mechanics (Schrödinger)

Schrödinger's starting point was that the Hamilton-Jacobi equations (8.49) for $S_0(\boldsymbol{x})$ should be the ray-optical approximation of a (mechanical) wave equation of a scalar field $u(\boldsymbol{x})$. Comparing (8.50) with (8.54) suggests that to a particle of momentum $\boldsymbol{p}$ there should be associated a wave with the (de Broglie) wave vector ($C = \hbar$)

$$\boldsymbol{p} = \hbar\boldsymbol{k} \tag{8.60}$$

The Hamilton-Jacobi equation (8.49)

$$(\boldsymbol{\nabla} S_0)^2 = 2m(E - V(\boldsymbol{x}))$$

is thus the ray-optical approximation to (8.56), i.e.,

$$\hbar^2 \Delta u(\boldsymbol{x}) + 2m(E - V(\boldsymbol{x}))u(\boldsymbol{x}) = 0\,. \tag{8.61}$$

Equation (8.61) is the *time-independent Schrödinger equation*: it describes in wave mechanics states u of energy E.

Analogously, the time-dependent Hamilton-Jacobi equation for the system (8.48),

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\boldsymbol{\nabla} S)^2 + V(\boldsymbol{x}) = 0 \tag{8.62}$$

is the ray-optical approximation of the phase $\tilde{S} = S/\hbar$ of a complex wave $\psi(\boldsymbol{x}, t) = A(\boldsymbol{x}, t)e^{iS(\boldsymbol{x},t)/\hbar}$ that satisfies the *time-dependent Schrödinger equation*

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi \tag{8.63}$$

This is the *equation of motion of wave mechanics* for the system (8.48). It determines $\psi(\boldsymbol{x}, t)$ from the initial state $\psi(\boldsymbol{x}, 0)$. Equation (8.61) arises from (8.63) by the ansatz

$$\psi(\boldsymbol{x}, t) = u(\boldsymbol{x})e^{-i\omega t} \qquad \text{with } E = \hbar\omega. \tag{8.64}$$

For the phase $S/\hbar$ this corresponds exactly to the separation ansatz (8.41).

The statistical interpretation of the wave function $\psi(\boldsymbol{x}, t)$ is due to Born (1926): one normalizes $\psi(\boldsymbol{x}, t)$ (at a time t_0) so that

$$\int |\psi(\boldsymbol{x}, t_0)|^2 \, d^3x = 1.$$

Then

$$P(\Omega, t) = \int_\Omega |\psi(\boldsymbol{x}, t)|^2 \, d^3x$$

is the probability that the particle is found in the region $\Omega \subset \mathbb{R}^3$ at time t.

Chapter 9

Relativistic Mechanics

9.1 Einstein's Relativity Principle

The Michelson-Morley experiment led Einstein (1905) to postulate, in addition to the Galilean principle of relativity—that the laws of physics are the same in all inertial frames—the following *Einstein postulate*:

> *The speed of light has the same magnitude* $c = 299\,792\,458\,m/s$ *in all inertial frames.*

This principle is the foundation of the *Special Theory of Relativity*. Inertial frames are defined so that free particles, as in classical mechanics, obey the law of inertia $\ddot{\boldsymbol{x}} = 0$. One sees immediately that the invariance of physical laws can no longer be demanded under Galilean transformations, since these lead to the elementary velocity-addition formulas. In Chapter 9.2 we will examine which transformation group is compatible with the Einstein postulate. For now, however, we first wish to become acquainted with a few simple consequences of the Einstein postulate.

Since in Chapter 1.3 we saw that assuming an absolute time leads directly from the relativity principle to the Galilean transformations as the general transformations between inertial frames, we must abandon the notion of absolute time and require in general that $t' \neq t$. The absolute value of the speed of light makes it possible to synchronize clocks 1 and 2 in two reference frames S and S' and thereby to introduce, at least in part, a common notion of time. The protocol for synchronizing two clocks in two distinct inertial frames S and S', which measure time by the same periodic process, is as follows: We imagine that in frame S a light flash is emitted from clock 1 at time t_0. The beam is reflected by clock 2 (at rest in frame S') and detected by clock 1 at time $t_0 + \Delta t$. Because the speed of light is absolute, we can identify the arrival time of the light ray at clock 2 with the time $t_0 + \Delta t/2$ in frame S and thereby synchronize the clocks. Furthermore, the universality of the speed of light allows us to reduce length measurements to time measurements and to define the meter as the 1/299792458-th part of the distance a light ray travels in one second. The problem

with clock synchronization is that the procedure is not transitive. If clocks 1 and 2 are synchronized and clocks 2 and 3 are synchronized, it does not follow that clocks 1 and 3 are synchronized. This leads us to the concept of the *relativity of simultaneity*.

Relativity of simultaneity: From the notion of absolute time in Galilean space-time, two events that occur simultaneously in frame S' also occur simultaneously in S. This fact is incompatible with Einstein's postulate of the universality of the speed of light. To see this, consider two frames moving relative to each other with velocity $\boldsymbol{w} = (w, 0, 0)$, $w > 0$. We place three clocks A, B, C along the 1-axis, at rest in S'. Clock B lies between clocks A and C so that the distance from A to B equals the distance from B to C. An observer in frame S' will find that a light flash emitted from B at time t'_{B} arrives simultaneously ($t'_{\mathrm{A}} = t'_{\mathrm{C}}$) at A and C.

From the viewpoint of an observer in the (moving) frame S, the light flash is emitted from B at time t_{B} and propagates with speed c in all directions. Since A moves toward the light flash while C moves away from it, from the perspective of S the inequality $t_{\mathrm{A}} < t_{\mathrm{C}}$ holds. Thus clocks synchronized in frame S' are not synchronized in frame S, and therefore the notion of "simultaneity" depends on the reference frame. The absence of an absolute notion of simultaneity is the origin of many of the initially paradoxical consequences of the relativity principle.

9.2 Lorentz Transformation

In the previous chapter we saw that Einstein's postulate, supported by the Michelson-Morley experiment, forces us to rethink our conceptions of time and space. In particular, we must abandon the idea of an absolute time. In this chapter we develop Einstein's postulate into a mathematical theory that allows us to formulate new laws of nature compatible with it. To do so, we must first determine what the new transformations between inertial frames are.

Just as Galilean transformations leave Newton's equations invariant, the *Lorentz transformations* are the group of transformations that connect systems satisfying Einstein's postulate. Thus the Lorentz transformations in special relativity allow one to pass from one inertial frame to any other inertial frame. Having established these transformations, the next task is to find a relativistic (i.e. Lorentz-invariant) formulation of mechanics, which we will take up in the following chapter.

9.2.1 Transformation between inertial frames

At first glance it may seem as though Einstein's postulate contradicts the law of inertia. This "feeling," however, stems from our deep entrenchment in the classical worldview; as seen in the previous chapter, Einstein's postulate entails the loss of the absolute notion of simultaneity and thereby introduces a separate time for each frame. To obtain the Lorentz-transformation group, consider two inertial frames S and S' moving relative to each other with velocity $\boldsymbol{w}$.

We consider a light flash emitted at $(t_1, \boldsymbol{x}_1)$ and arriving later at $(t_2, \boldsymbol{x}_2)$. By Einstein's postulate we have both

$$(\Delta \boldsymbol{x})^2 - c^2(\Delta t)^2 = 0 \qquad \text{and} \qquad (\Delta \boldsymbol{x}')^2 - c^2(\Delta t')^2 = 0 \tag{9.1}$$

with $\Delta t = t_2 - t_1$ and $\Delta \boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1$. The motion of a force-free particle in an inertial frame is given by straight lines. From the requirement that a Lorentz transformation map inertial frames to one another, together with the law of inertia $\ddot{\boldsymbol{x}} = \ddot{\boldsymbol{x}}' = 0$, it follows that straight lines are mapped to straight lines. Hence a Lorentz transformation is a linear map from $(\Delta t, \Delta \boldsymbol{x})$ to $(\Delta t', \Delta \boldsymbol{x}')$. By linearity, (9.1) is equivalent to[1]

$$(\Delta \boldsymbol{x}')^2 - c^2(\Delta t')^2 = \kappa(\boldsymbol{w})[(\Delta \boldsymbol{x})^2 - c^2(\Delta t)^2]. \tag{9.2}$$

Now consider a further frame S'' moving with velocity $-\boldsymbol{w}$ relative to S'. Then we obtain

$$(\Delta \boldsymbol{x}'')^2 - c^2(\Delta t'')^2 = \kappa(-\boldsymbol{w})\kappa(\boldsymbol{w})[(\Delta \boldsymbol{x})^2 - c^2(\Delta t)^2].$$

Since S'' is at rest relative to S, $\Delta \boldsymbol{x} = \Delta \boldsymbol{x}''$ and $\Delta t = \Delta t''$, so $\kappa(-\boldsymbol{w})\kappa(\boldsymbol{w}) = 1$. By the isotropy of space, $\kappa(\boldsymbol{w})$ can depend only on the magnitude w of the relative velocity, not on its direction. We conclude that $\kappa(-\boldsymbol{w})\kappa(\boldsymbol{w}) = \kappa(w)^2 = 1$ and hence $\kappa(w) = 1$.[2]

The results are best summarized by combining space and time into the *four-vector coordinates*

$$x \equiv (x^0, x^1, x^2, x^3) = (ct, \boldsymbol{x}) \tag{9.3}$$

of a particle. The Lorentz transformations are then defined as those that leave invariant the squared interval

$$\Delta s^2 = \big(\Delta x', \Delta x'\big) = \big(\Delta x, \Delta x\big) = \eta_{\mu\nu}(\Delta x)^\mu(\Delta x)^\nu \tag{9.4}$$

with the Minkowski metric

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{9.5}$$

We employ the *Einstein summation convention*: repeated indices are summed over $\{0, 1, 2, 3\}$.

A general affine transformation between frames S and S' can be written $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$ with $a \in \mathbb{R}^4$ and $\Lambda \in \mathrm{GL}(4, \mathbb{R})$. The condition (9.4) leads to the restriction

$$\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta (\Delta x)^\alpha (\Delta x)^\beta = \eta_{\mu\nu}(\Delta x')^\mu(\Delta x')^\nu = \eta_{\mu\nu}(\Delta x)^\mu(\Delta x)^\nu$$

[1] A linear map sends a homogeneous polynomial of degree 2 to a homogeneous polynomial of degree 2. Since $(\Delta \boldsymbol{x})^2 - c^2(\Delta t)^2 = (\Delta \boldsymbol{x}')^2 - c^2(\Delta t')^2 = 0$, homogeneity immediately yields (9.2).

[2] The alternative solution $\kappa(w) = -1$ can be discarded because $\kappa = 1$ at $w = 0$.

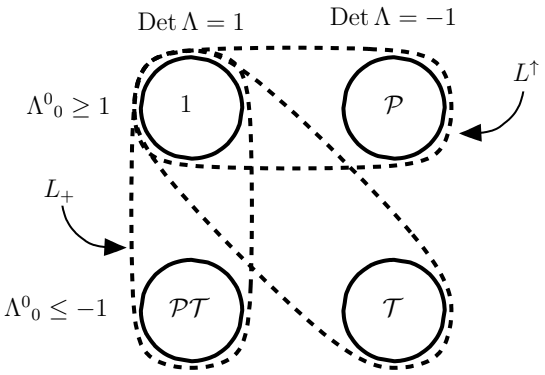


Figure 9.1: Structure of the Lorentz group: the Lorentz group L decomposes into four disjoint components characterized by $\mathrm{Det}\,\Lambda = \pm 1$ and $\mathrm{sgn}(\Lambda^0{}_0)$.

which is equivalent to

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}\,, \tag{9.6}$$

or in matrix notation $\Lambda^t\eta\Lambda = \eta$.[3] Transformations satisfying (9.6) are called elements of the *Lorentz group*, denoted $L = \mathrm{O}(1,3)$. The group of transformations connecting inertial frames with fixed scales is the group of inhomogeneous Lorentz transformations with $x' = \Lambda x + a$, also called the *Poincaré group*. Vectors v that transform under the homogeneous part of the Poincaré group, i.e. as $v' = \Lambda v$, are called *contravariant four-vectors*.

9.2.2 The Lorentz Group

We consider an element $\Lambda \in L$. From the relation (9.6) one can derive a series of properties of Λ immediately. Taking the determinant yields at once $(\mathrm{Det}\,\Lambda)^2 = 1$. Thus L decomposes into two components characterized by $\mathrm{Det}\,\Lambda = \pm 1$. Moreover, one obtains for the (00)-component the property $(\Lambda^0{}_0)^2 - \sum_{k=1}^{3}(\Lambda^k{}_0)^2 = 1$, i.e. $(\Lambda^0{}_0)^2 \geq 1$. The Lorentz group therefore has two disjoint components characterized by the sign of $\Lambda^0{}_0$.

That all four cases occur is shown by the reflections

$$1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{9.7}$$

[3] The inverse of Λ is obtained by multiplying from the left by η and from the right by Λ^{-1} as $\Lambda^{-1} = \eta\Lambda^t\eta$.

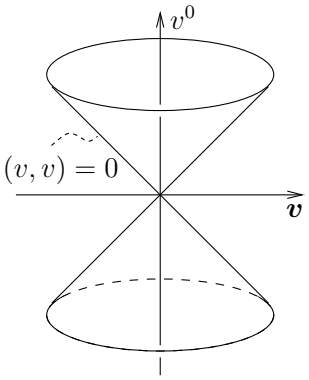


Figure 9.2: Vectors on the light cone with $(v,v)=0$ are called null vectors. Outside the light cone lie the spacelike vectors. The timelike vectors inside the cone split into two components depending on whether they point toward the future (above) or the past (below).

$$\mathcal{T}=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\qquad \mathcal{PT}=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}, \tag{9.8}$$

with $\mathcal{P}$ the reflection at the origin of coordinates (parity) and $\mathcal{T}$ time reversal. The reflections form a subgroup of L. Further subgroups are, for example,

$$\begin{aligned} L_+ &= \{\Lambda\in L \mid \operatorname{Det}\Lambda=1\}, && \text{the proper Lorentz transformations,}\\ L^\uparrow &= \{\Lambda\in L \mid \Lambda^0{}_0\geq 1\}, && \text{, the orthochronous Lorentz transformations,}\\ L_+^\uparrow &= L_+\cap L^\uparrow. \end{aligned} \tag{9.9}$$

A Lorentz transformation maps, by definition, the light cone with $\Delta s^2=0$ onto itself. Four-vectors on the light cone with $(v,v)=0$ are also called *null vectors*, see Figure 9.2. We further define vectors pointing to the exterior of the light cone as *spacelike vectors* v with $(v,v)<0$ and those pointing to the interior as *timelike vectors* with $(v,v)>0$. Timelike vectors can be further subdivided into vectors pointing toward the timelike future ($v^0>0$) and those pointing toward the timelike past ($v^0<0$).

That $L^\uparrow$ is a group can be seen geometrically. A Lorentz transformation $\Lambda\in L$ maps the interior of the light cone onto itself. In doing so, the two subcones (future and past) $V^\pm=\{x\mid (x,x)>0,\pm x^0\geq 0\}$ either remain invariant or are exchanged. The decisive factor here is the sign $\operatorname{sgn}(\Lambda^0{}_0)$, since $[\Lambda(1,\mathbf{0})]^0=\Lambda^0{}_0$. Thus $\operatorname{sgn}(\Lambda^0{}_0)$ and naturally also $\operatorname{Det}\Lambda$ behave multiplicatively[4] under the group operation. The orthochronous transformations $L^\uparrow$ map the future onto the future and the past onto the past.

Because of the multiplicity of $\operatorname{sgn}(\Lambda^0{}_0)$ and $\operatorname{Det}\Lambda$, every $\Lambda\in L$ can be written as the product of an element of the proper orthochronous Lorentz group $L_+^\uparrow$ with a

[4]The latter follows from the general relation $\operatorname{Det}(AB)=(\operatorname{Det}A)(\operatorname{Det}B)$ for arbitrary square matrices A and B.

reflection $\{1, \mathcal{P}, \mathcal{T}, \mathcal{PT}\}$. We therefore restrict ourselves in the following to the proper orthochronous subgroup.

The Proper Orthochronous Lorentz Group The 4×4-matrix Λ has in total 16 real entries. The equation (9.6) yields 10 independent equations.[5] Therefore elements $\Lambda \in L_+^\uparrow$ are determined by 6 real parameters.

Three of the parameters can be identified with three-dimensional rotations. Indeed, with

$$\Lambda(R) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right) \tag{9.10}$$

one immediately obtains $\Lambda^t \eta \Lambda = \eta$ where $R \in \mathrm{SO}(3)$ with $R^t R = 1$. Thus the rotations $\Lambda(R)$ form a subgroup of $L_+^\uparrow$.

Of greater interest are the remaining three-parameter transformations, which we expect to connect reference frames moving relative to one another. To this end we investigate whether special solutions with the block form

$$\Lambda = \left(\begin{array}{cc|cc} a & b & 0 & 0 \\ c & d & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \tag{9.11}$$

exist, i.e. x^2, x^3 are not transformed. Inserting into (9.6) yields the conditions

$$a^2 - c^2 = 1, \qquad ab - cd = 0, \qquad b^2 - d^2 = -1.$$

Since we are interested only in solutions from $L_+^\uparrow$, we additionally require $a > 0$ and $ad - bc = 1$.

We can satisfy the first equation by introducing the parameter χ with $a = \cosh\chi$ and $c = -\sinh\chi$. From the second equation we obtain $b = cd/a$. Inserting into $ad - bc = 1$ yields $1 = d/a$, i.e. $d = \cosh\chi$ and $b = -\sinh\chi$. Thus we obtain the special Lorentz transformations (boosts)

$$\Lambda(\chi) = \left(\begin{array}{cc|cc} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = \exp\left(\begin{array}{cc|c} 0 & -\chi & 0 \\ -\chi & 0 & \\ \hline & 0 & 0 \end{array}\right). \tag{9.12}$$

From the last identity it follows immediately that the boosts form a subgroup with the multiplication law

$$\Lambda(\chi_1 + \chi_2) = \Lambda(\chi_1)\Lambda(\chi_2). \tag{9.13}$$

From this one can directly conclude that $\Lambda(\chi)^{-1} = \Lambda(-\chi)$.

[5] Transposing (9.6) leads to the same equation system. Hence only 10 of the 16 equations are independent.

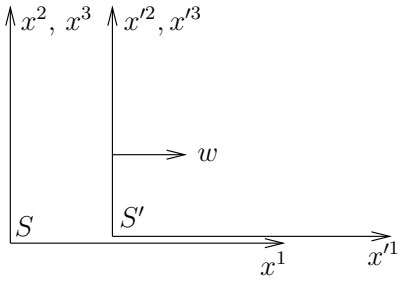


Figure 9.3: Relation between systems S and S' under a boost along the 1-axis.

Next we wish to explain the physical meaning of $\Lambda(\chi)$. To this end we write the transformation law $x = \Lambda(\chi)^{-1}x'$ of a boost in components

$$\begin{aligned} ct &= (\cosh\chi)ct' + (\sinh\chi)x'^1, & x^2 &= x'^2, \\ x^1 &= (\sinh\chi)ct' + (\cosh\chi)x'^1, & x^3 &= x'^3. \end{aligned} \tag{9.14}$$

An object which is at rest in the S'-system at position $\boldsymbol{x}' = \boldsymbol{x}'(t')$ moves therefore in the S-system with the velocity[6]

$$w = \frac{dx^1}{dt} = \frac{dx^1}{dt'}\frac{dt'}{dt} = \frac{dx^1}{dt'}\left(\frac{dt}{dt'}\right)^{-1} = c\tanh\chi\,. \tag{9.15}$$

Thus we obtain the interpretation that $\Lambda(\chi)$ transforms to a new inertial system which moves with velocity w along the 1-axis relative to S, see Figure 9.3. Simple hyperbolic relations yield the expressions

$$\begin{aligned} \cosh\chi &= \frac{1}{\sqrt{1-\tanh\chi^2}} = \frac{1}{\sqrt{1-(w/c)^2}} \equiv \gamma, \\ \sinh\chi &= \tanh\chi\cosh\chi = \frac{w/c}{\sqrt{1-(w/c)^2}} = \gamma w/c. \end{aligned} \tag{9.16}$$

From (9.14) it follows that

$$\begin{aligned} t &= \gamma(t' + wx'^1/c^2), & x^2 &= x'^2, \\ x^1 &= \gamma(x'^1 + wt'), & x^3 &= x'^3. \end{aligned} \tag{9.17}$$

For $w/c \to 0$ one obtains in the limit the Galilei transformation

$$t = t', \quad x^1 = x'^1 + wt, \quad x^2 = x'^2, \quad x^3 = x'^3.$$

A transformation to a system S' moving with velocity $\boldsymbol{w}$ in an arbitrary spatial direction relative to S can be obtained in principle by concatenating a rotation (so that $\boldsymbol{w}'$ is along the 1-direction), a boost, and an inverse rotation. Much simpler,

[6] In the literature $\chi = \operatorname{artanh}(w/c)$ is called the rapidity.

however, is the idea that one can decompose the vector $\boldsymbol{x}$ into a component $\boldsymbol{x}_\parallel$ along $\boldsymbol{w}$ and a component $\boldsymbol{x}_\perp$ orthogonal to it. Analogous to (9.17) one then obtains

$$t = \gamma(t' + (\boldsymbol{w}\cdot\boldsymbol{x}'_\parallel)/c^2), \qquad \boldsymbol{x}_\parallel = \gamma(\boldsymbol{x}'_\parallel + \boldsymbol{w}t'), \qquad \boldsymbol{x}_\perp = \boldsymbol{x}'_\perp. \tag{9.18}$$

Thus a boost $x' = \Lambda(\boldsymbol{w})x$ in general depends on the three real parameters $\boldsymbol{w}$. The Lorentz transformations become singular for $w \to c$. The speed of light represents a maximal speed. No object (and no information) can move faster than light.

9.2.3 Addition of Velocities

It is immediately clear that Galilean velocity addition $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{v}'$ leads to contradictions with the absoluteness of the speed of light. This requires namely that if $|\boldsymbol{v}| = c$ then $|\boldsymbol{v}'| = c$, independent of $\boldsymbol{w}$. The relativistic formulas for velocity addition can be derived immediately from the equations (9.18) for a general boost. We introduce two (coordinate-parallel) reference frames S and S' whose coordinates are related by (9.18) (i.e. they move with $\boldsymbol{w}$ relative to each other). Consider now an object moving along the path $\boldsymbol{x}'(t') = \boldsymbol{x}'_0 + \boldsymbol{v}'t'$ in S'. The same object will move in the S-system along the path $\boldsymbol{x}(t)$. In 4-vector notation we have the coordinates $x(t) = (ct, \boldsymbol{x}(t))$ and $x'(t') = (ct', \boldsymbol{x}'(t'))$, which are related by $x' = \Lambda(\boldsymbol{w})x$. With the general formula (9.18) for a boost, we obtain

$$\frac{dt}{dt'} = \gamma(1 + (\boldsymbol{w}\cdot\boldsymbol{v}'_\parallel)/c^2) = \gamma(1 + (\boldsymbol{w}\cdot\boldsymbol{v}')/c^2)$$

and

$$\frac{d\boldsymbol{x}_\parallel}{dt'} = \gamma(\boldsymbol{v}'_\parallel + \boldsymbol{w}),$$
$$\frac{d\boldsymbol{x}_\perp}{dt'} = \boldsymbol{v}'_\perp$$

The object therefore moves in the S-system with the velocity (*addition of velocities*)

$$\boldsymbol{v} = \frac{d(\boldsymbol{x}_\parallel + \boldsymbol{x}_\perp)}{dt} = \frac{\boldsymbol{w} + \boldsymbol{v}'_\parallel + \boldsymbol{v}'_\perp\sqrt{1-(w/c)^2}}{1 + (\boldsymbol{w}\cdot\boldsymbol{v}')/c^2}. \tag{9.19}$$

The inverse relation is obtained by replacing $\boldsymbol{w}$ with $-\boldsymbol{w}$. In the limit $|\boldsymbol{w}|, |\boldsymbol{v}'| \ll c$ one recovers, as desired, the nonrelativistic formula $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{v}'$.

9.3 Relativistic Mechanics

As we have seen, Newtonian mechanics is not compatible with the relativity postulate. In special relativity we must therefore render the laws of mechanics *covariant* (i.e. invariant under Lorentz transformations). As a side condition the old mechanics should be recovered in the limit $|\dot{\boldsymbol{x}}| \ll c$.

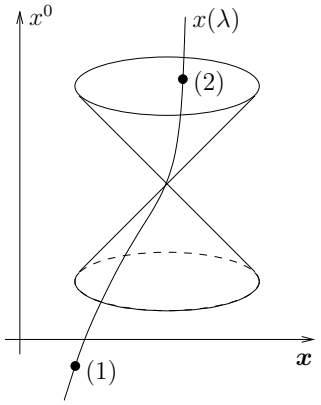


Figure 9.4: World line of a particle parametrized by λ. The motion takes place between the events $x^{(1)}$ and $x^{(2)}$.

9.3.1 Four-Velocity

The motion of a particle is described by the trajectory $\boldsymbol{x}(t)$ with the velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{x}}(t)$. In relativistic notation the trajectory becomes a world line

$$x(\lambda) = (ct(\lambda), \boldsymbol{x}(t(\lambda))) \tag{9.20}$$

with λ an arbitrary parameter that parametrizes the world line, see Figure 9.4. As explained in Chapter 9.2, transformations between different inertial systems are implemented by a Poincaré transformation Λ with $x' = \Lambda x + a$.

Man now wishes to generalize the concept of velocity to a four-vector. To achieve this, $x(\lambda)$ must be parametrized by a Lorentz scalar τ that coincides with time in the limit $|\boldsymbol{v}| \ll c$. Due to the invariance of the distance (9.4), the proper time (starting from an arbitrary time $t^{(1)}$)

$$\tau(t^{(2)}) = \frac{1}{c}\int_{(1)}^{(2)} ds = \int_{(1)}^{(2)} \sqrt{\eta_{\mu\nu}\, dx^\mu\, dx^\nu}/c = \int_{t^{(1)}}^{t^{(2)}} dt\, \sqrt{1 - \boldsymbol{v}^2(t)/c^2} \tag{9.21}$$

between the events $x^{(1)}$ and $x^{(2)}$ is itself a Lorentz scalar, i.e. τ is independent of the reference frame. To render the parametrization in (9.20) independent of the reference frame, it is therefore often expedient to set $\lambda = \tau$ by solving $\tau(t)$ from (9.21) for t. Differentiating the world line with respect to proper time yields the contravariant four-vector,

$$u(\tau) \equiv \frac{dx(\tau)}{d\tau} = \frac{dt}{d\tau}\frac{d(ct, \boldsymbol{x}(t))}{dt} = \gamma(t)(c, \boldsymbol{v}(t)) \tag{9.22}$$

with $\gamma(t) = (1 - \boldsymbol{v}^2/c^2)^{-1/2}$, which we term the *four-velocity*. For the square of the "length" of the four-vector one has $(u, u) = \gamma^2(c^2 - \boldsymbol{v}^2) = c^2$. Thus u is always normalized to c and only three parameters are independent.

9.3.2 Energy-Momentum Relation

With the four-velocity we can immediately generalize the momentum $\boldsymbol{p}$ to a contravariant four-momentum

$$p \equiv (p^0, \boldsymbol{p}) = mu = \gamma m(c, \boldsymbol{v}) \tag{9.23}$$

. In the nonrelativistic limit $|\boldsymbol{v}| \ll c$, $\gamma \to 1$, the spatial components of p^μ agree with the nonrelativistic momentum. The spatial part therefore remains conserved for a force-free particle in an inertial frame. Since p^μ is a contravariant four-vector, the entire four-momentum p^μ is conserved in every inertial frame. Analogously one may conclude that the total momentum $P^\mu = \sum_{j=1}^{N} (p_j)^\mu$ of N particles with four-momenta p_j is conserved in every inertial frame.

The question now arises which conserved quantity in nonrelativistic physics corresponds to the time component p^0. Expanding p^0 in $|\boldsymbol{v}|/c$ gives

$$p^0 = \frac{mc}{\sqrt{1 - \boldsymbol{v}^2/c^2}} = mc + \frac{1}{2} m\boldsymbol{v}^2/c + \cdots \tag{9.24}$$

One immediately recognizes that $p^0 c$, up to the constant mc^2, agrees with the kinetic energy T of Newtonian mechanics. Hence the (relativistic) *kinetic energy* E is given by

$$E = p^0 c = E_0 + T \quad \text{with} \quad E_0 = mc^2. \tag{9.25}$$

The rest energy $E_0 = mc^2$ is precisely Einstein's famous formula. In contrast to nonrelativistic physics, relativistic energy contains a rest-energy contribution that depends only on the object's mass. Unlike in nonrelativistic physics, mass can be annihilated in relativity theory and the corresponding energy converted into other forms of energy.

Consider as an example the (symmetric) decay of a particle into two parts. In the rest frame of the particle one starts with the four-momentum $P = (Mc, \mathbf{0})$ with M of the total mass of the particle.

After the symmetric decay the total momentum P is composed of the momenta of two particles of mass m moving with velocity $\pm\boldsymbol{v}$. The particles have four-momenta $\gamma m(c, \pm\boldsymbol{v})$ with $\gamma = (1 - \boldsymbol{v}^2/c^2)^{-1/2}$, which sum to the total momentum $P = 2\gamma m(c, 0)$. From conservation of the four-momentum we can infer that

$$2m = M\sqrt{1 - \boldsymbol{v}^2/c^2} < M, \tag{9.26}$$

i.e. the total mass is not conserved. The rest energy of the mass defect is given precisely by

$$(M - 2m)c^2 = 2mc^2(\gamma - 1) = 2\frac{1}{2} m\boldsymbol{v}^2 + \cdots \tag{9.27}$$

and hence for $|\boldsymbol{v}| \ll c$ equals the nonrelativistic energy of the decay products. In the decay of the particle, rest energy is therefore converted into kinetic energy of the decay products.

From the fact that $(u,u) = c^2$, one obtains the relativistic energy-momentum relation

$$(p,p) = \eta_{\mu\nu}\, p^\nu\, p^\mu = (E/c)^2 - \boldsymbol{p}^2 = m^2c^2, \tag{9.28}$$

which, by construction, holds in all inertial frames.

9.3.3 Equation of Motion

With this preparation it is now possible to formulate mechanics in a Lorentz-invariant way. The simplest route is via the Lagrangian formalism. In this formalism one assigns to each world line $x(\lambda)$ an action

$$S[x(\lambda)] = \int_{(1)}^{(2)} d\lambda\, L(x, \partial_\lambda x, \lambda) \tag{9.29}$$

with L the Lagrangian. Hamilton's principle demands that the trajectory of a particle be characterized by an extremum of the action for fixed endpoints in $\mathbb{R}^4$. From the invariant condition $\delta S = 0$ on the trajectory it is directly evident that the equation of motion becomes covariant provided the action S is a Lorentz scalar.[7]

Force-free Particle

In nonrelativistic mechanics a force-free particle is described by the Lagrangian $L_0 = \frac{1}{2}m\boldsymbol{v}^2$. To obtain a relativistic generalization of the equations of motion for a force-free particle we need an action that is a Lorentz scalar and, in the limit $|\boldsymbol{v}| \ll c$, reproduces the nonrelativistic behavior. As we saw in (9.21), proper time assigns a Lorentz scalar to a world line. A natural ansatz is therefore $S_0 = -mc^2\tau$, where the prefactor E_0 is chosen so that S_0 has the dimension of an action. Indeed we obtain

$$S_0 = -mc^2\tau = -mc^2 \int_{(1)}^{(2)} dt\, \sqrt{1 - \boldsymbol{v}^2/c^2} = \int_{(1)}^{(2)} dt \left(-mc^2 + \frac{1}{2}m\boldsymbol{v}^2 + \cdots\right),$$

so that the relativistic Lagrangian

$$L_0 = -mc^2\sqrt{1 - \boldsymbol{v}^2/c^2} \tag{9.30}$$

in the limit $|\boldsymbol{v}| \ll c$ agrees, up to the constant mc^2, with the nonrelativistic Lagrangian of a force-free particle. The relativistic equations of motion

$$\frac{d}{dt}(\gamma m\boldsymbol{v}) = \frac{d\boldsymbol{p}}{dt} = 0 \tag{9.31}$$

[7] In principle the actions in different inertial frames could also differ by a factor.

are obtained as the Euler-Lagrange equations for L_0. In this form it is not immediately obvious that the equation (9.31) is covariant. To make the covariance explicit, observe that by the energy-momentum relation (9.28) with (9.31) $\dot{p}^0$ is also determined. Indeed, differentiating (9.28) with respect to t

$$c\frac{dp^0}{dt} = c\frac{\boldsymbol{p}}{p^0}\cdot\frac{d\boldsymbol{p}}{dt} = \boldsymbol{v}\cdot\frac{d\boldsymbol{p}}{dt} = 0, \tag{9.32}$$

so that we arrive at the (manifestly) covariant equation $dp^\mu/d\tau = 0$.

Particle in a Potential

In Galilean physics one often considers a particle in a scalar potential $V(\boldsymbol{x}, t)$. The corresponding problem is not truly fruitful in relativistic mechanics. The issue is that, starting from a scalar potential V, the potential in another reference frame automatically becomes velocity dependent $\boldsymbol{v}$. It is therefore better to start directly from a vector potential that couples to the velocity. A further point is that there is no physical force that can be described by a scalar potential, since the gravitational force is not formulated covariantly and hence only the electromagnetic force fits into the relativistic framework. Electrodynamics comes equipped directly with a scalar potential $\varphi(\boldsymbol{x}, t)$ and a vector potential $\boldsymbol{A}(\boldsymbol{x}, t)$. The action of electromagnetic fields on a particle of charge e is described in classical mechanics by the Lagrangian, see (4.39),

$$L = L_0 + \frac{e}{c}\boldsymbol{v}\cdot\boldsymbol{A} - e\varphi = L_0 - \frac{e}{c}\eta_{\mu\nu}\,u^\mu A^\nu\sqrt{1-\boldsymbol{v}^2/c^2}\,. \tag{9.33}$$

with $A = (\varphi, \boldsymbol{A})$. Replacing the free Lagrangian $L_0 = m\boldsymbol{v}^2/2$ by the relativistic generalization (9.30), one sees that the action

$$S = \int_{(1)}^{(2)} dt\, L = -\int_{(1)}^{(2)} d\tau \left(mc^2 + \frac{e}{c}\,(u, A)\right) \tag{9.34}$$

is already covariant provided A^μ transforms like a four-vector.[8]

The equations of motion for a charged particle in an electromagnetic field are the Euler-Lagrange equations

$$\frac{d}{dt}\frac{m\boldsymbol{v}}{\sqrt{1-\boldsymbol{v}^2/c^2}} = e\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right), \tag{9.35}$$

for (9.34), since in the nonrelativistic calculation of Chapter 4.4 one merely has to replace

$$\frac{\partial}{\partial\boldsymbol{v}}\frac{m\boldsymbol{v}^2}{2} = m\boldsymbol{v} \quad\mapsto\quad -mc^2\frac{\partial}{\partial\boldsymbol{v}}\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}} = \frac{m\boldsymbol{v}}{\sqrt{1-\boldsymbol{v}^2/c^2}} \tag{9.36}$$

[8] This fact will be proved in electrodynamics.

i.e. $m\boldsymbol{v} \mapsto \boldsymbol{p}$. By (9.35) dp^0/dt is also determined. Differentiating (9.28) with respect to t yields

$$c\frac{dp^0}{dt} = c\frac{\boldsymbol{p}}{p^0} \cdot \frac{d\boldsymbol{p}}{dt} = \boldsymbol{v} \cdot \frac{d\boldsymbol{p}}{dt} = e\boldsymbol{v} \cdot \boldsymbol{E}\,. \tag{9.37}$$

This is the energy theorem, for on the right-hand side one has the power of the Lorentz force and on the left the change of kinetic energy.

Notation

$\boldsymbol{x}_i$	Cartesian coordinates of the i-th particle in an inertial frame
$\boldsymbol{y}_i$	body-fixed coordinates of the i-th particle
$\dot{\square}$	time derivative
q^α	(generalized) position coordinates
p_α	(generalized) momenta
x_i	coordinates in phase space
$\{\cdot,\cdot\}$	Poisson bracket
$\boldsymbol{X}$	center of mass
$\boldsymbol{P}$	total momentum
$\boldsymbol{F}$	force
$\boldsymbol{L}$	angular momentum with magnitude $l = \lvert\boldsymbol{L}\rvert$
$\boldsymbol{M}$	torque
$\boldsymbol{\omega}$	angular velocity in body-fixed coordinates
Θ	moment of inertia
$\boldsymbol{S}$	angular momentum in body-fixed coordinates
d^3y	infinitesimal volume element
E	(total) energy
T	kinetic energy
V	potential, potential energy
$L(q,\dot{q},t)$	Lagrangian
$H(q,p,t)$	Hamiltonian
S	action
$S_0 = S + Et$	reduced action