Analytical Mechanics

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Please report any typographical or other errors to F. Hassler mitteilen.

 The translation has been done with the help of GPT-4o mini. In case of ambiguities, it might be helpful to consult the German version.

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Chapter 1

Newtonian Mechanics

In this chapter, we will introduce the fundamentals of Newtonian mechanics. You have already encountered most of this in Experimental Physics 1. We will briefly summarize the basics in Section 1.1. We will then explain the Galilean principle of relativity, conservation laws, and the concept of fictitious forces in detail.

1.1 Fundamentals

Space and Time: After establishing the units of length (scales) and time (clocks), as well as a reference system, we characterize *events* by $(t, \boldsymbol{x}) \in \mathbb{R}^{1+3}$, where t is the time coordinate, and $\boldsymbol{x} = (x_1, x_2, x_3)$ are Cartesian coordinates of a three-dimensional Euclidean space. The quantities have an absolute, reference-system-independent meaning:

- $|t_1 - t_2|$: Time interval between two arbitrary events $(t_1, \boldsymbol{x}_1)$, $(t_2, \boldsymbol{x}_2)$
$$(\Longrightarrow \text{Simultaneity is absolute}); \tag{1.1}$$
- if $t_1 = t_2$:
$$|\boldsymbol{x}_1 - \boldsymbol{x}_2|: \text{Spatial distance between two simultaneous events.} \tag{1.2}$$

The general coordinate transformations that leave these quantities invariant are

$$\begin{aligned} t' &= \lambda t + a\,, && (\lambda = \pm 1; a \in \mathbb{R})\,, \\ \boldsymbol{x}' &= R(t)\boldsymbol{x} + \boldsymbol{b}(t)\,, && (R(t) \in \mathrm{O}(3); \boldsymbol{b}(t) \in \mathbb{R}^3)\,. \end{aligned} \tag{1.3}$$

Here, $R(t) \in \mathrm{O}(3)$ is a time-dependent orthogonal 3×3 matrix. These transformations allow us in particular: (i) to reverse the direction of time ($\lambda = -1$), (ii) to shift the time origin ($a \neq 0$), and (iii) to rotate and shift the spatial reference system (in a time-dependent manner) $(R(t), \boldsymbol{b}(t))$.

It can be easily verified that the transformations (1.3) form a group. This is essentially a consequence of the fact that the transformations are characterized by the invariants (1.1) and (1.2).

Inertial Systems: From the standpoint of mechanics, however, not all of these coordinate systems are equivalent. The special class of *inertial systems* is distinguished by the *law of inertia*. In an inertial system, a *free particle*[1] with coordinates $\boldsymbol{x}$ satisfies the equations of motion

$$\ddot{\boldsymbol{x}} = 0\,, \qquad \Big(\dot{f}(t) \equiv \frac{d}{dt}f(t)\Big)\,; \tag{1.4}$$

i.e., it moves uniformly in a straight line along the path

$$\boldsymbol{x}(t) = \boldsymbol{x}(0) + \dot{\boldsymbol{x}}(0)\,t\,. \tag{1.5}$$

In an inertial system, therefore, the path of a free particle $t \mapsto (t, \boldsymbol{x}(t))$ is a straight line in $\mathbb{R}^4$. The subgroup of coordinate transformations (1.3) that maps inertial systems to inertial systems are then the straight-line preserving mappings

$$\begin{aligned} t' &= \lambda t + a\,, && (\lambda = \pm 1; a \in \mathbb{R})\,, \\ \boldsymbol{x}' &= R\boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{b}\,, && (R \in \mathrm{O}(3); \boldsymbol{v},\, \boldsymbol{b} \in \mathbb{R}^3)\,. \end{aligned} \tag{1.6}$$

These mappings again form a group, known as the *Galilean group* or the group of *Galilean transformations*.

We postulate the existence of an inertial system (and thus infinitely many uniformly moving inertial systems). In practice, different reference systems are used as inertial systems. For example, in laboratory experiments on small time and length scales, one can approximately assume the Earth to be an inertial system. On the other hand, when considering planetary orbits in our solar system, its center of mass with axes oriented towards fixed stars (with high precision) is an inertial system. On even larger time and length scales, one would then have to transition to the center of our Milky Way, for example.

Unless stated otherwise, we will denote $(t, \boldsymbol{x})$ as the coordinates of an event in an (arbitrary) inertial system.

Center of Mass Theorem (Concept of Mass): Every point particle has an invariant mass $m > 0$, which (after establishing a unit of mass) is characterized by the *center of mass theorem*. For a closed (isolated) system of N particles with coordinates $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$, the center of mass theorem states

$$\sum_{i=1}^{N} m_i \ddot{\boldsymbol{x}}_i = \frac{d}{dt}\sum_{i=1}^{N} \boldsymbol{p}_i = 0\,. \tag{1.7}$$

[1] A free particle is a particle that is not subjected to any (external) force.

The definition of mass m_i is independent of the choice of inertial system, since $\ddot{\boldsymbol{x}}_i' = R\ddot{\boldsymbol{x}}_i$.

The quantity $\boldsymbol{p}_i = m_i\dot{\boldsymbol{x}}_i$ is called the *momentum* of the i-th particle. The above equation is then the conservation law of the total momentum $\boldsymbol{P} = \sum_{i=1}^N \boldsymbol{p}_i$. Defining the center of mass of the N-particle system by

$$\boldsymbol{X} = \frac{1}{M}\sum_{i=1}^N m_i\boldsymbol{x}_i\,, \qquad M = \sum_{i=1}^N m_i\,, \tag{1.8}$$

the center of mass theorem can also be written as $\ddot{\boldsymbol{X}} = 0$. The center of mass of a closed system thus moves along an inertial path

$$\boldsymbol{X}(t) = \boldsymbol{X}(0) + \frac{\boldsymbol{P}}{M}t\,. \tag{1.9}$$

In scattering processes, particles can be created or destroyed. However, the total momentum $\boldsymbol{P}$ must be conserved in every inertial system. Now consider two inertial systems that are related by $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{v}t$: The momentum of the i-th particle in the primed inertial system is then $\boldsymbol{p}_i' = \boldsymbol{p}_i + m_i\boldsymbol{v}$. Since both $\boldsymbol{P}$ and $\boldsymbol{P}'$ are constant, it follows from the relation $\boldsymbol{P}' = \boldsymbol{P} + M\boldsymbol{v}$ that the total mass M must also be constant. In particular, mass is additive if multiple particles combine into one.

1.2 The Newtonian Equations of Motion

We consider a system of N particles, whose position coordinates are described by $\boldsymbol{x}_i(t)$. The Newtonian principle of determinism requires that the paths $\boldsymbol{x}_i(t)$ are uniquely determined for all t, provided the positions $\boldsymbol{x}_i(t_0)$ and the velocities $\dot{\boldsymbol{x}}_i(t_0)$ are given at a time t_0. In particular, the accelerations $\ddot{\boldsymbol{x}}_i(t_0)$ are then determined, i.e., there exist functions $\boldsymbol{F}_i$, such that

$$m_i\ddot{\boldsymbol{x}}_i(t) = \underbrace{\boldsymbol{F}_i\big(\boldsymbol{x}_1(t),\ldots,\boldsymbol{x}_N(t),\dot{\boldsymbol{x}}_1(t),\ldots,\dot{\boldsymbol{x}}_N(t),t\big)}_{\text{Force law of the system}}\,. \tag{1.10}$$

These are the so-called *Newtonian equations of motion*. Conversely, it follows from the existence and uniqueness theorem for ordinary differential equations that (under weak conditions on $\boldsymbol{F}_i$) the paths $\boldsymbol{x}_i(t)$ are uniquely determined by (1.10) and the above initial data at $t = t_0$ for at least a small time interval around t_0. We call a system of N particles that satisfies a force law of the form (1.10) a *mechanical system*.

In physics, we usually investigate force laws of the form $\boldsymbol{F}_i = \boldsymbol{F}_i^{(\mathrm{in})} + \boldsymbol{F}_i^{(\mathrm{ex})}$ with an *external force* $\boldsymbol{F}_i^{(\mathrm{ex})} = \boldsymbol{F}^{(\mathrm{ex})}(\boldsymbol{x}_i,\dot{\boldsymbol{x}}_i,t)$ independent of $\boldsymbol{x}_j,\dot{\boldsymbol{x}}_j$ with $j \neq i$ and an *internal force* $\boldsymbol{F}_i^{(\mathrm{in})}$. The internal force $\boldsymbol{F}_i^{(\mathrm{in})}$ on the i-th particle depends only on the positions of the other particles,

$$\boldsymbol{F}_i^{(\mathrm{in})} = \boldsymbol{F}_i^{(\mathrm{in})}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)\,. \tag{1.11}$$

Since it does not depend on the velocities, it can be measured statically. In most cases, $\boldsymbol{F}_i^{(\mathrm{in})}$ is simply a superposition of two-body forces

$$\boldsymbol{F}_i^{(\mathrm{in})} = \sum_{\substack{k=1 \\ k\neq i}}^{3} \boldsymbol{F}_{ik}\,, \tag{1.12}$$

where $\boldsymbol{F}_{ik} = \boldsymbol{F}_{ik}(\boldsymbol{x}_i, \boldsymbol{x}_k)$ is the force exerted by particle k on particle i. For a (closed) two-particle system, it then follows from (1.7) that

$$\boldsymbol{F}_{ik} + \boldsymbol{F}_{ki} = 0 \tag{1.13}$$

(actio = reactio).

Examples of closed mechanical systems (with $\boldsymbol{F}^{(\mathrm{ex})} = 0$) include the solar system, whose force law is given by

$$m_i \ddot{\boldsymbol{x}}_i = -G \sum_{\substack{k=1 \\ k\neq i}}^{N} m_i m_k \frac{\boldsymbol{x}_i - \boldsymbol{x}_k}{|\boldsymbol{x}_i - \boldsymbol{x}_k|^3} \tag{1.14}$$

or a system of charged particles (with charge e_i), for which the force law

$$m_i \ddot{\boldsymbol{x}}_i = \sum_{\substack{k=1 \\ k\neq i}}^{N} e_i e_k \frac{\boldsymbol{x}_i - \boldsymbol{x}_k}{|\boldsymbol{x}_i - \boldsymbol{x}_k|^3} \tag{1.15}$$

holds.

Examples of non-closed mechanical systems include a charged particle in an external electromagnetic field (given by $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$)

$$m\ddot{\boldsymbol{x}} = e\boldsymbol{E}(\boldsymbol{x}, t) + \frac{e}{c}\dot{\boldsymbol{x}} \times \boldsymbol{B}(\boldsymbol{x}, t)\,, \tag{1.16}$$

or a one-dimensional oscillator with friction force and driving force $k(t)$

$$m\ddot{x} = -fx - r\dot{x} + k(t)\,. \tag{1.17}$$

In both cases, the particle is only subjected to an external force.

The description of these systems as mechanical systems requires approximations. For example, in (1.16), $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$ are given external electromagnetic fields, and the influence of the charged particle on them has been ignored. In (1.17), $-r\dot{x}$ is a summary description of the friction, without considering the dynamics of the damping medium.

Since friction is an effective description of a complicated interaction of a particle with a medium, we will hardly consider it in this lecture. You will find that friction overall plays a subordinate role in the standard lectures of the study. Thus, you will (probably) only encounter irreversibility and friction again in statistical mechanics.

1.3 The Galilean Principle of Relativity

The Galilean (or classical) principle of relativity states that *the laws of nature are the same in all inertial systems*, i.e., they are invariant under Galilean transformations.[2] The principle thus makes all inertial systems equal. For mechanics, the principle of relativity requires that the equations of motion of a closed system are invariant under Galilean transformations (1.6). We now want to show that then $\boldsymbol{F}_{ik}$ must have the form

$$\boldsymbol{F}_{ik} = f_{ik}(|\boldsymbol{x}_i - \boldsymbol{x}_k|)\frac{\boldsymbol{x}_i - \boldsymbol{x}_k}{|\boldsymbol{x}_i - \boldsymbol{x}_k|} \tag{1.18}$$

where $f_{ik}(r)$ is any scalar function of one variable.[3] This means that the force acts along the line connecting the two particles and depends only on the distance between the two particles.

To prove this, consider a mechanical two-particle system whose force law in an inertial system is of the form

$$m_1 \frac{d^2\boldsymbol{x}_1}{dt^2} = \boldsymbol{F}_{12}(\boldsymbol{x}_1, \boldsymbol{x}_2) \tag{1.19}$$

Let $\boldsymbol{x}'$ be the coordinates in another inertial system, which arises from the unprimed inertial system through a (time-independent) Galilean transformation with $t = t'$:

$$\boldsymbol{x}_i'(t) = R\boldsymbol{x}_i(t) + \boldsymbol{b}\,. \tag{1.20}$$

The principle of relativity then requires that the same equation of motion holds in the primed inertial system; i.e.,

$$m_1 \ddot{\boldsymbol{x}}_1' = \boldsymbol{F}_{12}(\boldsymbol{x}_1', \boldsymbol{x}_2') \tag{1.21}$$

should be equivalent to (1.19) with $\boldsymbol{F}_{12}$ being the same function. For the left side, we have $m_1\ddot{\boldsymbol{x}}_1' = m_1 R\ddot{\boldsymbol{x}}_1$ from (1.20).

Invariance under a pure translation ($R = 1$, $\boldsymbol{b} \neq \boldsymbol{0}$) thus leads to the condition $\boldsymbol{F}_{12}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{F}_{12}(\boldsymbol{x}_1 + \boldsymbol{b}, \boldsymbol{x}_2 + \boldsymbol{b})$. This immediately implies that $\boldsymbol{F}_{12}$ can only depend on $\boldsymbol{x}_1 - \boldsymbol{x}_2$.[4] Thus, we obtain that $\boldsymbol{F}_{12}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{F}_{12}(\boldsymbol{x}_1 - \boldsymbol{x}_2)$. Invariance under rotations $R \in \mathrm{SO}(3)$ then leads to the condition that

$$R\,\boldsymbol{F}_{12}(\boldsymbol{x}) = \boldsymbol{F}_{12}(R\,\boldsymbol{x}) \tag{1.22}$$

holds. If one specifically chooses a rotation R around the axis in the direction of the connecting line $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ ($R\,\boldsymbol{x} = \boldsymbol{x}$), it follows with $R\,\boldsymbol{F}_{12}(\boldsymbol{x}) = \boldsymbol{F}_{12}(\boldsymbol{x})$ the assertion

$$\boldsymbol{F}_{12}(\boldsymbol{x}) = f_{12}(\boldsymbol{x})\frac{\boldsymbol{x}}{|\boldsymbol{x}|} \tag{1.23}$$

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[2] This fundamental principle will later be replaced by Einstein's principle of relativity in special relativity.

[3] The factor $1/|\boldsymbol{x}_i - \boldsymbol{x}_k|$ could of course also have been absorbed into the definition of f_{ik}. However, we want f_{ik} to have the units of a force.

[4] This can be seen, for example, by setting $\boldsymbol{b} = -\boldsymbol{x}_2$.

about the direction of the force. Furthermore, (1.22) requires that $f_{12}(\boldsymbol{x}) = f_{12}(R\boldsymbol{x})$ for any rotation R, thus $f_{12}(\boldsymbol{x}) = f_{12}(|\boldsymbol{x}|)$. This proves the assertion (1.18).

The forces in (1.18) always possess a *potential*

$$\boldsymbol{F}_{ik} = -\frac{\partial}{\partial \boldsymbol{x}_i} V_{ik}(|\boldsymbol{x}_i - \boldsymbol{x}_k|)\,, \quad \text{with} \quad \frac{d}{dr} V_{ik}(r) = -f_{ik}(r)\,. \tag{1.24}$$

Accordingly, for the superposition of two-body forces, we have

$$\boldsymbol{F}_i^{(\text{in})} = -\frac{\partial}{\partial \boldsymbol{x}_i} V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)\,, \qquad V = \sum_{\substack{i,k=1 \\ i<k}}^{3} V_{ik}(|\boldsymbol{x}_i - \boldsymbol{x}_k|)\,. \tag{1.25}$$

So far, we have considered the ‘passive interpretation’ of the principle of relativity: if $\boldsymbol{x}(t)$ and $\boldsymbol{x}'(t')$ describe the *same* path in different inertial systems, then they must satisfy the equations of motion in both inertial systems (which must have the same form). However, one can also use an ‘active interpretation’: if $\boldsymbol{x}(t)$ is a solution of the equations of motion in one inertial system, then $\boldsymbol{x}(t)$ is also a solution with respect to any other inertial system.[5] As an example, consider the case of time reversal ($\lambda = -1$): if $\boldsymbol{x}(t)$ is a solution, then $\boldsymbol{x}(-t)$ is also a solution; the time-reversed motion is therefore always also a solution of the equations of motion.

Invariant versus Form Invariant: *Form invariance* of an equation requires that the equation has the same form in different reference systems. In this case, the potential $V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ must be replaced by $V'(\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_N)$ defined by

$$V'(\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_N) = V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N), \tag{1.26}$$

since the value of the potential energy of a particle configuration does not depend on the choice of coordinates. Fields that transform like (1.26) are called *scalar fields*. Starting from the potential, one finds directly the relation

$$\boldsymbol{F}'_i(\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_N) = -\frac{\partial V'}{\partial \boldsymbol{x}'_i} = -R\frac{\partial V}{\partial \boldsymbol{x}_i} = R\boldsymbol{F}_i(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \tag{1.27}$$

of the force laws in the two reference systems. Fields that transform like (1.27) are called *vector fields*. If one restricts the possible transformations to the Galilean transformations (1.20), one obtains that $\ddot{\boldsymbol{x}}' = R\ddot{\boldsymbol{x}}$, as already shown. Thus, the Newtonian equations (1.10) are form invariant under Galilean transformations, as long as $\boldsymbol{F}$ is a vector field (both sides are simply multiplied by R in the primed reference system). Form invariance simply describes the mathematical fact that an equation has a certain (nice) form in certain reference systems.

[5] The path $\boldsymbol{x}(t)$ then generally describes a different path, as this function now denotes the components with respect to a different coordinate system!

The *invariance* of an equation under Galilean transformations, on the other hand, requires that the equation is identical in two reference systems. In the case of the Newtonian equation, this means that $\boldsymbol{F}'(\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_N) = \boldsymbol{F}(\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_N)$ or also $V'(\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_N) = V(\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_N)$, i.e., V' and V are the same function. Due to (1.26), it follows that

$$V(R\boldsymbol{x}_1 + \boldsymbol{b}, \ldots, R\boldsymbol{x}_N + \boldsymbol{b}) = V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \tag{1.28}$$

for a time-independent Galilean transformation $\boldsymbol{x}'_i = R\boldsymbol{x}_i + \boldsymbol{b}$. The invariance of a closed system in classical mechanics under Galilean transformations is an experimental fact with physical consequences. The force law thus has the form (1.18) and we obtain, as discussed in detail in the next chapter, the 10 classical conservation laws.

1.4 Conservation Laws

We now want to make some general statements about mechanical systems. Mechanical systems are characterized by satisfying a force law of the form (1.10).

(a) Impulse Law: It directly follows from (1.10), that

$$\frac{d}{dt} \underbrace{\sum_{i=1}^{N} \boldsymbol{p}_i}_{\boldsymbol{P}} = \underbrace{\sum_{i=1}^{N} \boldsymbol{F}_i}_{\boldsymbol{F}:\ \text{resultant force}} . \tag{1.29}$$

(b) Angular Momentum Law: Since the time derivative of $\boldsymbol{x}_i$ is proportional to $\boldsymbol{p}_i$, it follows that

$$\frac{d}{dt} \underbrace{\sum_{i=1}^{N} \boldsymbol{x}_i \times \boldsymbol{p}_i}_{\boldsymbol{L}} = \underbrace{\sum_{i=1}^{N} \boldsymbol{x}_i \times \boldsymbol{F}_i}_{\boldsymbol{M}:\ \text{resultant torque with respect to } \boldsymbol{x}=0} . \tag{1.30}$$

The angular momentum $\boldsymbol{L}$ can be decomposed, $\boldsymbol{L} = \boldsymbol{X} \times \boldsymbol{P} + \boldsymbol{L}_\mathrm{S}$, into the part of the center of mass motion $\boldsymbol{X} \times \boldsymbol{P}$ and a part of the internal motion relative to the center of mass

$$\boldsymbol{L}_\mathrm{S} = \sum_{i=1}^{N} (\boldsymbol{x}_i - \boldsymbol{X}) \times (\boldsymbol{p}_i - m_i \dot{\boldsymbol{X}}) . \tag{1.31}$$

In the decomposition, we have used that

$$\sum_{i=1}^{N} m_i \boldsymbol{x}_i \times \dot{\boldsymbol{X}} = \boldsymbol{X} \times \boldsymbol{P} = \boldsymbol{X} \times \sum_{i=1}^{N} \boldsymbol{p}_i = \boldsymbol{X} \times \sum_{i=1}^{N} m_i \dot{\boldsymbol{X}} , \tag{1.32}$$

since $\sum_{i=1}^{N} m_i \boldsymbol{x}_i = M\boldsymbol{X}$ and $M\dot{\boldsymbol{X}} = \mathbf{P}$.[6]

[6]Here, M denotes the total mass, $M = \sum_{i=1}^{N} m_i$.

(c) Energy Law: Multiplying (1.10) by $\dot{\boldsymbol{x}}_i$ and summing over i leads to

$$\frac{d}{dt}\underbrace{\sum_{i=1}^{N}\frac{1}{2}m_i\dot{\boldsymbol{x}}_i^2}_{T} = \underbrace{\sum_{i=1}^{N}\boldsymbol{F}_i\cdot\dot{\boldsymbol{x}}_i}_{\text{power of the forces.}}, \tag{1.33}$$

where we have introduced the *kinetic energy* T. As with angular momentum, the kinetic energy can also be decomposed

$$T = \frac{1}{2}M\dot{\boldsymbol{X}}^2 + T_{\mathrm{S}}\,, \qquad \text{where} \qquad T_{\mathrm{S}} = \sum_{i=1}^{N}\frac{1}{2}m_i\big(\dot{\boldsymbol{x}}_i - \dot{\boldsymbol{X}}\big)^2 \tag{1.34}$$

describes the kinetic energy of the internal motion with respect to the center of mass. In the decomposition, we have used that

$$\frac{1}{2}\sum_{i=1}^{N}m_i\dot{\boldsymbol{x}}_i\dot{\boldsymbol{X}} = \frac{1}{2}M\dot{\boldsymbol{X}}^2. \tag{1.35}$$

Closed Systems: Closed systems, which only interact with each other through internal forces, are described by a potential $V(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)$ with $\boldsymbol{F}_i^{(\mathrm{in})} = -\partial_{\boldsymbol{x}_i}V$, see Eq. (1.25). The invariance of the equations of motion under Galilean transformations (Galilean principle of relativity) requires that

$$V(R\boldsymbol{x}_1+\boldsymbol{b},\ldots,R\boldsymbol{x}_N+\boldsymbol{b}) = V(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N), \qquad (R\in \mathrm{O}(3);\boldsymbol{b}\in\mathbb{R}^3). \tag{1.36}$$

Thus, for closed, Galilean invariant systems, we have:

(i) the resultant force $\boldsymbol{F}=0$ vanishes ($\boldsymbol{P}$ is conserved),

(ii) the resultant torque $\boldsymbol{M}=0$ vanishes ($\boldsymbol{L}$ is conserved),

(iii) the power reduces the potential,

$$\sum_{i=1}^{N}\boldsymbol{F}_i\cdot\dot{\boldsymbol{x}}_i = -\frac{dV}{dt} \tag{1.37}$$

(the energy $T+V$ is conserved).

To prove this, consider an arbitrary unit vector $\boldsymbol{e}$:

(i) It holds

$$\boldsymbol{e}\cdot\boldsymbol{F} = -\frac{d}{d\lambda}V(\boldsymbol{x}_1+\lambda\boldsymbol{e},\ldots,\boldsymbol{x}_N+\lambda\boldsymbol{e})\Big|_{\lambda=0} = 0\,, \tag{1.38}$$

since the potential is invariant under translations. Since this holds for arbitrary unit vectors $\boldsymbol{e}$, it follows that $\boldsymbol{F}=0$.

(ii) Consider a rotation $R(\varphi)$ of angle φ around $\boldsymbol{e}$. Then we have

$$\frac{d}{d\varphi} R(\varphi)\boldsymbol{x}\Big|_{\varphi=0} = \frac{d}{d\varphi}(\varphi \boldsymbol{e}) \times \boldsymbol{x}\Big|_{\varphi=0} = \boldsymbol{e} \times \boldsymbol{x}\,, \tag{1.39}$$

and therefore

$$\begin{aligned} \boldsymbol{e} \cdot \boldsymbol{M} &= \sum_{i=1}^{N} \boldsymbol{e} \cdot (\boldsymbol{x}_i \times \boldsymbol{F}_i) = \sum_{i=1}^{N} (\boldsymbol{e} \times \boldsymbol{x}_i) \cdot \boldsymbol{F}_i \\ &= -\frac{d}{d\varphi} V(R(\varphi)\boldsymbol{x}_1, \ldots, R(\varphi)\boldsymbol{x}_N)\Big|_{\varphi=0} = 0\,. \end{aligned} \tag{1.40}$$

Since $\boldsymbol{e}$ is arbitrary, it follows that $\boldsymbol{M} = 0$.

(iii) follows directly from the chain rule.

A closed system that satisfies the Galilean principle of relativity thus possesses the ten classical conservation laws (integrals of motion):

$$\boldsymbol{P} \quad \text{and} \quad M\boldsymbol{X} - \boldsymbol{P}t \qquad \text{(6 center of mass integrals)}, \tag{1.41}$$

$$\boldsymbol{L} \quad (\text{or } \boldsymbol{L}_{\mathrm{S}}) \qquad \text{(3 angular momentum integrals)}, \tag{1.42}$$

$$T + V \quad (\text{or } T_{\mathrm{S}} + V) \qquad \text{(energy integral)}. \tag{1.43}$$

These 10 conservation laws are related to the 10 continuous parameters $\boldsymbol{b}$, $\boldsymbol{v}$, R and a $(3+3+3+1)$ of the Galilean group (see Chapter 4.5.3).

1.5 Accelerated Reference Systems

The Newtonian equations of motion have the simple form

$$m\ddot{\boldsymbol{x}} = \boldsymbol{F}, \tag{1.44}$$

which is valid in all inertial systems. Sometimes, however, it is convenient to work in coordinate systems that are not inertial systems. In such accelerated reference systems, so-called 'fictitious forces' arise.

Let $\boldsymbol{y}$ denote the coordinates of an arbitrary reference system, which is related to an inertial system $\boldsymbol{x}$ by

$$\boldsymbol{x} = R(t)\boldsymbol{y} + \boldsymbol{b}(t)\,, \qquad (R(t) \in \mathrm{SO}(3); \boldsymbol{b}(t) \in \mathbb{R}^3) \tag{1.45}$$

Then it holds

$$\dot{\boldsymbol{x}} = \dot{R}\,\boldsymbol{y} + R\,\dot{\boldsymbol{y}} + \dot{\boldsymbol{b}}\,, \qquad \ddot{\boldsymbol{x}} = \ddot{R}\,\boldsymbol{y} + 2\dot{R}\,\dot{\boldsymbol{y}} + R\,\ddot{\boldsymbol{y}} + \ddot{\boldsymbol{b}}. \tag{1.46}$$

Multiplying the equation for $\ddot{\boldsymbol{x}}$ by R^t (where R^t is the transpose matrix of R) leads to

$$m\ddot{\boldsymbol{y}} = R^t\,\boldsymbol{F} - 2m\,R^t\dot{R}\,\dot{\boldsymbol{y}} - mR^t\,\ddot{R}\,\boldsymbol{y} - m\,R^t\,\ddot{\boldsymbol{b}}\,, \tag{1.47}$$

since $R^tR = 1$.

The first term on the right side is the force vector

$$\boldsymbol{K} \equiv R^t \boldsymbol{F} \tag{1.48}$$

in y-coordinates. The additional terms are the fictitious forces that arise. The last term contains the acceleration of the point $\boldsymbol{y} = 0$ in y-coordinates

$$\boldsymbol{a} \equiv R^t \ddot{\boldsymbol{b}}. \tag{1.49}$$

The remaining terms can be simplified by introducing the mapping $\Omega = R^t\,\dot{R}$. One can easily see that the mapping Ω is antisymmetric; for taking the time derivative of the relation $R^t\,R = 1$, we obtain

$$R^t\,\dot{R} + \dot{R}^t\,R = \Omega + \Omega^t = 0\,. \tag{1.50}$$

Thus, Ω generally has the form

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} : \qquad \Omega\,\boldsymbol{y} = \boldsymbol{\omega} \times \boldsymbol{y}, \text{ with } \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3). \tag{1.51}$$

A point at rest in the $\boldsymbol{y}$-system has in the $\boldsymbol{x}$-system the velocity $\dot{\boldsymbol{x}} = \dot{R}\,\boldsymbol{y} + \dot{\boldsymbol{b}}$. The components of this velocity in the $\boldsymbol{y}$-system are therefore $R^t\dot{\boldsymbol{x}} = \boldsymbol{\omega} \times \boldsymbol{y} + R^t\,\dot{\boldsymbol{b}}$. In particular, $\boldsymbol{\omega}$ describes the $\boldsymbol{y}$-components of the (instantaneous) angular velocity of the $\boldsymbol{y}$-system relative to the $\boldsymbol{x}$-system; the vector $\boldsymbol{\omega}$ points in the direction of the instantaneous axis of rotation.

Furthermore, we have

$$R^t\,\ddot{R} = \dot{\Omega} - \dot{R}^t\,\dot{R} = \dot{\Omega} - \dot{R}^t\,R\,R^t\,\dot{R} = \dot{\Omega} + \Omega^2\,, \tag{1.52}$$

$$R^t\,\ddot{R}\,\boldsymbol{y} = \dot{\boldsymbol{\omega}} \times \boldsymbol{y} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{y})\,. \tag{1.53}$$

Overall, equation (1.47) thus reads

$$m\,\ddot{\boldsymbol{y}} = \boldsymbol{K} - 2m\,(\boldsymbol{\omega} \times \dot{\boldsymbol{y}}) - m\,(\dot{\boldsymbol{\omega}} \times \boldsymbol{y}) - m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{y}) - m\boldsymbol{a}\,. \tag{1.54}$$

The terms appearing to the right of $\boldsymbol{K}$ are called fictitious forces, in particular,

$$\begin{array}{rl} -2\,m\,\boldsymbol{\omega} \times \dot{\boldsymbol{y}} & \quad \textit{the Coriolis force} \\ \text{and } \; -\,m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{y}) & \quad \textit{the centrifugal force.} \end{array} \tag{1.55}$$

These are the only fictitious forces in uniform rotation of the $\boldsymbol{y}$-system ($\boldsymbol{\omega}$ = constant, $\boldsymbol{a} = 0$). The additional term $-m\boldsymbol{a}$, which appears for $\boldsymbol{a} \neq 0$, is called the *guiding force.*

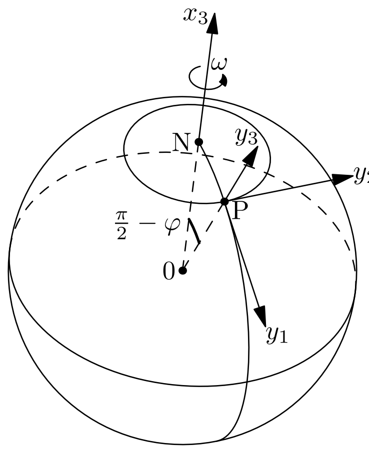


Figure 1.1: The point P is located at geographical latitude φ. The Earth's rotation occurs around the axis from the center of the Earth to the North Pole ($\overrightarrow{0N}$). The $\boldsymbol{y}$-system is fixed at point P and rotates with angular velocity 2π/day around the Earth's axis.

It should be noted that the above fictitious forces are proportional to the *inertial mass* m_T of the particle, whereas the gravitational force $m_S\,\boldsymbol{g}$ is proportional to the *gravitational mass* m_S.[7] Experimentally, it is found that the two are equal:

$$m_S = m_T\,, \qquad \text{Experiment: } m_S/m_T = 1 \pm 10^{-12}. \tag{1.56}$$

Due to the equality of gravitational and inertial mass, the equations of motion of a particle in a homogeneous gravitational field $\boldsymbol{g}$ relative to a freely falling, non-rotating ($\boldsymbol{\omega} = 0$) reference system are given by

$$m_T\ddot{\boldsymbol{y}} = m_S\boldsymbol{g} - m_T\boldsymbol{g} = 0\,, \tag{1.57}$$

i.e., the weight force is completely transformed away. In an inhomogeneous gravitational field, this only holds locally. The equivalence of inertial and gravitational mass is an important foundation of general relativity.

1.5.1 Example: Free Fall to the Earth's Surface

As an application, we consider free fall to the Earth's surface. Let P be a point on the Earth's surface, whose geographical latitude is described by φ. Let $(t, \boldsymbol{x})$ be a 'space-fixed' inertial system with origin at the center of the Earth, whose x_3-axis points towards the North Pole N. (We neglect here the Earth's motion around the Sun, axial precessions, etc.) The rotating coordinate system is denoted by $\boldsymbol{y}$ with

[7] The inertial mass is the constant that appears in the Newtonian equations of motion $\boldsymbol{F} = m_T\ddot{\boldsymbol{x}}$. The gravitational mass describes the Newtonian gravitational force $\boldsymbol{F} = -\boldsymbol{\nabla}(Gm_{S1}m_{S2}/x_{12})$.

$\boldsymbol{x} = R(t)\boldsymbol{y} + \boldsymbol{b}(t)$. The vector $\boldsymbol{b}$ is equal to $\boldsymbol{b} = R\boldsymbol{p}$, where $\boldsymbol{p} = \overrightarrow{0P}$ is the fixed vector from the center of the Earth to P (at time $t = 0$), see Fig. 1.1.

Using (1.54), we find the equations of motion

$$\ddot{\boldsymbol{y}} = \boldsymbol{g}' - 2\,\boldsymbol{\omega} \times \dot{\boldsymbol{y}} \tag{1.58}$$

of a mass point near P, where

$$\boldsymbol{\omega} = (-\omega\cos\varphi, 0, \omega\sin\varphi)\,, \qquad \omega = 2\pi/\text{day}\,,$$

and

$$\begin{aligned} \boldsymbol{g}' &= \boldsymbol{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{y}) - \overbrace{\boldsymbol{a}}^{=\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{p})} = \boldsymbol{g} - \boldsymbol{\omega}\, \times \big(\boldsymbol{\omega} \times (\boldsymbol{p} + \boldsymbol{y})\big) \\ &\equiv (0, 0, -g)\,, \qquad\qquad (g \approx 9.81\,\mathrm{m/s^2}). \end{aligned} \tag{1.59}$$

In the last line, we have defined the y_3-direction as the vertical direction along which the resultant of the gravitational acceleration, the centrifugal force, and the guiding force acts at P.[8]

In components, equation (1.58) reads

$$\begin{aligned} \ddot{y}_1 &= 2\omega\sin(\varphi)\,\dot{y}_2\,, \\ \ddot{y}_2 &= -2\omega\sin(\varphi)\,\dot{y}_1 - 2\omega\cos(\varphi)\,\dot{y}_3\,, \\ \ddot{y}_3 &= -g + 2\omega\cos(\varphi)\,\dot{y}_2\,. \end{aligned} \tag{1.60}$$

We want to describe a mass point that is released from rest at $t = 0$ ($\dot{\boldsymbol{y}} = 0$) at height h ($y_1 = y_2 = 0$, $y_3 = h > 0$). The first and third equations then yield

$$\dot{y}_1 = 2\omega\sin(\varphi)y_2 \qquad \text{and} \qquad \dot{y}_3 = -gt + 2\omega\cos(\varphi)y_2\,. \tag{1.61}$$

Substituting into the second equation leads to the differential equation

$$\ddot{y}_2 + 4\omega^2\,y_2 = 2gt\omega\cos\varphi \tag{1.62}$$

for y_2. The general solution of this is

$$y_2 = \frac{g\cos\varphi}{2\omega}t + A\sin(2\omega t) + B\cos(2\omega t)\,. \tag{1.63}$$

The initial condition $y_2(0) = 0$ gives $B = 0$, and $\dot{y}_2(0) = 0$ requires $A = -(2\omega)^{-2}g\cos\varphi$. Thus, we find the result

$$y_2 = \frac{g\cos\varphi}{2\omega}\left(t - \frac{1}{2\omega}\sin 2\omega t\right)\,. \tag{1.64}$$

---

[8] The gravitational acceleration is approximately the same everywhere on Earth. We have further assumed that $|\boldsymbol{y}| \ll |\boldsymbol{p}|$, so that the gravitational acceleration is independent of $\boldsymbol{y}$.

For $t > 0$, $y_2 > 0$, i.e., there is an *eastward deflection*. Since $\omega t \simeq$ fall time/day $\ll 1$, we expand in powers of ωt and obtain

$$y_2 = \frac{gt^2}{3}\cos(\varphi)\omega t\,. \tag{1.65}$$

In principle, we can now also determine y_1, y_3 using (1.61). However, the corrections to free fall for these components are only of the order $(\omega t)^2$. The impact occurs at time T with $gT^2/2 = h$. We obtain the total eastward deflection

$$y_2(T) = \frac{1}{3}\omega\Big(\frac{8h^3}{g}\Big)^{1/2}\cos\varphi\,. \tag{1.66}$$

For example, for $\varphi = 45°$, $h = 100\,\mathrm{m}$, the eastward deflection is just $y_2 \approx 1.6\,\mathrm{cm}$.

1.6 Mechanical Similarity

In part, it is possible to find certain properties of the solution of the Newtonian equations of motion (1.10) without explicitly solving them. An important example is the case when the forces are generated by a potential V that is a homogeneous function (of degree k) of the coordinates, i.e.,

$$V(\boldsymbol{x}'_1,\dots,\boldsymbol{x}'_N) = V(\lambda\boldsymbol{x}_1,\dots,\lambda\boldsymbol{x}_N) = \lambda^k\,V(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N) \tag{1.67}$$

with $\lambda > 0$, $\boldsymbol{x}'_i = \lambda\boldsymbol{x}_i$ and $k \in \mathbb{R}$. If one differentiates this relation with respect to $\boldsymbol{x}_i$, one obtains

$$\lambda\frac{\partial V}{\partial\boldsymbol{x}'_i}(\boldsymbol{x}'_1,\dots,\boldsymbol{x}'_N) = \lambda^k\frac{\partial V}{\partial\boldsymbol{x}_i}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N), \tag{1.68}$$

i.e., the forces $\boldsymbol{F}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N) = -\partial_{\boldsymbol{x}_i}V$ are homogeneous of degree $k-1$.

Given a solution $\boldsymbol{x}_i(t)$ of the equations of motion

$$m_i\ddot{\boldsymbol{x}}_i = -\frac{\partial V}{\partial\boldsymbol{x}_i}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N) \tag{1.69}$$

one can obtain further (scaled) solutions $\boldsymbol{x}'_i(t') = \lambda\,\boldsymbol{x}_i(t)$ using the transformation $\boldsymbol{x}'_i = \lambda\boldsymbol{x}_i$ and $t' = \beta t$ as long as

$$\beta = \lambda^{1-k/2} \tag{1.70}$$

is chosen. With this choice, it holds

$$m_i\frac{d^2\boldsymbol{x}'_i}{dt'^2} = \frac{\lambda}{\beta^2}m_i\ddot{\boldsymbol{x}}_i \overset{(1.69)}{=} -\frac{\lambda}{\beta^2}\frac{\partial V}{\partial\boldsymbol{x}_i}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N) \overset{(1.68)}{=} -\underbrace{\frac{\lambda^{2-k}}{\beta^2}}_{=1}\frac{\partial V}{\partial\boldsymbol{x}'_i}(\boldsymbol{x}'_1,\dots,\boldsymbol{x}'_N)$$

and thus $\boldsymbol{x}'_i(t')$ is also a solution of (1.69).

The transition from $\boldsymbol{x}_i(t)$ to $\boldsymbol{x}'_i(t')$ corresponds to the transition to new particle paths, where only spatial and temporal distances are scaled by the corresponding factors λ and β (geometric similarity). Since the new coordinates again satisfy the equations of motion, it holds for two paths (which are similar to each other) that the ratio of the times along the paths behaves like

$$\frac{t'}{t} = \beta = \left(\frac{l'}{l}\right)^{1-k/2} \tag{1.71}$$

where l, l' describes the spatial extent of the paths.[9] We also obtain the scaling relations

$$\frac{|\dot{\boldsymbol{x}}'|}{|\dot{\boldsymbol{x}}|} = \frac{\lambda}{\beta} = \left(\frac{l'}{l}\right)^{k/2}, \qquad \frac{|\boldsymbol{L}'|}{|\boldsymbol{L}|} = \frac{\lambda^2}{\beta} = \left(\frac{l'}{l}\right)^{1+k/2}, \qquad \frac{E'}{E} = \frac{\lambda^2}{\beta^2} = \left(\frac{l'}{l}\right)^{k}, \tag{1.72}$$

for the velocity $\dot{\boldsymbol{x}}$, the angular momentum $|\boldsymbol{L}|$, and the total energy $E = T + V$.

As an example, we consider the gravitational attraction between two point masses ($k = -1$). We will show in Chapter 2.2 that the corresponding paths are ellipses around the common center of mass. Two similar paths then have the same eccentricity but generally a different semi-major axis a. From (1.71), we obtain

$$\frac{T'}{T} = \left(\frac{a'}{a}\right)^{3/2}, \tag{1.73}$$

i.e., the squares of the orbital periods T behave like the cubes of the semi-major axes (*3rd Kepler's Law*).

Virial Theorem: In the case that the potential is a homogeneous function and the motion of the particles occurs in a bounded spatial region (with bounded momentum), there is a simple relation between the time average of the kinetic and potential energy. The theorem is of a statistical nature, as it does not make a statement about instantaneous (exact) relations between different mechanical quantities, but only deals with the time average.

To obtain the relation, we rewrite the kinetic energy a bit,

$$2T = \sum_{i=1}^{N} \boldsymbol{p}_i \cdot \dot{\boldsymbol{x}}_i = \frac{d}{dt}\underbrace{\left(\sum_{i=1}^{N} \boldsymbol{p}_i \cdot \boldsymbol{x}_i\right)}_{=G} - \sum_{i=1}^{N} \boldsymbol{x}_i \cdot \underbrace{\dot{\boldsymbol{p}}_i}_{=\boldsymbol{F}_i}. \tag{1.74}$$

We now want to average this expression over time. The time average of a function $f(t)$ is defined by

$$\overline{f} = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau dt\, f(t). \tag{1.75}$$

[9] For example, one could choose $l = |\boldsymbol{x}_1(0)|$ and $l' = |\boldsymbol{x}'_1(0)|$.

It is easy to see that for a bounded function $F(t)$

$$\overline{\frac{dF}{dt}} = \lim_{\tau\to\infty} \frac{F(\tau) - F(0)}{\tau} = 0 \tag{1.76}$$

holds. By assumption, $G(t)$ is bounded and from (1.74) follows the *virial theorem*

$$2\overline{T} = -\overline{\sum_{i=1}^{N} \boldsymbol{x}_i \cdot \boldsymbol{F}_i}; \tag{1.77}$$

the right side of this equation is referred to as the *virial* of the system.

For a homogeneous potential, the *Euler's theorem*[10]

$$kV = \sum_{i=1}^{N} \boldsymbol{x}_i \cdot \frac{\partial V}{\partial \boldsymbol{x}_i}$$

holds, and thus we find ($\boldsymbol{F}_i = -\partial_{\boldsymbol{x}_i} V$)

$$2\overline{T} = k\overline{V}. \tag{1.78}$$

For the gravitational potential ($k = -1$), it follows that $2\overline{T} = -\overline{V}$ (V is negative). For an isotropic oscillator, we have $k = 2$ and therefore $\overline{T} = \overline{V}$.

[10]Note: Differentiating $V(\lambda x) = \lambda^k V(x)$ with respect to λ yields $\sum_i \partial_{\boldsymbol{x}_i} V \cdot \boldsymbol{x}_i = k\lambda^{k-1} V$ (by the chain rule). One then sets $\lambda = 1$.

Chapter 2

The Two-Body Problem

Before we want to further investigate the structure of classical mechanics, it is instructive to first study a few examples. We begin with the simplest case, an isolated mechanical system of two particles. We will see that this case can generally be reduced to the problem of a particle in an effective potential. This insight is the basis for the general solution (integration) of the two-particle problem.

2.1 The General Case

As we have shown in Chapter 1.3, the force between two particles acts along the line connecting the two particles and depends only on the distance between the two particles. The equations of motion are therefore generally of the form

$$\ddot{\boldsymbol{x}}_1 = -\frac{1}{m_1}\frac{\partial V}{\partial \boldsymbol{x}_1}(|\boldsymbol{x}_1 - \boldsymbol{x}_2|)\,, \qquad \ddot{\boldsymbol{x}}_2 = \frac{1}{m_2}\frac{\partial V}{\partial \boldsymbol{x}_1}(|\boldsymbol{x}_1 - \boldsymbol{x}_2|)\,, \tag{2.1}$$

where $V(r)$ is the potential.

For two particles with masses m_1 and m_2, which interact only through gravitational force, the potential is, for example, given by $V(r) = -Gm_1m_2/r$. In this case, equations (2.1) also apply to extended, spherically symmetric bodies (e.g., in good approximation for celestial bodies), since outside such a body, the gravitational force is exactly as if the entire mass were located at the center (*Newton's theorem*).

2.1.1 Relative Coordinates and Conserved Quantities

To solve the above problem, it is helpful to separate the center of mass motion. For this purpose, we introduce the center of mass and relative coordinates

$$\boldsymbol{X} = \frac{1}{M}(m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2)\,, \qquad (M = m_1 + m_2)\,, \tag{2.2}$$

$$\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2 \tag{2.3}$$

In terms of these variables, the equations of motion are

$$M\ddot{\boldsymbol{X}} = 0 \tag{2.4}$$

$$\mu\ddot{\boldsymbol{x}} = -\frac{\partial V}{\partial \boldsymbol{x}}(|\boldsymbol{x}|)\,, \qquad \left(\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}\right), \tag{2.5}$$

i.e., the center of mass motion $\boldsymbol{X}(t)$ and the relative motion $\boldsymbol{x}(t)$ are decoupled. The center of mass motion $\boldsymbol{X}(t)$ is an inertial motion (see Chapter 1.1), and the equation for the relative motion $\boldsymbol{x}(t)$ is that of *a single* particle with reduced mass μ under the influence of the external potential $V(|\boldsymbol{x}|)$. The relative system still possesses two conserved quantities.

Angular Momentum: The (relative) angular momentum

$$\boldsymbol{L} = \mu\,\boldsymbol{x} \times \dot{\boldsymbol{x}} \ (= \text{constant}) \tag{2.6}$$

is conserved, since $\dot{\boldsymbol{L}} = -\boldsymbol{x} \times \partial_{\boldsymbol{x}} V(|\boldsymbol{x}|) = 0$. Thus, the trajectory $\boldsymbol{x}(t)$ lies in the plane perpendicular to $\boldsymbol{L}$, the *orbital plane*. To describe the motion in the orbital plane, we use polar coordinates (r, φ) with the associated unit vectors $\boldsymbol{e}_r = (\cos\varphi, \sin\varphi, 0)$ and $\boldsymbol{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$ in radial and azimuthal directions.[1] With $\boldsymbol{x} = r\boldsymbol{e}_r$ and $\dot{\boldsymbol{e}}_r = \dot{\varphi}\boldsymbol{e}_\varphi$, we find for the velocity

$$\dot{\boldsymbol{x}} = \dot{r}\,\boldsymbol{e}_r + r\,\dot{\varphi}\,\boldsymbol{e}_\varphi. \tag{2.7}$$

Thus, the magnitude of the angular momentum is given by

$$l \ \equiv \ |\boldsymbol{L}| = \mu\, r^2\,\dot{\varphi} = \text{constant}\,. \tag{2.8}$$

This is the area law (*2nd Kepler's Law*): If $F(t)$ denotes the area swept out by the vector $\boldsymbol{x}(t)$ in the orbital plane, then

$$\dot{F}(t) = \frac{1}{2}r^2\dot{\varphi} = \frac{l}{2\mu} = \text{constant}\,, \tag{2.9}$$

see Fig. 2.1.

Energy: The other conserved quantity is the relative energy

$$T + V = \frac{\mu}{2}\dot{\boldsymbol{x}}^2 + V(r) = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + V(r) = E = \text{constant}\,. \tag{2.10}$$

[1] We have set the coordinate system such that $\boldsymbol{L}$ points along the (positive) x_3-axis.

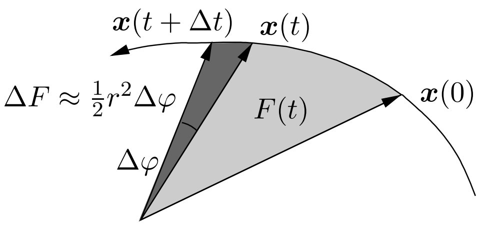


Figure 2.1: Illustration of the area law: $F(t)$ denotes the area in the orbital plane swept out by the vector $\boldsymbol{x}(t)$ in time t. Its change ΔF is given by the area $r^2\Delta\varphi/2$ of the dark gray triangle. We thus obtain for the area change per time $\dot{F} = r^2\dot{\varphi}/2$, which is proportional to the angular momentum $l = \mu r^2 \dot{\varphi}$.

The constancy of the energy E follows from $\dot{E} = \dot{\boldsymbol{x}} \cdot \mu\, \ddot{\boldsymbol{x}} + \dot{\boldsymbol{x}} \cdot \partial_{\boldsymbol{x}} V(|\boldsymbol{x}|) = 0$.

The combination of the two conserved quantities allows us to solve the two-body problem. Substituting $\dot{\varphi}$ from (2.8) into the energy equation (2.10) yields

$$\frac{1}{2}\mu \dot{r}^2 + U(r) = E\,, \qquad U(r) = \frac{l^2}{2\mu r^2} + V(r)\,. \tag{2.11}$$

The term $l^2/2\mu r^2$ is called the *centrifugal potential*. The radial motion $r(t)$ on $0 < r < \infty$ is that of a mass point with mass μ under the influence of the effective potential $U(r)$.

The remaining differential equation (2.11) is separable with the solution

$$t(r) - t(r_0) = \pm \int_{r_0}^{r} \frac{dx}{\sqrt{\frac{2}{\mu}(E - U(x))}}\,; \tag{2.12}$$

the square root changes sign at each turning point. For the region where the radius increases (decreases), the positive (negative) sign must be chosen. Additionally, we obtain with

$$\frac{d\varphi}{dr} = \frac{\dot{\varphi}}{\dot{r}} = \pm \frac{l}{\mu\, r^2 \sqrt{\frac{2}{\mu}(E - U(r))}}\,, \tag{2.13}$$

also

$$\varphi(r) - \varphi(r_0) = \pm \int_{r_0}^{r} \frac{l\, dx}{x^2 \sqrt{2\,\mu\,(E - U(x))}}\,. \tag{2.14}$$

The function $\varphi(r)$ describes the trajectory in the plane without reference to the temporal progression. To find the trajectory $\boldsymbol{x}(t)$, one must solve (2.12) for r. This gives $r(t)$ and, using (2.14), also $\varphi(t)$.

Through (2.12) and (2.14), the determination of the trajectory is reduced to the calculation of integrals: the problem is integrable, see also Chapter 8.2. The trajectory

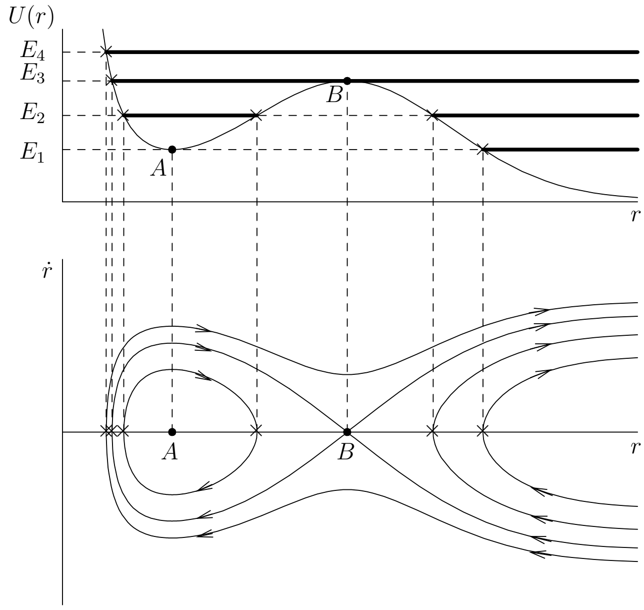


Figure 2.2: Potential and phase portrait for four different energies.

depends on four *motion integrals* E, l, r_0 and $\varphi_0 = \varphi(r_0)$. Together with the orbital plane (direction of $\boldsymbol{L}$), these uniquely determine the trajectory.[2]

2.1.2 The Different Types of Orbits

Using an example, we illustrate possible types of motion of the relative motion. Figure 2.2 shows the motion at four different energies in phase space with the coordinates $(r, \dot{r})$ (phase portrait). The trajectories always lie in the region

$$E - U(r) = \frac{\mu}{2}\,\dot{r}^2 \geq 0\,. \tag{2.15}$$

At the boundaries of this region, $\dot{r} = 0$; these are the turning points of the trajectory (marked with $\times$ in the diagram). For a point with $\dot{r} = 0$ to actually be a turning point, it must hold that

$$\mu\ddot{r} = -\frac{dU}{dr} \neq 0 \tag{2.16}$$

must hold. If $dU/dr = 0$ for $r = r_0$ (these are the points A and B in the diagram), there is an equilibrium solution with $r = r_0$. The solution at A is stable (i.e., a trajectory with initial conditions $(r, \dot{r})$ close to A will always remain close to A),

[2] The connection to the 'elementary' integration constants $r_0, \varphi_0, \dot{r}_0, \dot{\varphi}_0$ is obtained through the equations (2.8) and (2.10).

whereas B is unstable. In general, we distinguish *bound orbits*, which run entirely in the finite region, and *scattering orbits* with $r(t) \to \infty$ for $t \to \pm\infty$.

2.1.3 Bound Orbits

For bound orbits, $r(t)$ is periodic with the period

$$T(E) = 2 \int_{r_{\min}}^{r_{\max}} \frac{dx}{\sqrt{\frac{2}{\mu}(E - U(x))}}\,, \tag{2.17}$$

where $r_{\min}$ and $r_{\max}$ are the zeros of $E - U(r)$ that limit the allowed r-interval. This formula follows directly from (2.12).

If both $r_{\min}$ and $r_{\max}$ are turning points, then $T < \infty$. In this case, one can locally (near $x^* = r_{\min}$ and $x^* = r_{\max}$) expand the function $E - U(x)$ with $E - U(x^* + \epsilon) \approx \alpha\,\epsilon$. The resulting integral $\propto \int d\epsilon\,/\sqrt{\epsilon}$ converges. If one of the interval boundaries is not a turning point but an equilibrium point (as for $E = E_3$, where $r_{\max} = r_{\mathrm{B}}$), then $T(E_3) = \infty$.

For a periodic orbit, the azimuth φ increases during one period T by the angle (*perihelion precession*)

$$\Delta\varphi = 2 \int_{r_{\min}}^{r_{\max}} \frac{l\,dx}{x^2\sqrt{2\mu(E - U(x))}}, \tag{2.18}$$

which follows directly from (2.14). The trajectory is generally a 'rose curve' in the ring $r_{\min} \le r \le r_{\max}$, which only closes if $\Delta\varphi/2\pi$ is rational, see Fig. 2.3(a).

2.1.4 Scattering Orbits

We now consider the case that $V(r) \to 0$ for $r \to \infty$. Then there are scattering orbits only for energies $E \ge 0$ (with $r(t) \to \infty$ for $t \to \pm\infty$).[3] If $E > 0$, the scattering orbits have straight asymptotes (for $t \to \pm\infty$). A scattering orbit is determined by the energy $E > 0$, the direction $\boldsymbol{e}$ ($|\boldsymbol{e}| = 1$) of the incoming asymptote, and the impact parameter $\boldsymbol{b} \perp \boldsymbol{e}$, see Fig. 2.3(b). For $t \to -\infty$ (due to $V \to 0$) the asymptotic relation $\mu|\dot{\boldsymbol{x}}| \to \sqrt{2\mu E}$ holds, and thus $\mu\dot{\boldsymbol{x}} \to \sqrt{2\mu E}\,\boldsymbol{e}$. With $l = \mu|\boldsymbol{x} \times \dot{\boldsymbol{x}}|$, we obtain the relation

$$l = b\sqrt{2\mu E} \tag{2.19}$$

between the impact parameter $b = |\boldsymbol{b}|$ and the angular momentum l. Therefore, b and E determine the *scattering angle*

$$\chi = \pi - 2\theta \tag{2.20}$$

[3] In the limit $r \to \infty$, $V(r) - U(r) = 0$.

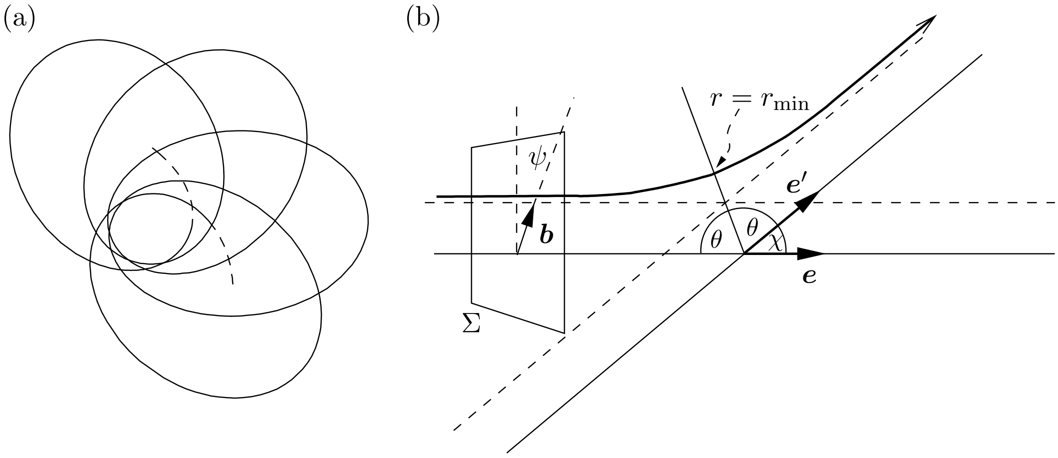


Figure 2.3: (a) Rose curve (b) Scattering orbit with the impact parameter $\boldsymbol{b}$ in the plane Σ. The particle starts at $t \to -\infty$ at the position $r \to \infty$ and approaches the scattering center in the direction $\boldsymbol{e} \perp \boldsymbol{b}$. It passes at $r_{\min}$ the closest distance to the scattering center before it moves towards $r \to \infty$ in the direction $\boldsymbol{e}'$ (deflected by the scattering angle χ).

with

$$\theta = \int_{r_{\min}}^{\infty} \frac{l\,dx}{x^2\sqrt{2\mu(E - U(x))}} = \int_{r_{\min}}^{\infty} \frac{b\,dx}{x^2\sqrt{1 - V(x)E^{-1} - b^2x^{-2}}}\,, \tag{2.21}$$

cf. with equation (2.14).

For fixed E and $\boldsymbol{e}$, the vector $\boldsymbol{b}$ determines the direction vector $\boldsymbol{e}'$ ($|\boldsymbol{e}'| = 1$) of the outgoing asymptote. This relationship defines a mapping $\Sigma \ni \boldsymbol{b} \mapsto \boldsymbol{e}' \in S^2$. Since $\boldsymbol{e}'$ is described by the polar angle (χ, ψ) with respect to $\boldsymbol{e}$, under this mapping the area element $d\sigma = b\,db\,d\psi$ transforms into the solid angle element $d\Omega = \sin\chi\,d\chi\,d\psi$. The *differential scattering cross-section* (with dimension of an area) is defined as the ratio

$$\frac{d\sigma}{d\Omega}(\chi) = \left|\frac{b}{\sin\chi}\frac{db}{d\chi}\right| = \frac{b}{\sin\chi}\left|\left(\frac{d\chi}{db}\right)^{-1}\right|\,, \tag{2.22}$$

where $\chi = \chi(b)$ is given by (2.20) and (2.21). It may happen that different b_k lead to the same scattering angle χ; then (2.22) must be replaced by

$$\frac{d\sigma}{d\Omega}(\chi) = \sum_k \frac{b_k}{\sin\chi}\left|\left(\frac{d\chi}{db_k}\right)^{-1}\right| \tag{2.23}$$

The latter case occurs, for example, whenever $-\infty < V(0) \leq 0$ is true. Then it holds that $\chi(0) = 0$, since for $b = 0$ the trajectory passes directly through the origin. Since in general $\lim_{b\to\infty}\chi(b) = 0$, it follows from the intermediate value theorem that every (non-extremal) value $\chi(b_k)$ is assumed at least at two points b_k.

In a *scattering experiment*, one bombards the scattering center with a homogeneous flux density j (particle number per area and time unit) of incoming particles with fixed E and $\boldsymbol{e}$. Then $j\,(d\sigma/d\Omega)\,d\Omega$ is the number of scattered particles per unit time with outgoing asymptote direction $\boldsymbol{e}' \equiv (\chi, \psi)$ in the solid angle element $d\Omega$.

The total number of scattered particles $j\,\sigma_{\text{tot}}$ is therefore proportional to the *total scattering cross-section*

$$\sigma_{\text{tot}} = \int_{S^2} \frac{d\sigma}{d\Omega} d\Omega. \tag{2.24}$$

It is equal to the area of the impact parameters $\boldsymbol{b} \in \Sigma$ that lead to a scattering (i.e., exhibit $\chi(b) \neq 0$); for integrating (2.24) we immediately obtain the result

$$\sigma_{\text{tot}} = \pi a^2 \,, \tag{2.25}$$

where $a = \sup_{r\geq 0}\{r \mid V(r) \neq 0\} \leq \infty$ is the range of the potential.

2.2 The Kepler Problem

The Kepler problem is an important special case of the two-body problem, in which two particles with masses m_1 and m_2 attract each other gravitationally. In this case, the potential takes the form

$$V(r) = -\frac{1}{r} G m_1 m_2 \,, \tag{2.26}$$

where $G = 6.67 \cdot 10^{-11}\,\mathrm{m^3/kg\,s^2}$ denotes the Newtonian *gravitational constant*. With the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}\,, \qquad m_1 m_2 = M\mu\,, \quad (M = m_1 + m_2) \tag{2.27}$$

one finds

$$U(r) = \frac{l^2}{2\mu r^2} - \frac{GM\mu}{r}. \tag{2.28}$$

The trajectory is given by the general relationship (2.14). Substituting $x = 1/s$ (with $-dx/x^2 = ds$) leads to

$$\varphi(r) - \varphi_0 = \mp \int_{r_0^{-1}}^{r^{-1}} \frac{l\,ds}{\sqrt{2\mu(E - U(1/s))}} = \mp \int_{r_0^{-1}}^{r^{-1}} \frac{ds}{\sqrt{\alpha + 2\beta s - s^2}}\,, \tag{2.29}$$

where we have introduced the parameters $\alpha = 2\mu E l^{-2}$ and $\beta = GM\mu^2 l^{-2}$. The last integral has the antiderivative ($\alpha + \beta^2 \geq 0$)

$$\int \frac{ds}{\sqrt{\alpha + 2\beta s - s^2}} = -\arccos\Big(\frac{s - \beta}{\sqrt{\alpha + \beta^2}}\Big). \tag{2.30}$$

Thus, (2.29) can be explicitly integrated with the result

$$\varphi(r) = \pm \arccos\Big(\frac{r^{-1} - \beta}{\sqrt{\alpha + \beta^2}}\Big); \tag{2.31}$$

here we have chosen the free constant φ_0 in such a way that it cancels the contribution from the lower limit of integration. By solving for r, the trajectory in polar coordinates is obtained as

$$r = \frac{d}{1 + \varepsilon \cos\varphi}\,, \tag{2.32}$$

with the parameters

$$d = \frac{1}{\beta} = \frac{l^2}{GM\mu^2}\,, \text{ and} \tag{2.33}$$

$$\varepsilon = \frac{\sqrt{\alpha + \beta^2}}{\beta}\,, \quad \text{that is,} \quad 1 - \varepsilon^2 = -\frac{\alpha}{\beta^2} = -\frac{2El^2}{G^2M^2\mu^3}\,. \tag{2.34}$$

The equation (2.32) for the trajectory defines a conic section with a focus at $r = 0$; depending on the value of the eccentricity ε, it can be one of the following:

Circle	:	$\varepsilon = 0$,	that is,	$E = -G^2M^2\mu^3/2l^2$,
Ellipse	:	$\varepsilon < 1$,	that is,	$E < 0$,
Parabola	:	$\varepsilon = 1$,	that is,	$E = 0$,
Hyperbola	:	$\varepsilon > 1$,	that is,	$E > 0$.

The angle $\varphi = 0$ corresponds to the perihelion (minimum r). We will show below that (2.32) corresponds to an ellipse for $\varepsilon < 1$.

2.2.1 Elliptical Orbits

From Fig. 2.4(a) and the equation of the ellipse $r + \bar{r} = 2a$, we obtain the relationship

$$\bar{r}^2 = (r\cos\varphi + 2a\varepsilon)^2 + r^2\sin^2\varphi = (2a - r)^2. \tag{2.35}$$

From this, we obtain after factoring the result

$$4ar(1 + \varepsilon\cos\varphi) = 4a^2(1 - \varepsilon^2)\,. \tag{2.36}$$

This is precisely the trajectory (2.32) with

$$d = a(1 - \varepsilon^2). \tag{2.37}$$

The fact that Newton's law of gravitation with $V(r) \propto r^{-1}$ predicts elliptical orbits is historically significant: Johannes Kepler derived from the observations of Tycho Brahe that the orbit of Mars is an ellipse with a focus at the Sun (or in the center of

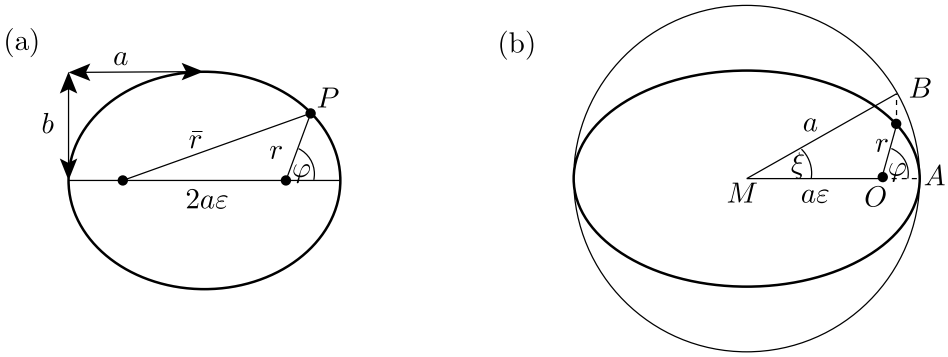


Figure 2.4: (a) Ellipse with the major and minor axes a and b. The two foci are separated by the distance $2a\varepsilon$ with the eccentricity $\varepsilon = \sqrt{1 - b^2/a^2}$. (b) Graphical relationship between the true anomaly (azimuth) φ and the eccentric anomaly ξ. For the construction of the eccentric anomaly, the circumference from the radius a is needed. The point B on the circumference (and thus ξ) is characterized by having the same projection on the axis MA as the point $P \equiv (r, \varphi)$ on the ellipse.

mass), and extended this to the other planets (*1st Kepler's Law*). Newton was able to theoretically explain this result with his theory of gravitation.

Another important feature of an ellipse is the semi-minor axis

$$b = a\sqrt{1 - \varepsilon^2} = \sqrt{ad} \tag{2.38}$$

with which the area of the ellipse $F = \pi ab = \pi d^{1/2} a^{3/2}$ can be determined. According to the area law (2.9) (*2nd Kepler's Law*), it holds that

$$F = T\dot{F} = \frac{Tl}{2\mu}\,, \tag{2.39}$$

where T is the orbital period. With this relation, one can solve for the orbital period

$$T = \frac{2\mu}{l}F = \frac{2\mu}{l}\pi d^{1/2}\, a^{3/2} = \frac{2\pi}{\sqrt{GM}}a^{3/2} \tag{2.40}$$

If $m_1 \ll m_2$, then $M \approx m_2$, and the ratio $T^2 \colon a^3$ is the same for all planets (*3rd Kepler's Law*).

To describe the temporal progression of the orbit, it is useful to use the *eccentric anomaly* ξ (i.e., the angle relative to the center of the ellipse, see Fig. 2.4(b)) instead of the true anomaly (azimuth) φ.[4] By considering the projection of MB onto MA, one finds the relationship

$$a\cos\xi = a\varepsilon + r\cos\varphi, \tag{2.41}$$

[4]The eccentric anomaly was introduced by Kepler.

which links the eccentric anomaly ξ with the true anomaly φ. Regarding the eccentric anomaly, the *Kepler's equations* hold

$$r = a(1-\varepsilon\cos\xi), \qquad t = \sqrt{\frac{a^3}{GM}}(\xi - \varepsilon\sin\xi), \tag{2.42}$$

where $t = 0$ corresponds to a perihelion passage. The equations (2.41) and (2.42) together provide an (implicit) expression for the orbit $\boldsymbol{x}(t)$.

Proof of Kepler's Equations: The equation for r follows directly from (2.41) with

$$\begin{aligned} \varepsilon a\cos\xi &= \varepsilon(a\varepsilon + r\cos\varphi) \\ &= a - a(1-\varepsilon^2) + \varepsilon r\cos\varphi \\ &= a - r(1+\varepsilon\cos\varphi) + \varepsilon r\cos\varphi \\ &= a - r\,, \end{aligned} \tag{2.43}$$

where in the second transformation the trajectory (2.36) was used.

The proof of the time equation is obtained directly from the area law $F(t) = l\,t/2\mu$, see equation (2.9). For this, one uses that a stretching of the ellipse by the factor $a/b = \sqrt{a/d} \geq 1$ in the direction of the minor axis results in the circumcircle. The area swept out by the radius vector $F(t)$ transforms into the area

$$\tilde{F}(t) = \frac{a}{b}F(t) = \sqrt{\frac{a}{d}}\frac{l\,t}{2\mu} = \frac{1}{2}\sqrt{GMa}\,t \tag{2.44}$$

of AOB (multiplied by a/b due to the stretching). However, the area $\tilde{F}$ of AOB is precisely the difference of the circular sector AMB with area $a^2\xi/2$ and the triangle OMB with area $(a\varepsilon)(a\sin\xi)/2$. Thus, we obtain

$$\tilde{F}(t) = \frac{1}{2}a^2(\xi - \varepsilon\sin\xi) \tag{2.45}$$

and by comparing with (2.44) the second equation in (2.42).

2.2.2 Hyperbolic Orbits

As already shown at the beginning of the chapter, we obtain in the Kepler problem scattering orbits (hyperbolas) for positive energies. To determine the differential scattering cross-section, we need the relationship of the scattering angle $\chi = \pi - 2\theta$ with the impact parameter $b = l/\sqrt{2\mu E}$.

As seen in Fig. 2.5, the angle θ corresponds to the limit $\theta = \pi - \lim_{r\to\infty}\varphi(r)$. According to (2.32), it holds that $\cos\varphi \to -1/\varepsilon$ for $r \to \infty$. Expressed in terms of the angle θ, this is the condition $\cos\theta = 1/\varepsilon$ and therefore $\tan^2\theta = \varepsilon^2 - 1$. On the other hand, according to (2.34), it also holds that

$$\varepsilon^2 - 1 = \frac{2E\,\overbrace{(2\mu E b^2)}^{=l^2}}{G^2M^2\mu^3}\,. \tag{2.46}$$

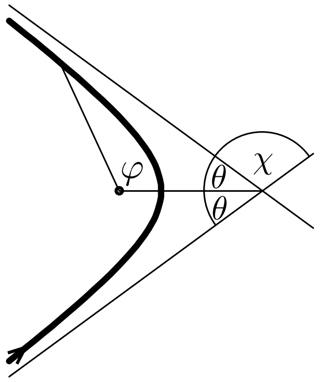


Figure 2.5: Scattering orbit (hyperbola with $\varepsilon > 1$) for the Kepler problem.

Thus, we obtain the result

$$\tan\theta = \frac{2E}{GM\mu} b \, . \tag{2.47}$$

To determine the differential scattering cross-section, we need the intermediate results

$$\frac{d\chi}{db} = -2\frac{d\theta}{db} = -\frac{4E}{GM\mu}\cos^2\theta \tag{2.48}$$

and

$$\frac{b}{\sin\chi} = \frac{b}{2\sin\theta\cos\theta} = \frac{GM\mu}{4E}\frac{1}{\cos^2\theta} \, . \tag{2.49}$$

Thus, we obtain the scattering cross-section

$$\frac{d\sigma}{d\Omega} = \left| \frac{b}{\sin\chi}\left(\frac{d\chi}{db}\right)^{-1} \right| = \left(\frac{GM\mu}{4E\sin^2\frac{\chi}{2}} \right)^2 , \tag{2.50}$$

where we have used $\cos\theta = \cos(\pi/2 - \chi/2) = \sin(\chi/2)$. For charged particles, the product of the charges $e_1 e_2$ appears instead of $GM\mu$ (*Rutherford scattering formula*). Surprisingly, the same result also holds in quantum mechanics.

2.2.3 Laplace–Runge–Lenz Vector

The Kepler problem has an additional integral of motion compared to the general two-body problem, namely the Laplace–Runge–Lenz (LRL) vector

$$\boldsymbol{A} = \mu\dot{\boldsymbol{x}} \times \boldsymbol{L} - GM\mu^2\frac{\boldsymbol{x}}{r} \, . \tag{2.51}$$

This conserved quantity allows for a completely algebraic determination of the orbit.

To show that $\boldsymbol{A}$ is indeed conserved, we first determine a vectorial form of the area law

$$\begin{aligned}\mu \frac{d}{dt}\frac{\boldsymbol{x}}{r} &= \mu\Big(\frac{\dot{\boldsymbol{x}}}{r} - \frac{\boldsymbol{x}}{r^2}\overbrace{\dot{r}}^{=\boldsymbol{x}\cdot\dot{\boldsymbol{x}}/r}\Big) = \frac{\mu}{r^3}\left(\dot{\boldsymbol{x}}\, r^2 - \boldsymbol{x}(\boldsymbol{x}\cdot\dot{\boldsymbol{x}})\right)\\ &= -\frac{\mu}{r^3}\boldsymbol{x}\times(\boldsymbol{x}\times\dot{\boldsymbol{x}}) = -\frac{1}{r^3}\,\boldsymbol{x}\times\boldsymbol{L}.\end{aligned} \tag{2.52}$$

Thus, we directly obtain the result

$$\frac{d\boldsymbol{A}}{dt} = -\frac{GM\mu}{r^3}(\boldsymbol{x}\times\boldsymbol{L} - \boldsymbol{x}\times\boldsymbol{L}) = 0\,, \tag{2.53}$$

where we have used the equations of motion

$$\mu\ddot{\boldsymbol{x}} = -\frac{\partial V}{\partial \boldsymbol{x}}(|\boldsymbol{x}|) = -\frac{\boldsymbol{x}}{r^3}GM\mu \tag{2.54}$$

as well as the time independence of $\boldsymbol{L}$.

The LRL vector does not provide three additional conserved quantities, as it is related to the angular momentum $\boldsymbol{L}$ and the energy E. On the one hand, the LRL vector lies in the orbital plane, as it is orthogonal to $\boldsymbol{L}$ with

$$\boldsymbol{A}\cdot\boldsymbol{L} = 0, \tag{2.55}$$

as can be easily verified by explicit calculation.[5] On the other hand, the square of its length is related to the energy

$$\begin{aligned}\boldsymbol{A}^2 &= (GM\mu^2)^2 + \mu^2\dot{\boldsymbol{x}}^2\, l^2 - \frac{2GM\mu^2}{r}\overbrace{\mu\boldsymbol{x}\cdot(\dot{\boldsymbol{x}}\times\boldsymbol{L})}^{=l^2}\\ &= (GM\mu^2)^2 + 2\mu E l^2\,,\end{aligned} \tag{2.56}$$

where we have used the formula (2.10) for E. Therefore, the LRL vector provides only one additional conserved quantity.

To understand the significance of this conserved quantity, we measure the azimuth φ with respect to $\boldsymbol{A}$ with $\boldsymbol{x}\cdot\boldsymbol{A} = r|\boldsymbol{A}|\cos\varphi$ ($\boldsymbol{A}$ lies in the orbital plane). On the other hand, we can directly calculate $\boldsymbol{x}\cdot\boldsymbol{A}$ from (2.51) with the result

$$\boldsymbol{x}\cdot\boldsymbol{A} = \mu(\boldsymbol{x}\times\dot{\boldsymbol{x}})\cdot\boldsymbol{L} - GM\mu^2 r = l^2 - GM\mu^2 r\,. \tag{2.57}$$

Thus, we obtain (purely algebraically) the trajectory

$$r = \frac{l^2}{GM\mu^2 + |\boldsymbol{A}|\cos\varphi} = \frac{d}{1+\varepsilon\cos\varphi} \tag{2.58}$$

[5] In fact, both terms in (2.51) are individually perpendicular to $\boldsymbol{L}$.

with d as in (2.33) and the eccentricity

$$\varepsilon = \frac{|\boldsymbol{A}|}{GM\mu^2} \tag{2.59}$$

proportional to the length of $\boldsymbol{A}$. The direction of $\boldsymbol{A}$ is the additional conserved quantity. Since in the parametrization (2.58) φ corresponds to the perihelion, the vector $\boldsymbol{A}$ points from the origin to the perihelion. The conservation of $\boldsymbol{A}$ thus corresponds to the absence of perihelion precession.

From the conservation of $\boldsymbol{A}$, we can also easily determine the trajectory of $\boldsymbol{p}(t) = \mu\dot{\boldsymbol{x}}(t)$. Since $\boldsymbol{A}$ lies in the orbital plane, it holds that $\boldsymbol{L}\cdot\boldsymbol{p} = 0$. The calculation of

$$\begin{aligned}
\boldsymbol{L}\times\boldsymbol{A} &= \boldsymbol{L}\times(\boldsymbol{p}\times\boldsymbol{L}) - GM\mu^2\,\boldsymbol{L}\times\frac{\boldsymbol{x}}{r} \\
&= \big(l^2\boldsymbol{p} - (\boldsymbol{L}\cdot\boldsymbol{p})\boldsymbol{L}\big) - GM\mu^2\,\boldsymbol{L}\times\frac{\boldsymbol{x}}{r} \\
&= l^2\boldsymbol{p} - GM\mu^2\,\boldsymbol{L}\times\frac{\boldsymbol{x}}{r}
\end{aligned}$$

allows us to solve for $\boldsymbol{p}$ with the result

$$\boldsymbol{p} = \frac{1}{l^2}\,\boldsymbol{L}\times\boldsymbol{A} + \frac{GM\mu^2}{l}\frac{\boldsymbol{L}}{l}\times\frac{\boldsymbol{x}}{r}. \tag{2.60}$$

Now, $(\boldsymbol{L}/l)\times(\boldsymbol{x}/r)$ is simply the unit vector in the orbital plane, which is perpendicular to $\boldsymbol{x}$ (in the direction of the orbit). Thus, $\boldsymbol{p}(t)$ describes a circle over the course of one orbit around $\boldsymbol{L}\times\boldsymbol{A}/l^2$ with radius $GM\mu^2/l$.

Chapter 3

Selected Three-Body Problems

In contrast to the two-body problem, which we discussed in Chapter 2, the three-body problem is no longer integrable. However, one can find special solutions (partly approximate).

3.1 Equilibrium Positions and Their Stability

The Sun and Jupiter are the heaviest bodies in the solar system. We want to investigate the motion of a light object (asteroid) under their influence. We assume that the Sun S and Jupiter J move in circular orbits with angular velocity $\omega = 2\pi/T_\mathrm{J}$ around the common center of mass 0. The distance $R = a_\mathrm{J}$ between the two bodies remains constant over time. The asteroid A is attracted by the gravitational force of the Sun and Jupiter but is assumed to have no influence on the motion of S and J. Due to this approximation, this formulation is also called the *restricted three-body problem.* We choose the units as follows:

$$\omega = 1, \qquad R = 1, \qquad G = 1 \, . \tag{3.1}$$

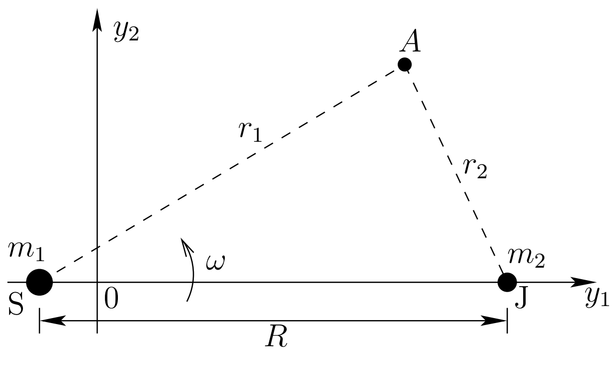


Figure 3.1: Motion of an asteroid A under the influence of the Sun and Jupiter.

Thus, we define the three mechanical units (length, time, mass) as follows:

$$\begin{aligned}
&\text{Length:} && 1 \mathrel{\hat{=}} a_{\mathrm{J}} \approx 5.2\, a_{\mathrm{E}} = 7.8 \cdot 10^8\,\mathrm{km},\\
&\text{Time:} && 1 \mathrel{\hat{=}} \frac{T_{\mathrm{J}}}{2\pi} \approx 1.9\,\text{years},\\
&\text{Mass:} && 1 \mathrel{\hat{=}} \frac{4\pi^2 a_{\mathrm{J}}^3}{G T_{\mathrm{J}}^2} \overset{(2.40)}{\approx} 1.0 M_{\mathrm{S}} = 2.0 \cdot 10^{30}\,\mathrm{kg}.
\end{aligned}$$

Since the y-system defined in Figure 3.1 is rotating, the equations of motion (1.54) for an accelerated reference frame are to be used. In the center of mass system, J has a distance $r_{\mathrm{J}} = R m_1/(m_1 + m_2)$ from the origin, while the distance from S is exactly $r_{\mathrm{S}} = R m_2/(m_1 + m_2)$. Since S and J are in an equilibrium position, the gravitational force and centrifugal force at their centers of mass $(-r_{\mathrm{S}}\boldsymbol{e}_1,\, r_{\mathrm{J}}\boldsymbol{e}_1)$ cancel each other out; that is, we have

$$\frac{G}{R^2} m_1 m_2 = \frac{m_1 m_2}{m_1 + m_2} R\omega^2 \,. \tag{3.2}$$

In the chosen units, we obtain the relationship[1]

$$m_1 + m_2 = 1\,, \tag{3.3}$$

and thus S and J have the coordinates $(-m_2, 0, 0)$, $(m_1, 0, 0)$.

Since all forces on A are proportional to its mass m, it drops out of the equations of motion for A. Therefore, we can measure all forces by accelerations. With these preliminary considerations, we have to consider the following accelerations acting on body A:

$$\begin{aligned}
&\text{Gravitational Acceleration:} && \boldsymbol{G} = -\frac{m_1}{r_1^3}(y_1 + m_2, y_2, y_3) - \frac{m_2}{r_2^3}(y_1 - m_1, y_2, y_3),\\
&\text{Centrifugal Acceleration:} && \boldsymbol{Z} = (y_1, y_2, 0),\\
&\text{Coriolis Acceleration:} && \boldsymbol{C} = 2(\dot{y}_2, -\dot{y}_1, 0).
\end{aligned} \tag{3.4}$$

We are looking for equilibrium solutions with $\boldsymbol{y}$ = constant. The Coriolis force disappears in this case, and we obtain the conditions $(\boldsymbol{G} + \boldsymbol{Z} = 0)$

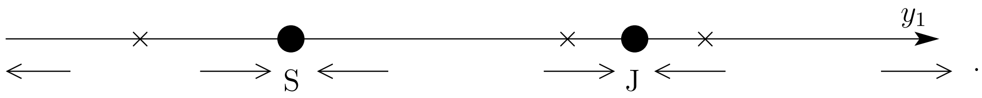
$$\begin{aligned}
&\Big(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} - 1\Big) y_1 + m_1 m_2 \Big(\frac{1}{r_1^3} - \frac{1}{r_2^3}\Big) = 0\,,\\
&\Big(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} - 1\Big) y_2 = 0\,, \qquad y_3 = 0\,.
\end{aligned} \tag{3.5}$$

For the solution of the second equation, there are 2 possibilities. Either $y_2 = 0$ (Euler's special case) or

$$\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} = 1 \qquad \text{(Lagrange's special case)}. \tag{3.6}$$

[1] The result also follows directly from the 3rd Kepler's Law (2.40).

In the *Euler case*, A lies on the y_1-axis. On this axis, the resulting acceleration $\boldsymbol{G}+\boldsymbol{Z}$ is directed as follows:

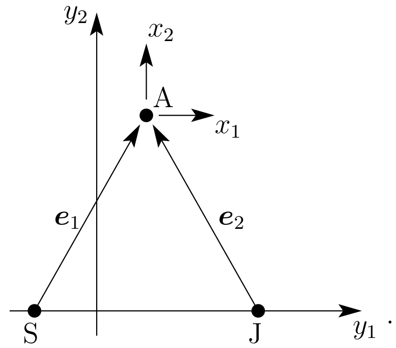


Thus, there are 3 equilibrium positions ($\times$ in the diagram), which we do not want to determine further.

In the *Lagrange case*, it follows from (3.5) and (3.6)

$$r_1 = r_2 = 1\,, \tag{3.7}$$

i.e., SJA forms an equilateral triangle in the 12-plane;



Therefore, there are two Lagrange equilibrium positions (with $y_2 > 0$ or $y_2 < 0$), whose stability we now want to investigate.

For the stability analysis, we consider small deviations $\boldsymbol{x} \ll 1$ from the rest position $\boldsymbol{x} = 0$, i.e., we set $\boldsymbol{y} = \left(\frac{1}{2} - m_2, \sqrt{3}/2, 0\right) + \boldsymbol{x}$ and linearize the gravitational force $\boldsymbol{G}$; $\boldsymbol{Z}$ is already linear. Due to $\partial_{\boldsymbol{y}} r_i^{-3}\big|_{\mathrm{A}} = -3\boldsymbol{e}_i$ with

$$\boldsymbol{e}_1 = \frac{1}{2}(1, \sqrt{3}, 0)\,, \qquad \boldsymbol{e}_2 = \frac{1}{2}(-1, \sqrt{3}, 0) \tag{3.8}$$

in linear approximation we have

$$r_i^{-3} = 1 - 3\boldsymbol{e}_i \cdot \boldsymbol{x}. \tag{3.9}$$

Thus, we obtain the linearized acceleration ($m_1 + m_2 = 1$)

$$\begin{aligned}
\boldsymbol{G} &\approx -m_1(1 - 3\boldsymbol{e}_1 \cdot \boldsymbol{x})\Big(\frac{1}{2} + x_1, \frac{\sqrt{3}}{2} + x_2, x_3\Big) \\
&\qquad - m_2(1 - 3\boldsymbol{e}_2 \cdot \boldsymbol{x})\Big(-\frac{1}{2} + x_1, \frac{\sqrt{3}}{2} + x_2, x_3\Big) \\
&\approx \boldsymbol{G}^0 + \frac{3}{2} m_1 \boldsymbol{e}_1 \cdot \boldsymbol{x}\,(1, \sqrt{3}, 0) + \frac{3}{2} m_2 \boldsymbol{e}_2 \cdot \boldsymbol{x}\,(-1, \sqrt{3}, 0) - (x_1, x_2, x_3) \\
&\equiv \boldsymbol{G}^0 + (G_1, G_2, G_3)\,;
\end{aligned} \tag{3.10}$$

here, $\boldsymbol{G}^0 = (\frac{m_2-m_1}{2}, -\frac{\sqrt{3}}{2}, 0)$ are the zeroth-order terms and

$$\begin{aligned} G_1 &= \Big(\frac{3}{4}-1\Big)x_1 + \frac{3\sqrt{3}}{4}(m_1-m_2)x_2\,, \\ G_2 &= \frac{3}{4}\sqrt{3}(m_1-m_2)x_1 + \Big(\frac{9}{4}-1\Big)x_2\,, \\ G_3 &= -x_3 \end{aligned} \tag{3.11}$$

are the first-order corrections in $\boldsymbol{x}$. Furthermore, we have

$$\boldsymbol{Z} = \boldsymbol{Z}^0 + (x_1, x_2, 0)\,, \qquad \boldsymbol{C} = 2(\dot{x}_2, -\dot{x}_1, 0)\,, \tag{3.12}$$

where $\boldsymbol{Z}^0 = (\frac{1}{2} - m_2, \frac{\sqrt{3}}{2}, 0)$ again represents the zeroth-order contribution. By construction, we have $\boldsymbol{G}^0 + \boldsymbol{Z}^0 = 0$ (equilibrium position), and we obtain the (linearized) equations of motion

$$\begin{aligned} \ddot{x}_1 &= \frac{3}{4}x_1 + \frac{3\sqrt{3}}{4}(m_1-m_2)x_2 + 2\dot{x}_2, \\ \ddot{x}_2 &= \frac{3\sqrt{3}}{4}(m_1-m_2)x_1 + \frac{9}{4}x_2 - 2\dot{x}_1, \\ \ddot{x}_3 &= -x_3\,. \end{aligned} \tag{3.13}$$

The x_3 motion is decoupled and performs a harmonic oscillation with frequency $\omega = 1$ (synchronized with the circular motion of S and J).

To solve the two remaining equations, we make the exponential ansatz

$$x_k(t) = a_k\, e^{i\lambda t}\,, \qquad (k = 1, 2)\,. \tag{3.14}$$

Since the coefficients of the linear differential equation (3.13) are real, the real and imaginary parts of the complex solution (3.14) form the sought real solutions. Substituting the ansatz yields the homogeneous system of equations

$$a_1\Big(\lambda^2 + \frac{3}{4}\Big) + a_2\Big(\frac{3\sqrt{3}}{4}(m_1-m_2) + 2i\lambda\Big) = 0, \tag{3.15}$$

$$a_1\Big(\frac{3\sqrt{3}}{4}(m_1-m_2) - 2i\lambda\Big) + a_2\Big(\lambda^2 + \frac{9}{4}\Big) = 0\,. \tag{3.16}$$

For the system of equations to have a nontrivial solution $(a_1, a_2) \neq 0$, its determinant must vanish, i.e.,

$$\Big(\lambda^2 + \frac{3}{4}\Big)\Big(\lambda^2 + \frac{9}{4}\Big) = \frac{27}{16}(m_1-m_2)^2 + 4\lambda^2\,. \tag{3.17}$$

After a few elementary transformations using $(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2 = 1 - 4m_1m_2$, we can bring this condition into the form

$$\Big(\lambda^2 - \frac{1}{2}\Big)^2 = \frac{1}{4}(1 - 27m_1m_2)\,. \tag{3.18}$$

For $27m_1m_2 < 1$ or in general mass units for

$$\frac{m_1 m_2}{(m_1+m_2)^2} < \frac{1}{27} \tag{3.19}$$

there are 4 different real solutions $\pm\lambda_1$, $\pm\lambda_2$ to (3.18), and the general solution of (3.13) is any superposition of the 4 eigenoscillations (3.14). Since all λ_i are real, these solutions behave oscillatory and are bounded. At least in linear approximation, the equilibrium position is therefore stable if (3.19) is satisfied. If the opposite inequality holds ($27m_1m_2 > 1$), complex eigenfrequencies arise (including always those with $\operatorname{Im}\lambda < 0$). Thus, there are exponentially growing solutions, and the equilibrium position $\boldsymbol{x} = 0$ is unstable.

The stability condition (3.19) can be rewritten as a condition for the mass ratio $r = m_1/m_2$; $r > 1$, since without loss of generality $m_1 > m_2$. Substituting $m_1 = r\,m_2$ directly yields the condition $r/(1+r)^2 < 1/27$. By solving for r, we obtain the result

$$r = \frac{m_1}{m_2} > \frac{1}{2}(25 + 3\sqrt{69}) \approx 25.0. \tag{3.20}$$

In the case of the Sun and Jupiter, (3.20) is certainly satisfied ($m_{\mathrm{J}} \approx 10^{-3} m_{\mathrm{S}}$). In fact, there are numerous asteroids in the vicinity of the Lagrange equilibrium positions, known as Trojan asteroids.[2] In the Lagrange equilibrium position concerning the Earth and the Moon, the positioning of Tracking and Data Relay Satellites is planned.

3.2 Motion of the Moon

The Moon M orbits the Earth E in an orbital plane that is slightly tilted to the ecliptic E (the orbital plane of the Earth around the Sun S). As shown in Figure 3.2, we introduce the auxiliary plane Σ. This allows us to distinguish the following events for the motion of the Moon:

$\mathrm{M} \in \Sigma^+$,	Full Moon (Moon is in opposition to the Sun),
$\mathrm{M} \in \Sigma^-$,	New Moon (Moon is in conjunction with the Sun),
$\mathrm{M} \in E \cap \Sigma^+$,	Lunar Eclipse,
$\mathrm{M} \in E \cap \Sigma^-$,	Solar Eclipse.

One can now also distinguish two periods for the Moon's orbit around the Earth:

$$\begin{aligned} 2\pi\mu \quad &= \text{synodic month} \\ &= \text{period between successive, similarly directed} \\ &\ \text{passages through } \Sigma \text{ (from New Moon to New Moon)} \\ &= 29.53059\,\text{days}; \end{aligned}$$

[2]Without proof: For sufficiently small values of the ratio (3.19), stability also holds outside the linear approximation. The Euler equilibrium positions are, however, obviously unstable.

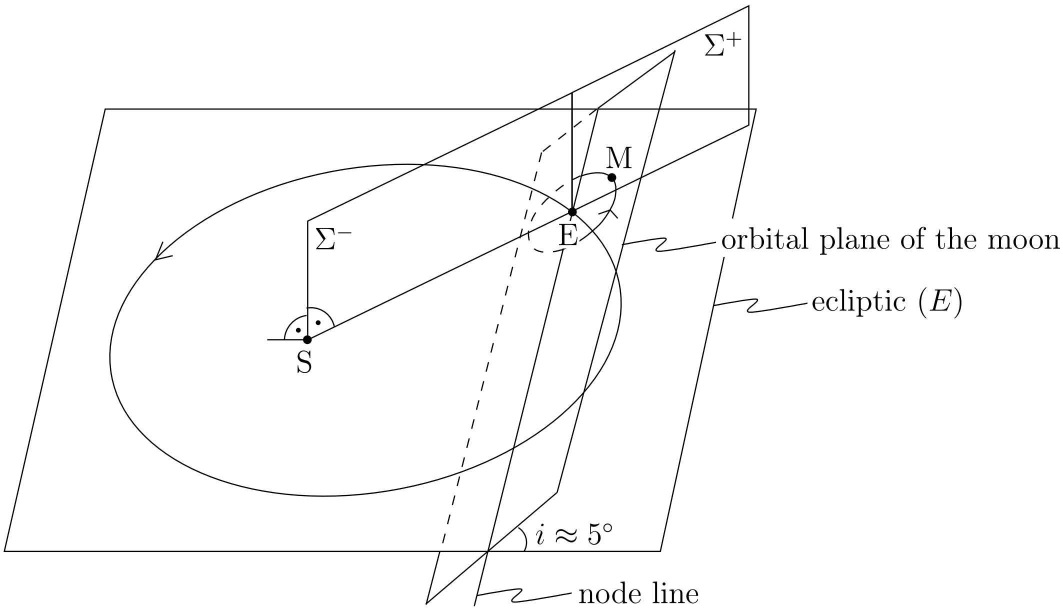


Figure 3.2: Orbit of the Moon relative to the Earth and the Sun. The orbital plane of the Moon is inclined by $i \approx 5°$ with respect to the ecliptic. We additionally define the (auxiliary) plane Σ, which is perpendicular to the ecliptic and connects the Sun with the Earth. We further divide this plane into two areas: The points that lie between the Earth and the Sun (conjunction) are denoted by Σ^-, and the points that are (from the perspective of the Earth) in the opposite direction to the Sun (opposition) are denoted by Σ^+.

$$\begin{aligned} 2\pi\tilde{\mu} \quad &= \text{draconic month} \\ &= \text{period between successive, similarly directed} \\ &\quad\ \text{passages through the ecliptic (from ascending node to ascending node)} \\ &= 27.21222\,\text{days.} \end{aligned}$$

The two periods are related by

$$r = \frac{\mu}{\tilde{\mu}} = 1.08520\ . \tag{3.21}$$

This ratio can mostly be explained purely kinematically. When we project the Moon's orbit onto the ecliptic, we approximately obtain (since i is small) a circular orbit. Comparing the position after a synodic month (after the time $2\pi\mu$), we obtain from Figure 3.3 the relationship

$$(2\pi\mu)\tilde{\omega} = 2\pi + (2\pi\mu)\omega_0 \tag{3.22}$$

with $\tilde{\omega} = \tilde{\mu}^{-1}$, where the additional term on the right side arises due to the Earth's motion around the Sun with angular velocity $\omega_0 = J^{-1}$, $2\pi J = 1\,\text{year} = 365.242\,\text{days}$.

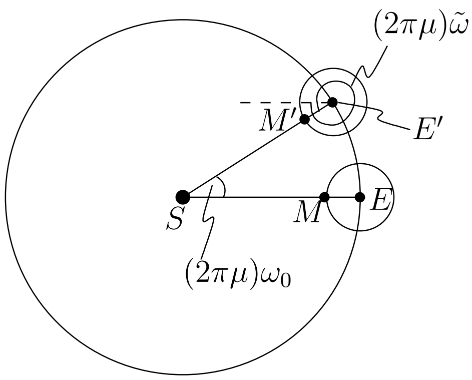


Figure 3.3: Projection of Fig. 3.2 onto the ecliptic. Since i is very small, the motion of the Moon is (almost) a circular orbit.

Thus, we would have

$$\frac{\mu}{\tilde{\mu}} = 1 + \frac{\mu}{J} = 1.08085 \approx 1 + \frac{1}{12.3}\,. \tag{3.23}$$

The deviation from the observed value (3.21) suggests a slow rotation of the Moon's orbital plane. We will return to this later.

The kinematic ratio in (3.23) can also be simply explained as follows: Due to the Earth's motion around the Sun, the plane Σ is 'running away' from the Moon. Since the plane Σ rotates once in a year, we find that during this time the Moon has passed through E once more than Σ, $N_E = N_\Sigma + 1$. By definition, we have $N_E = J/\tilde{\mu}$ and $N_\Sigma = J/\mu$, and therefore

$$\frac{\mu}{\tilde{\mu}} = \frac{N_E}{N_\Sigma} = 1 + \frac{1}{N_\Sigma} = 1 + \frac{\mu}{J}. \tag{3.24}$$

3.2.1 Eclipse Cycles

The ratio (3.21) determines the cadence of solar eclipses (or lunar eclipses): Starting from an exact conjunction SME (solar eclipse), the next solar eclipse can only occur after the time

$$\Delta t = 2\pi\mu p \tag{3.25}$$

with $p \in \mathbb{Z}$, since only then $M \in \Sigma^-$. For a solar eclipse, however, the Moon must additionally be in the ecliptic. Thus, one of the conditions

$$2\pi\mu p = 2\pi\tilde{\mu}q, \qquad \text{(in the same node)}, \tag{3.26}$$

or

$$2\pi\mu p = 2\pi\tilde{\mu}\left(q + \frac{1}{2}\right), \qquad \text{(in the opposite node)}, \tag{3.27}$$

must also be satisfied. Thus, the determination of eclipses reduces to finding rational approximations, so that[3]

$$r = \frac{\mu}{\tilde{\mu}} \approx \frac{q(+\frac{1}{2})}{p}. \tag{3.28}$$

Due to the finite extent of S, E, and M, these conditions need only be satisfied within a 'tolerance' $\Delta \approx \sin i = 0.087$,[4] i.e.,

$$\operatorname{dist}\left(p\,r, \left\{ \begin{array}{c} \mathbb{Z} \\ \mathbb{Z}+\frac{1}{2} \end{array} \right\}\right) < \Delta/2. \tag{3.29}$$

Since $r \approx 1 + 1/12$, see Eq. (3.23), after 12 synodic months (= one lunar year) an eclipse occurs again in the same node. However, this periodicity is not exact; already after

$$\frac{\Delta}{|12r - 13|} \approx 3.88 \tag{3.30}$$

cycles, or the integer part thereof, the tolerance is exhausted and the periodicity breaks down.

On the other hand, starting from an exact conjunction (opposition) after the time $\Delta t = 2\pi\mu(p + 1/2)$, an opposition (conjunction) occurs if

$$2\pi\mu\left(p + \frac{1}{2}\right) = 2\pi\tilde{\mu}q, \qquad \text{(in the same node)}, \tag{3.31}$$

or

$$2\pi\mu\left(p + \frac{1}{2}\right) = 2\pi\tilde{\mu}\left(q + \frac{1}{2}\right), \qquad \text{(in the opposite node)}. \tag{3.32}$$

Thus, in this case, we need a rational approximation

$$r = \frac{\mu}{\tilde{\mu}} \approx \frac{q(+1)}{2p+1} \tag{3.33}$$

with an odd denominator.

Good rational approximations of the (irrational) number $2r$ can be obtained using the method of continued fractions.[5] Among others, we obtain the approximations

$$2r = 2, \frac{11}{5}, \frac{13}{6}, \frac{102}{47}, \frac{191}{88}, \frac{293}{135}, \frac{484}{223}. \tag{3.34}$$

[3] The same result is obtained for determining another lunar eclipse, starting from a perfect opposition.

[4] More information can be found at www.math.ualberta.ca/pi/issue6/page17-18.pdf.

[5] We treat $2r$, since the denominator in (3.28) is generally half-integer. The continued fraction expansion of $2r$ is given by $[2; 5, 1, 6, 1, 1, 1, 1]$. This corresponds to the convergents $2r = 2, (11/5), 13/6, (89/41), 102/47, 191/88, 293/135, 484/223$, see de.wikipedia.org/wiki/Kettenbruch.

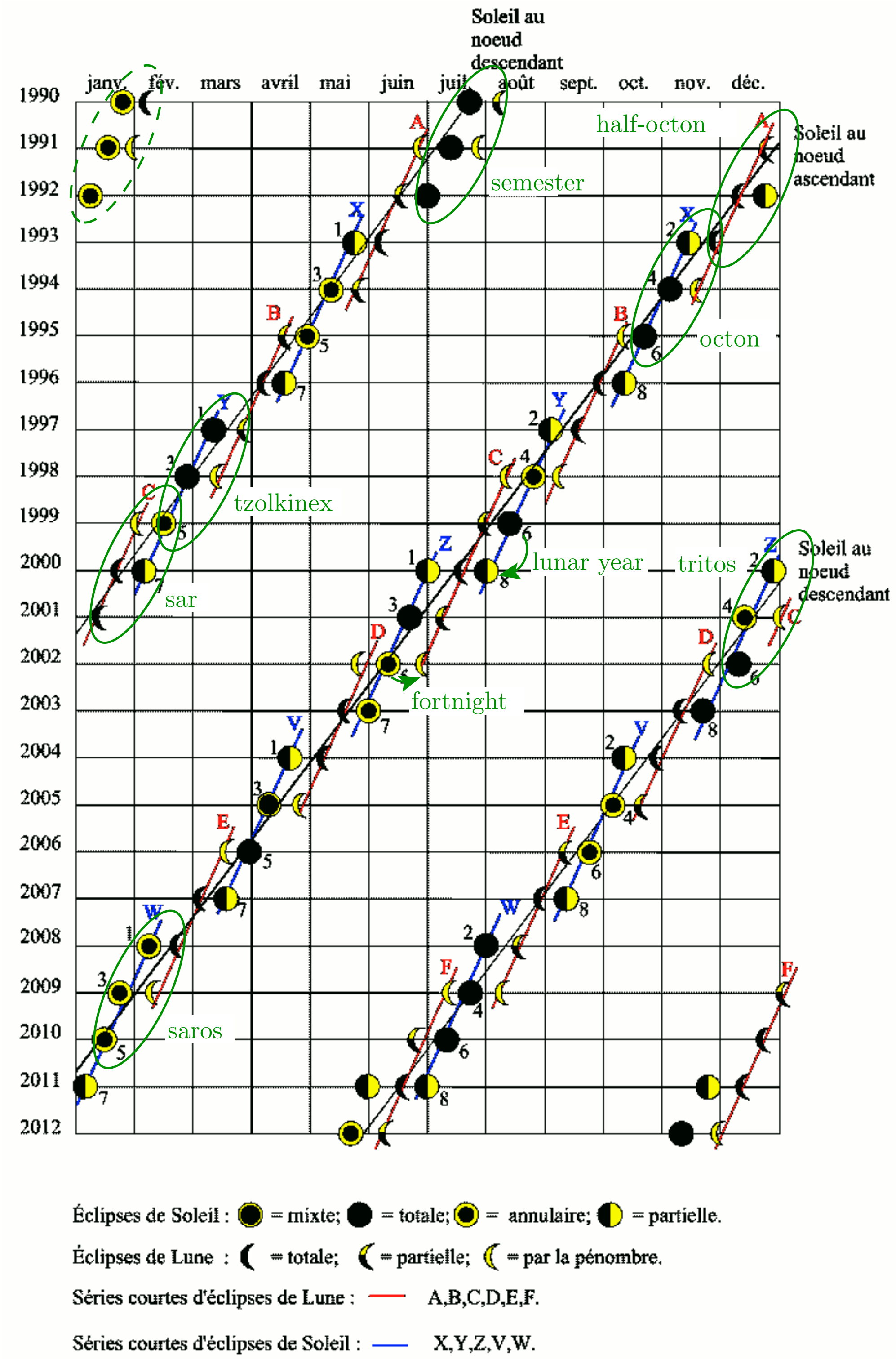


Figure 3.4: Canon of eclipses, adapted from www.imcce.fr (Patrick Rocher); starting from the situation in the upper left (green dashed) with three solar and two lunar eclipses, the same configuration (with a small error) is obtained after certain times (green bordered).

With (3.28) we obtain the following eclipse cycles (from solar eclipse to solar eclipse, or from lunar eclipse to lunar eclipse):

q:	p:	Name:	Δt/year	q:	p:	Name:	Δt/year
Moon in the same node:				*Moon in the opposite node:*			
13	12	lunar year	0.970	6	6	semester	0,485
51	47	octon	3.800	95	88	tzolkinex	7,115
242	223	saros	18.030	146	135	tritos	10,915

Additionally, with (3.28) for the transition from a solar eclipse to a lunar eclipse (or vice versa), we obtain the cycles[6]

q:	p:	Name:	Δt/year	q:	p:	Name:	Δt/year
Moon in the same node:				*Moon in the opposite node:*			
121	111	sar	9.015	0	0	fortnight	0,040
		(half saros)		25	23	half octon	1.900

In Fig. 3.4 we have illustrated the various cycles. The most famous cycle is the *Saros*(cycle) ($223r = 241.999 \approx 242$) with the period[7]

$$\Delta t = (2\pi\mu)223 \equiv 1 \ \textbf{Saros} \approx 18\,\text{years} + 11\,\text{days} + 8\,\text{hours}, \tag{3.35}$$

which was already known to the Chaldeans. The cycle lasts over 1200 years (over 70 eclipses) until it breaks down. The total solar eclipse that was visible on August 11, 1999 in southern Germany, therefore repeated itself on August 21, 2017, 8 hours later (i.e., over the USA).[8] Due to the time shift, the next solar eclipse in Europe (in the same cycle) will not occur again until 3 Saros = 1 *Exeligmos* on September 12, 2053.

3.2.2 Restricted Three-Body Problem

We now want to determine the important ratio (3.21) for the determination of eclipses as accurately as possible theoretically as a function of μ/J. To do this, we treat the system S, E, M (with masses $M \gg m \gg m'$) as a restricted three-body problem, see Chapter 3.1; that is, we consider the Moon as a body in the external gravitational field of the Sun and Earth. The Earth describes a circular orbit (radius R) around the Sun. Conversely, the Sun describes a circular orbit in the rotating reference frame

[6] Only approximations $r \approx q/p$ with p odd yield an entry.

[7] In this formula (and only here), 18 years refers to the time of $(18 \cdot 365 + 4)$ days. The 4 additional days correspond to the (minimal) number of leap years in the 18 years.

[8] The corresponding events are cataloged as solar eclipse No. 21 and 22 in the 145th Saros cycle. The discrepancy $11 + 11 = 22 \neq 21$ in the date arises because there are 5 leap years (2000, 2004, 2008, 2012, 2016) in the 18 years.

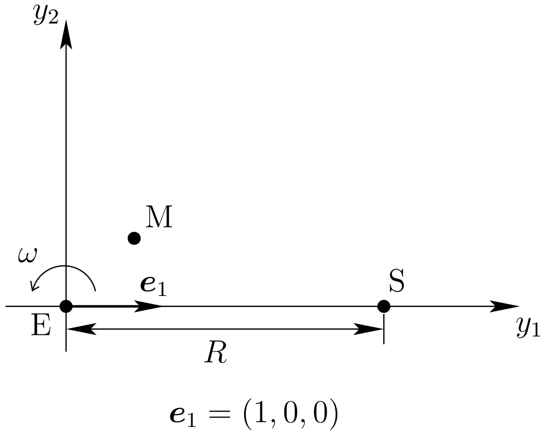


Figure 3.5: Accelerated coordinate system for the restricted three-body problem Sun (S), Earth (E), Moon (M). The system rotates with angular velocity $\omega = J^{-1}$ around E.

of the Earth with origin E. We choose the units[9]

$$\omega = J = 1, \qquad m = 1, \qquad G = 1. \tag{3.36}$$

Thus, the synodic month

$$\mu \approx \frac{29.53\,\text{days}}{1\,\text{year}} = 0.08085\,. \tag{3.37}$$

In the 2-body system Sun-Earth, the gravitational and centrifugal forces must cancel as in (3.2),

$$G\frac{mM}{R^2} = \frac{mM}{M+m}R\omega^2 \approx mR\omega^2. \tag{3.38}$$

In our units, we thus obtain the relationship

$$M = R^3. \tag{3.39}$$

We choose the accelerated coordinate system $\boldsymbol{y}$, as shown in Figure 3.5. The following accelerations act on the Moon (forces divided by m'). The *gravitational accelerations* from the Earth

$$\boldsymbol{G}_E = \frac{\partial}{\partial \boldsymbol{y}}\frac{1}{|\boldsymbol{y}|} = -\frac{1}{|\boldsymbol{y}|^3}\boldsymbol{y}\,, \tag{3.40}$$

[9]Thus, the three mechanical units are given by $2.2 \cdot 10^6$ km, 58 days and $6.0 \cdot 10^{24}$ kg.

and from the Sun

$$\begin{aligned}\boldsymbol{G}_S &= M\,\frac{\partial}{\partial \boldsymbol{y}}\,\frac{1}{|R\,\boldsymbol{e}_1 - \boldsymbol{y}|} \\ &= M\Big(\frac{\boldsymbol{e}_1}{R^2} + \frac{3(\boldsymbol{e}_1\cdot\boldsymbol{y})\boldsymbol{e}_1 - \boldsymbol{y}}{R^3} + O(y^2)\Big) \\ &= R\,\boldsymbol{e}_1 + 3\,y_1\,\boldsymbol{e}_1 - \boldsymbol{y} + \ldots\,. \end{aligned} \tag{3.41}$$

The linearization in $\boldsymbol{y}/R$ is appropriate, since for the Moon's orbit $|\boldsymbol{y}|/R \approx 1/390$ holds. The first term compensates exactly the *leading acceleration*[10] (see (1.54))

$$-\boldsymbol{a} = -\boldsymbol{\omega}\times(\boldsymbol{\omega}\times(-R\boldsymbol{e}_1)) = -R\,\boldsymbol{e}_1\,. \tag{3.42}$$

Furthermore, the *centrifugal acceleration*

$$\boldsymbol{Z} = (y_1, y_2, 0), \tag{3.43}$$

compensates the $1,2$-components of $-\boldsymbol{y}$ in (3.41). Finally, the *Coriolis acceleration* is given by

$$\boldsymbol{C} = 2(\dot{y}_2, -\dot{y}_1, 0)\,. \tag{3.44}$$

We thus obtain the equations of motion of the Moon (with $r = |\boldsymbol{y}|$)

$$\begin{aligned}\ddot{y}_1 - 2\dot{y}_2 &= \Big(3 - \frac{1}{r^3}\Big)\,y_1\,, \\ \ddot{y}_2 + 2\dot{y}_1 &= -\frac{1}{r^3}y_2\,, \\ \ddot{y}_3 &= -\Big(\frac{1}{r^3}+1\Big)y_3\,. \end{aligned} \tag{3.45}$$

Since the inclination of the Moon's orbit with respect to the ecliptic is small ($i \approx 5°$), we first determine periodic orbits in the 12-plane and then, as their perturbation, the actual Moon's orbit.

3.2.3 Periodic Orbits

Let $y_3 \equiv 0$: We construct a 1-parameter family of periodic solutions of (3.45), which rotate counterclockwise around $\boldsymbol{y} = 0$, see Fig. 3.6. We obtain exactly one orbit (up to a shift in time) through every point of the (y_1, y_2)-plane near $\boldsymbol{y} = 0$. Initially, it is evident from (3.45) that with each solution $(y_1(t), y_2(t))$, the pairs $(y_1(-t), -y_2(-t))$ and $(-y_1(-t), y_2(-t))$ are also solutions. They arise from time reversal and subsequent reflection at the 1- or 2-axis.

For every starting point $(y_1(0) > 0, y_2(0) = 0)$ on the 1-axis, one can always choose a vertical initial velocity $(\dot{y}_1(0) = 0, \dot{y}_2(0) > 0)$ such that the orbit intersects the 2-axis

[10] The coordinate origin, which is located in the Earth, is moved around the Sun with angular velocity ω.

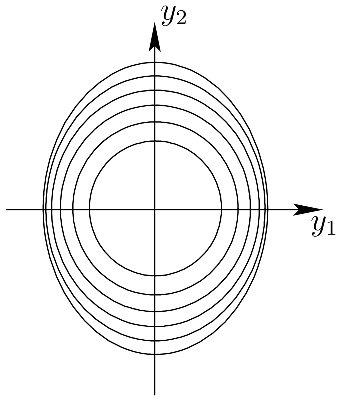


Figure 3.6: Family of possible Moon orbits in the ecliptic with $y_3 = 0$.

(at time $\pi\mu/2$) horizontally, i.e., $y_1(\pi\mu/2) = 0$, $\dot{y}_2(\pi\mu/2) = 0$ with $\mu > 0$. From the aforementioned symmetries, we obtain the relationships

$$\begin{aligned} y_1(t) &= y_1(-t)\,, & y_2(t) &= -y_2(-t)\,, \\ y_1(\pi\mu/2 + t) &= -y_1(\pi\mu/2 - t)\,, & y_2(\pi\mu/2 + t) &= y_2(\pi\mu/2 - t)\,. \end{aligned} \tag{3.46}$$

Thus, with the uniqueness of the solution for given initial conditions, one can extend any solution from $t \in [0, \pi\mu/2]$ to all of $\mathbb{R}$. In particular, the orbits are periodic, $\boldsymbol{y}(t + 2\pi\mu) = \boldsymbol{y}(t)$, and oval (mirror-symmetric with respect to the 1- and 2-axes).

We choose the orbital period $2\pi\mu$ as the parameter and introduce the new time unit

$$\tau = \frac{t}{\mu} \tag{3.47}$$

so that the synodic month (in units of τ) $= 2\pi$. Then, (3.45) with $' = d/d\tau$ becomes

$$\begin{aligned} y_1'' - 2\mu y_2' &= -\frac{\mu^2}{r^3} y_1 + \underline{3\mu^2 y_1}\,, \\ y_2'' + 2\mu y_1' &= -\frac{\mu^2}{r^3} y_2\,, \\ y_3'' &= -\mu^2 \Big(\frac{1}{r^3} + 1\Big) y_3\,. \end{aligned} \tag{3.48}$$

Without the underlined term (tidal forces), which we treat as a perturbation, the first two equations

$$(y_1 + iy_2)'' + 2i\mu(y_1 + iy_2)' = -\frac{\mu^2}{r^3}(y_1 + iy_2).$$

The sought (now 2π-)periodic solution is a circular orbit

$$y_1(\tau) + iy_2(\tau) = r_0 e^{i\tau}\,, \qquad y_3(\tau) = 0 \tag{3.49}$$

with

$$r_0 = \frac{\mu^{\frac{2}{3}}}{(1 + 2\mu)^{\frac{1}{3}}}\,. \tag{3.50}$$

Subsequently, we can now justify the neglect of the tidal perturbation in (3.48); if we substitute the solution (3.49) into the correction term, it is smaller by a factor of μ^2 than the leading term.

We now want to consider the influence of tidal forces in perturbation theory in $\mu \ll 1$. Instead of $\boldsymbol{y}$, we introduce new coordinates $(z, \zeta) \in \mathbb{C} \times \mathbb{R}$ through

$$\begin{aligned} y_1 + iy_2 &= \mu^{2/3}(1+2\mu)^{-1/3} e^{i\tau} z\,, \\ y_3 &= \mu^{2/3}(1+2\mu)^{-1/3}\zeta \end{aligned}$$

('Variation of Constants'). They are adapted to the above approximate solution (3.49) $(z(\tau) = 1, \zeta(\tau) = 0)$ and correspond to a reference frame that additionally rotates with one revolution per month around the y_3-axis. The equations (3.48) now read

$$z'' + 2i(1+\mu)z' = (1+2\mu)\Big(1 - \frac{1}{\rho^3}\Big)z + \underline{\frac{3}{2}\mu^2\left(z + e^{-2i\tau}z^*\right)}\,, \tag{3.51}$$

$$\zeta'' = -\Big(\frac{1+2\mu}{\rho^3} + \mu^2\Big)\zeta\,, \tag{3.52}$$

with $\rho^2 = |z|^2 + \zeta^2$. Of course, without the perturbation term, the previous approximate solution (i.e., $z(\tau) = 1$, $\zeta(\tau) = 0$) is still a solution.

We now assume the sought 2π-periodic solution as a power series in μ [11]

$$z(\tau) = 1 + \sum_{k=2}^{\infty} \mu^k z_k(\tau) \;\equiv\; 1 + \tilde{z}(\tau) \tag{3.53}$$

with

$$\begin{aligned} z_k(\tau + 2\pi) &= z_k(\tau) \\ z_k(\tau)^* &= z_k(-\tau) \\ z_k\Big(\frac{\pi}{2} + \tau\Big)^* &= z_k\Big(\frac{\pi}{2} - \tau\Big)\,. \end{aligned} \tag{3.54}$$

To calculate $z_2(\tau)$, we first develop $z\rho^{-3}$ in μ. In leading order, $\zeta = 0$, and therefore we obtain

$$\begin{aligned} \rho^2 &= |z|^2 = (1+\tilde{z})(1+\tilde{z}^*) = 1 + (z_2 + z_2^*)\mu^2 + O(\mu^4) \\ \rho^{-3} &= 1 - \frac{3}{2}(z_2 + z_2^*)\mu^2 + O(\mu^4) \\ z\rho^{-3} &= 1 + \tilde{z} - \frac{3}{2}(\tilde{z} + \tilde{z}^*) + O(\mu^4) = 1 - \Big(\frac{1}{2}z_2 + \frac{3}{2}z_2^*\Big)\mu^2 + O(\mu^4)\,. \end{aligned} \tag{3.55}$$

Comparing the terms $\propto \mu^2$ in (3.51) yields

$$z_2'' + 2iz_2' = \frac{3}{2}(z_2 + z_2^*) + \underline{\frac{3}{2}(1 + e^{-2i\tau})}\,. \tag{3.56}$$

[11] The perturbation term is $O(\mu^2)$, so the expansion (3.53) starts with $k = 2$; (3.54) follows from (3.46).

The ansatz $z_2(\tau) = a_0 + a_- e^{-2i\tau} + a_+ e^{2i\tau}$ (with $a_{0,\pm} \in \mathbb{R}$ due to (3.54)) yields[12]

$$z_2(\tau) = -\frac{1}{2} - \frac{19}{16} e^{-2i\tau} + \frac{3}{16} e^{2i\tau} \,. \tag{3.57}$$

Equation (3.57) describes, in leading order, the oval shape of the periodic orbit. The difference of the 'semi-axes' is given by

$$\Delta = \mu^2 \Big(z_2 \left(\frac{\pi}{2} \right) - z_2(0) \Big) \approx 2\mu^{2+2/3} \,. \tag{3.58}$$

For $\mu = 0.08085$, this is approximately $\Delta = 1.3 \cdot 10^{-2}$ (with $z = 1$ the distance from Earth to Moon). In reality, the change in distance is greater, and the Moon's orbit is not perfectly periodic.

3.2.4 Variations of the Orbit

The ansatz for orbits that are close to the periodic solution $(z(\tau), \zeta(\tau) \equiv 0)$ is

$$\big(z(\tau) + \hat{z}(\tau), \hat{\zeta}(\tau) \big) \,, \tag{3.59}$$

with $\hat{z}$ and $\hat{\zeta}$ small. We substitute this ansatz into the equations of motion (3.51) and (3.52), and linearize in $\hat{z}, \hat{\zeta}$. This yields the *variation equations* of the periodic solution.

Since $\frac{\partial \rho}{\partial \zeta}\big|_{\zeta=0} = 0$, the variation equation from (3.51) contains no $\hat{\zeta}$, i.e., only $\hat{z}$ appears. In particular, this equation is solved by $\hat{z} \equiv 0$ and we can consider a variation that lies above the periodic orbit. The variation of (3.52) reads

$$\hat{\zeta}'' = -\Big(\frac{1 + 2\mu}{\rho^3} + \mu^2 \Big) \hat{\zeta} \,, \tag{3.60}$$

where $\rho = \rho|_{\zeta=0}$ refers only to the periodic orbit. Substituting (3.57) into (3.55) leads to

$$\rho^{-3} = 1 + \frac{3}{2}\mu^2 (1 + e^{2i\tau} + e^{-2i\tau}) + O(\mu^4) \,. \tag{3.61}$$

Thus, (3.60) becomes

$$\begin{aligned} \hat{\zeta}'' &= -\Big(1 + 2\mu + \frac{3}{2}\mu^2 (1 + 2\cos 2\tau) + \mu^2 + \dots \Big) \hat{\zeta} \\ &= -\Big(1 + 2\mu + \frac{5}{2}\mu^2 + 3\mu^2 \cos 2\tau + \dots \Big) \hat{\zeta} \,. \end{aligned} \tag{3.62}$$

If we initially neglect the term $\propto 3\mu^2 \cos 2\tau$, we obtain the differential equation of a harmonic oscillator in $\hat{\zeta}$ with the angular frequency

$$r = \Big(1 + 2\mu + \frac{5}{2}\mu^2 \Big)^{1/2} = 1 + \mu + \frac{3}{4}\mu^2 + \dots \,. \tag{3.63}$$

[12] The solution $z_2^0(\tau) = a e^{i\tau} - 3a^* e^{-i\tau}$ of the homogeneous part of the equation (3.56) does not satisfy (3.54). It therefore does not contribute.

in τ-units (with $\mu = 1$). In any units, this is precisely the ratio (3.21) of the synodic to the draconic month. In first order, we have $r = 1 + \mu$ in agreement with (3.23). For the Moon with $\mu = 0.08085$ (see (3.37)), we obtain (including the second-order correction)

$$r = 1.08575 \tag{3.64}$$

in very good agreement with (3.21).

The term $\propto 3\mu^2 \cos 2\tau$ implies that $\hat{\zeta}(\tau)$ is no longer periodic, but has an average frequency r. Due to the oscillatory behavior, r is only influenced by this in order μ^4. By considering further terms of the perturbation series for ζ, as well as for $\hat{z}(\tau) \neq 0$, the agreement with the observed value (3.21) improves even further; for example, in the next order one finds

$$r = 1.08517\,. \tag{3.65}$$

Chapter 4

Lagrangian Mechanics

In this chapter, we want to reformulate Newton's equations of motion into the Lagrangian formulation. This formulation has several significant advantages. Conceptually, *all* coordinate systems are treated equally, meaning that the formalism remains valid even in accelerated and curved reference frames. This leads to the automatic consideration of fictitious forces and constraints. Another advantage is that the Lagrangian function is a scalar, so one no longer has to worry about the direction of the forces. In a certain sense, this chapter is the centerpiece of the lecture. It is therefore worthwhile to study the material carefully. In particular, we will derive two of the most important results, the 'Hamilton's principle' and the 'Noether's theorem'.

4.1 Configuration Space and Position Coordinates

The reformulation of Newton's equations of motion is particularly useful for systems with (holonomic) *constraints*. In this perspective, part of the forces is described indirectly by restricting the motion possibilities of the mass points.

We will introduce the relevant terms using three examples:

(A) The *mathematical pendulum:* The motion of the mass point swinging in a vertical plane is restricted by the string maintaining a fixed distance from the suspension point (*constraint*). The permissible positions of the pendulum are then given by

$$\boldsymbol{x} = l(t)\boldsymbol{e} \equiv \boldsymbol{x}(\boldsymbol{e}, t) \tag{4.1}$$

with $l(t)$ being the (time-dependent) length of the string. The *configuration space* of the pendulum is S^1, and the motion of the pendulum is determined by a function $\mathbb{R} \ni t \mapsto \boldsymbol{e}(t) \in S^1$. On the configuration space, the *position coordinate* $\theta \in K \subset \mathbb{R}$ can be introduced with $\boldsymbol{e}(\theta) = (\cos\theta, \sin\theta)$. The open set K is called a *chart*. The mapping $\boldsymbol{e}(\theta)$ is not bijective, as $\boldsymbol{e}(2\pi) = \boldsymbol{e}(0)$. To represent the

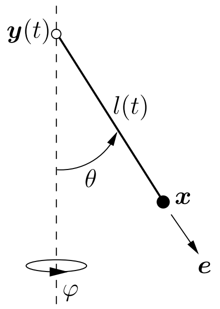


Figure 4.1: Spherical pendulum with variable suspension point $\boldsymbol{y}(t)$ and length $l(t)$.

entire S^1 bijectively, at least two charts (an atlas) are needed. However, this subtlety is not important in this course.

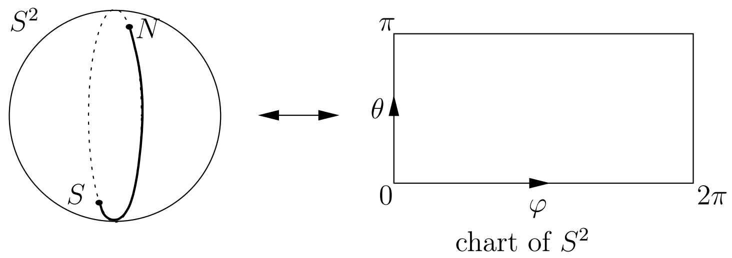
(B) The *spherical pendulum:* The suspension point $\boldsymbol{y}(t)$ and the pendulum length $l(t)$ are prescribed functions of time t, see Fig. 4.1. The permissible positions of the pendulum under these constraints are

$$\boldsymbol{x} = \boldsymbol{y}(t) + l(t)\boldsymbol{e} \equiv \boldsymbol{x}(\boldsymbol{e}, t)\,, \tag{4.2}$$

where $\boldsymbol{e}$ is a point on the unit sphere S^2 (configuration space of the pendulum). The motion of the pendulum is given by a function $t \mapsto \boldsymbol{e}(t) \in S^2$. Now, position coordinates can be introduced on S^2; for example, the spherical coordinates θ, φ with

$$\boldsymbol{e}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,. \tag{4.3}$$

This maps the configuration space to a piece of $\mathbb{R}^2$:



The mapping is not bijective on the boundary of the rectangle (or on the prime meridian of S^2).

(C) A *pearl on a rotating wire:* On a parabolically curved wire, which rotates around the 3-axis with angular velocity ω, sits a pearl (mass point). The two constraints are satisfied by the parametrization

$$\boldsymbol{x} = R(\omega t)\Big(r\boldsymbol{e}_1 + \frac{1}{2}\alpha r^2\boldsymbol{e}_3\Big) = R(\omega t)\Big(r, 0, \frac{1}{2}\alpha r^2\Big) \equiv \boldsymbol{x}(r, t) \tag{4.4}$$

with $R(\varphi)$ being the rotation around the 3-axis by angle φ and $r \in \mathbb{R}$ being the position coordinate. The configuration space of the problem is $\{\boldsymbol{x}(r,t) \mid r \in \mathbb{R}\}$, i.e., isomorphic to $\mathbb{R}$.

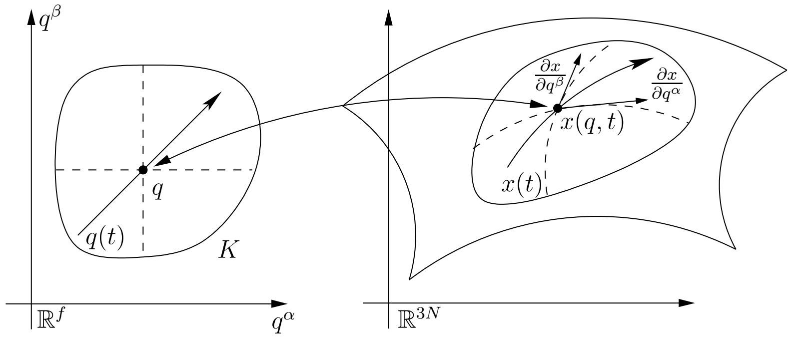


Figure 4.2: Mapping from the position coordinates (left) to the Cartesian coordinates (right). The tangential vectors $\partial x/\partial q^\alpha$ locally span the vector space of virtual displacements in $\mathbb{R}^{3N}$.

In general, we have a system of N particles, whose configuration space in Cartesian coordinates $x = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in \mathbb{R}^{3N}$ at time t forms a smooth f-dimensional surface in $\mathbb{R}^{3N}$; constraints of this type are called "*holonomic constraints*". The system then has f *degrees of freedom*.[1] We describe the system using the position coordinates $q = (q^1, \ldots, q^f)$; that is, we have (locally) a chart $q \in K \subset \mathbb{R}^f$ and (for each $t \in \mathbb{R}$) a mapping

$$x\colon K \to \mathbb{R}^{3N}, \text{ with } q \mapsto x(q,t)\,, \tag{4.5}$$

which is differentiable in (q,t). We additionally need the tangential mapping to have rank f; that is, the vectors

$$\frac{\partial x}{\partial q^\alpha} \in \mathbb{R}^{3N}\,, \qquad (\alpha = 1, \ldots, f)\,, \tag{4.6}$$

should be linearly independent. Thus, the tangential vectors (4.6) at each point (q,t) form an f-dimensional vector space, see Fig. 4.2.

Any permissible (compatible with the constraints) motion $x(t)$ of the system (within the chart K) is then induced by a function $t \mapsto q(t) \in K$ via

$$x(t) \equiv x(q(t), t) \tag{4.7}$$

To formulate the equations of motion of a system in the position coordinates $q = (q^1, \ldots, q^f)$, we need the following terms.

Velocities, Kinetic Energy: With (4.7), the permissible velocities at time t at position q are given by

$$\dot{x} = \sum_{\alpha=1}^{f} \frac{\partial x}{\partial q^\alpha}(q,t)\,\dot{q}^\alpha + \frac{\partial x}{\partial t}(q,t) \equiv \dot{x}(q,\dot{q},t) \tag{4.8}$$

[1] In the example B, we have $N = 1$ and $f = 2$. Thus, we have two degrees of freedom.

with arbitrary *generalized velocities* $\dot{q} = (\dot{q}^1, \dots, \dot{q}^f) \in \mathbb{R}^f$. The velocities $\dot{x} \in \mathbb{R}^{3N}$ thus lie (necessarily) in the vector space spanned by $\partial x/\partial q^\alpha$ and $\partial x/\partial t$, but not necessarily in the tangent space.

With (4.8), $\dot{x}$ becomes a function of the *independent* variables[2]

$$(q, \dot{q}, t) \in K \times \mathbb{R}^f \times \mathbb{R}.$$

Thus, the kinetic energy T can also be expressed as a new function

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i^2 \equiv T(q, \dot{q}, t) \tag{4.9}$$

Due to (4.8), we have

$$\frac{\partial \dot{x}}{\partial \dot{q}^\alpha} = \frac{\partial x}{\partial q^\alpha} \tag{4.10}$$

and thus

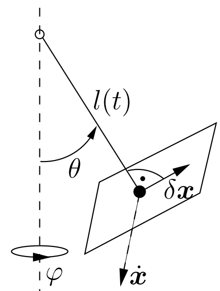
$$\frac{\partial T}{\partial \dot{q}^\alpha} = \sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \frac{\partial \dot{\boldsymbol{x}}_i}{\partial \dot{q}^\alpha} = \sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \frac{\partial \boldsymbol{x}_i}{\partial q^\alpha}. \tag{4.11}$$

Virtual Displacements: At a point (q, t), the (directional) *derivative* of the function $x(q, t)$ at *fixed* t is a linear mapping $\mathbb{R}^f \ni \delta q \mapsto \delta x \in \mathbb{R}^{3N}$, defined by

$$\delta x = \sum_{\alpha=1}^{f} \frac{\partial x}{\partial q^\alpha}\,\delta q^\alpha. \tag{4.12}$$

The permissible vectors of the image δx are the tangential vectors at the point (q, t). The permissible δx are also called the space of *virtual displacements* of the system from the position $x(q, t)$ at time t.

For examples A and C, the configuration space is one-dimensional. The space of virtual displacements is therefore simply the tangent to the circle (for A) or to the parabola (for C) at the point $x(q, t)$. For the spherical pendulum (example B), $\delta \boldsymbol{x}$ corresponds to a variation of the position coordinates θ, φ at fixed time and is thus perpendicular to the string $\boldsymbol{e}$.



The comparison of (4.12) with (4.8) shows that $\dot{x}$ is generally *not* a virtual displacement under time-dependent constraints.

[2] In the Lagrangian formalism, it is essential to consider $(q, \dot{q}, t)$ as independent variables. Here, $\dot{q}$ according to (4.8) are the coordinates of the velocity vector in the tangent space while (q, t) determine the point x at which the tangent space is formed.

Virtual Work, Generalized Forces and Momenta: Let $(\boldsymbol{F}_1, \ldots, \boldsymbol{F}_N)$ be the forces acting on the particles at position $x(q,t)$. Their *virtual work* is defined as[3]

$$\delta A = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{x}_i. \tag{4.13}$$

With (4.12), we can rewrite this relationship as

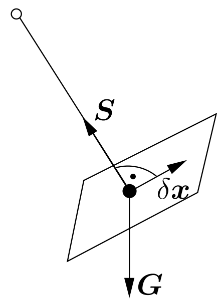
$$\delta A = \sum_{\alpha=1}^{f} K_\alpha \delta q^\alpha \equiv \langle K, \delta q \rangle \tag{4.14}$$

with the coefficients (*generalized forces*)

$$K_\alpha = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{x}_i}{\partial q^\alpha}(q,t). \tag{4.15}$$

By construction, the expression (4.14) does not depend on the choice of position coordinates.

The forces arising from the restriction (4.5) of the permissible configurations $x \in R^{3N}$ of the system are called *constraint forces*, while the remaining forces are called *driving forces*. For the example B of the spherical pendulum, the tension in the string $\boldsymbol{S}$ is a constraint force and the weight force $\boldsymbol{G}$ is a driving force. Note that $\boldsymbol{S} \perp \delta\boldsymbol{x}$. The *principle of d'Alembert* postulates that this statement generally holds: 'Constraint forces are perpendicular to the virtual displacements and do *not perform virtual work*.' Thus, constraint forces do not contribute to the generalized forces K_α and do not need to be calculated in the Lagrangian formalism. Although constraint forces do not perform virtual work, they can perform *real work* under time-dependent constraints with $\boldsymbol{S} \cdot \dot{\boldsymbol{x}} \neq 0$ (even though $\boldsymbol{S} \cdot \delta\boldsymbol{x} = 0$).



Analogous to the generalized forces K_α, one also defines the *generalized momenta* p_α via

$$\sum_{i=1}^{N} \underbrace{m_i \dot{\boldsymbol{x}}_i}_{=\boldsymbol{p}_i} \cdot \delta \boldsymbol{x}_i = \sum_{\alpha=1}^{f} p_\alpha \, \delta q^\alpha \equiv \langle p, \delta q \rangle, \tag{4.16}$$

i.e.,

$$p_\alpha = \sum_{i=1}^{N} \boldsymbol{p}_i \cdot \frac{\partial \boldsymbol{x}_i}{\partial q^\alpha} = \frac{\partial T}{\partial \dot{q}^\alpha}. \tag{4.17}$$

The last equation follows from (4.11). With (4.9), we thus obtain the relationship $p_\alpha = p_\alpha(q, \dot{q}, t)$.

[3]The virtual force δA is a mapping of the virtual displacement to the real numbers. Note that δA is generally not a derivative of a function $A(q,t)$; see also Chapter 4.3.

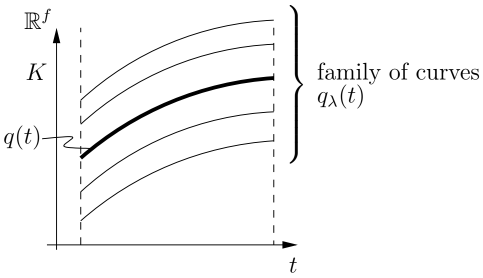


Figure 4.3: The path $q(t)$ is embedded in a family $q_\lambda(t)$ of curves such that $q_0(t) = q(t)$ (bold line).

Variation of a Path: As a final preparation, we introduce the concept of the variation of a path. The variation of a path generalizes the concept of directional derivative (partial derivative with respect to a variable) to the 'derivative' with respect to a function $\delta q^\alpha(t)$.

For a path $q(t)$, we form an arbitrary 1-parameter family of curves $q_\lambda(t)$ such that $q_0(t) = q(t)$, see Fig. 4.3. The *variation* of a function $F(q, \dot{q}, t)$ is then defined as

$$\delta F(t) = \frac{\partial}{\partial \lambda} F(q_\lambda(t), \dot{q}_\lambda(t), t)\Big|_{\lambda=0} \tag{4.18}$$

with

$$q^\alpha_\lambda(t) = q^\alpha(t) + \lambda\, \delta q^\alpha(t). \tag{4.19}$$

The f functions $\delta q^\alpha \colon \mathbb{R} \to K$ are freely selectable. They correspond to the variations of $q^\alpha(t)$. We will later need the relation

$$\delta \dot{q}^\alpha(t) = \frac{\partial}{\partial \lambda} \dot{q}^\alpha_\lambda(t)\Big|_{\lambda=0} = \frac{d}{dt}\, \delta q^\alpha(t), \tag{4.20}$$

which links the variations $\delta \dot{q}^\alpha$ with the variations δq^α.

It is easy to verify that the variation of the position δx corresponds exactly to the virtual displacements from (4.12). It follows that a variation δF of any function $F(x, t)$ is a virtual displacement.

4.2 Equations of Motion

With these preliminary remarks, we can formulate the equations of motion in a coordinate-free manner, so that they are valid in any reference frame.

Let $q(t)$ be a *mechanical path*, i.e., $x(q(t), t)$ is a solution of Newton's equations of motion

$$m_i \ddot{\boldsymbol{x}}_i = \boldsymbol{F}_i. \tag{4.21}$$

Then, for *any* variation of a mechanical path, we have

$$\begin{aligned}\frac{d}{dt}\langle p, \delta q\rangle &\overset{(4.16)}{=} \frac{d}{dt}\sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \delta \boldsymbol{x}_i = \sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \delta \dot{\boldsymbol{x}}_i + \sum_{i=1}^{N} m_i \ddot{\boldsymbol{x}}_i \cdot \delta \boldsymbol{x}_i \\ &= \delta\left(\frac{1}{2}\sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i^2\right) + \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{x}_i. \end{aligned} \tag{4.22}$$

Thus, we have transformed Newton's equations (4.21) for arbitrary position coordinates q into the form

$$\frac{d}{dt}\langle p, \delta q\rangle = \delta T + \delta A \tag{4.23}$$

The forces are now only contained in the form of virtual work. Since the constraint forces do not perform virtual work, it suffices to determine the driving forces, so that we will henceforth denote the driving forces by $\boldsymbol{F}_i$ only.

Starting from (4.23), we immediately obtain the explicit form

$$\sum_{\alpha=1}^{f}(\dot{p}_\alpha \delta q^\alpha + \underline{p_\alpha \delta \dot{q}^\alpha}) = \sum_{\alpha=1}^{f}\left(\frac{\partial T}{\partial q^\alpha}\delta q^\alpha + \underline{\frac{\partial T}{\partial \dot{q}^\alpha}\delta \dot{q}^\alpha} + K_\alpha \delta q^\alpha\right) \tag{4.24}$$

of the equations of motion. The underlined terms cancel out according to (4.17). Since for fixed t the variations $\delta q^\alpha(t)$ can be chosen arbitrarily, the prefactors must vanish, i.e.,

$$\dot{p}_\alpha = \frac{\partial T}{\partial q^\alpha} + K_\alpha. \tag{4.25}$$

With the relation (4.17), we thus obtain the equations of motion

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\alpha} - \frac{\partial T}{\partial q^\alpha} = K_\alpha, \qquad (\alpha = 1, \ldots, f). \tag{4.26}$$

for arbitrary position coordinates $q = (q^1, \ldots, q^f)$. These are f second-order differential equations for the functions $q^\alpha(t)$ given the generalized forces $K_\alpha = K_\alpha(q, \dot{q}, t)$. Note that here and in the following, the total time derivative (d/dt) refers to taking the derivative along a path $q(t)$.

Remark: The equations of motion (4.26) can also be derived without the concept of virtual displacements. Using (4.9) and (4.11), we find

$$\begin{aligned}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\alpha} &= \sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \frac{d}{dt}\frac{\partial \boldsymbol{x}_i}{\partial q^\alpha} + \sum_{i=1}^{N} m_i \ddot{\boldsymbol{x}}_i \cdot \frac{\partial \boldsymbol{x}_i}{\partial q^\alpha}, \\ \frac{\partial T}{\partial q^\alpha} &= \sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \frac{\partial \dot{\boldsymbol{x}}_i}{\partial q^\alpha} = \sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \frac{d}{dt}\frac{\partial \boldsymbol{x}_i}{\partial q^\alpha}. \end{aligned} \tag{4.27}$$

Together with (4.15) and (4.21), this directly leads to the final result (4.26).

4.3 Lagrangian Systems

The Lagrangian formalism becomes particularly elegant if the driving forces $\boldsymbol{F}_i$ possess a potential $V(x,t)$ with $\boldsymbol{F}_i = -\partial_{\boldsymbol{x}_i} V(x,t)$; such a force field is called *conservative.* Then it holds

$$\delta A = -\sum_{i=1}^{N} \frac{\partial V}{\partial \boldsymbol{x}_i} \cdot \delta \boldsymbol{x}_i = -\sum_{\alpha=1}^{f} \frac{\partial V}{\partial q^\alpha} \delta q^\alpha, \tag{4.28}$$

so that $-\delta A$ is the variation of $V(q,t) \equiv V(x(q,t),t)$. It immediately follows

$$K_\alpha = -\frac{\partial V}{\partial q^\alpha}, \tag{4.29}$$

so that $V(q,t)$ is the *potential* of the generalized forces. With the *Lagrangian function*

$$L(q,\dot{q},t) = T(q,\dot{q},t) - V(q,t) \tag{4.30}$$

the *Euler-Lagrange equations*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0, \qquad (\alpha = 1, \ldots, f) \tag{4.31}$$

for a mechanical path result from (4.26).

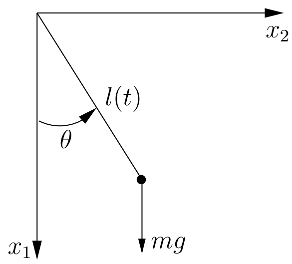
The equations of motion of a system with constraints can therefore generally be established as follows:

1. Write $T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i^2$, $V = V(x)$ (in an inertial system) in Cartesian components without considering the constraints.
2. Express the (permissible according to the constraints) configurations $x = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ in terms of independent position coordinates $q = (q^1, \ldots, q^f)$. With $x = x(q,t)$, compute $\dot{x} = \dot{x}(q,\dot{q},t)$.
3. Substitute $x, \dot{x}$ into $L = T - V$.
4. Set up the Euler-Lagrange equations (4.31).

This procedure will now be illustrated with a few examples.

Example 1: (see A)

A planar pendulum with a fixed suspension point and prescribed, time-dependent length $l(t)$. The position coordinate is θ. From



$$\boldsymbol{x} = l(\cos\theta, \sin\theta) = l\,\boldsymbol{e}(\theta)$$
$$\dot{\boldsymbol{x}} = \dot{l}(\cos\theta, \sin\theta) + l\dot{\theta}(-\sin\theta, \cos\theta)$$

one calculates

$$T = \frac{m}{2}\big(\dot{l}^2 + (l\dot{\theta})^2\big)\,, \qquad V = -mgx_1 = -mgl\cos\theta\,,$$

and from this

$$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}\,, \qquad \frac{\partial L}{\partial\theta} = -mgl\sin\theta\,.$$

Thus, the Euler-Lagrange equation (4.31) reads

$$\frac{d}{dt}(ml(t)^2\dot{\theta}) + mgl(t)\sin\theta = 0\,. \tag{4.32}$$

which corresponds to the angular momentum theorem.

Remark: The tension $\boldsymbol{S}$ as a constraint force is not considered in the formalism. However, it does perform work, as $\dot{\boldsymbol{x}}$ in this example is not a virtual displacement. Therefore, the energy $T+V$ is not conserved. One finds

$$\frac{d}{dt}(T+V) = m(\dot{l}\,\ddot{l} + l^2\dot{\theta}\ddot{\theta} + l\dot{l}\dot{\theta}^2) - mg\dot{l}\cos\theta + mgl\dot{\theta}\sin\theta\,. \tag{4.33}$$

Utilizing the equation of motion (4.32), one obtains

$$\frac{d}{dt}(T+V) = m\dot{l}(\ddot{l} - l\dot{\theta}^2 - g\dot{l}\cos\theta)\,. \tag{4.34}$$

The right side corresponds exactly to the power $\boldsymbol{S}\cdot\dot{\boldsymbol{x}}$ of the constraint force. Since we know that $\boldsymbol{S}$ points in the direction of the string, we can explicitly determine the tension in this case and obtain

$$\boldsymbol{S} = m(\ddot{l} - l\dot{\theta}^2 - g\cos\theta)\boldsymbol{e}(\theta)\,. \tag{4.35}$$

The last term thus corresponds to the counterforce of the weight along the string. The first term is the counterforce of the guiding force and the second term is the counterforce of the centrifugal force. These forces must all be balanced by the tension in the string to ensure that the constraint $|\boldsymbol{x}| = l$ remains satisfied. These forces are not needed in the Lagrangian formalism and are usually not calculated at all.

Example 2: (see B)

We consider the spherical pendulum with $l(t) \equiv l$ and $\boldsymbol{y}(t) \equiv 0$. The position of the pendulum is given by

$$\boldsymbol{x} = l(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

By differentiating with respect to t, we obtain the expression

$$\dot{\boldsymbol{x}} = l\dot{\theta}(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) + l\sin\theta\,\dot{\varphi}(-\sin\varphi, \cos\varphi, 0)$$

for the velocity vector. Thus, the kinetic energy becomes

$$T = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) \tag{4.36}$$

and the potential energy is $V = -mgx_3 = -mgl\cos\theta$. The Lagrangian function is given by

$$L = \frac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) + mgl\cos\theta. \tag{4.37}$$

We compute

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}, \qquad \frac{\partial L}{\partial \dot{\varphi}} = ml^2\sin^2\theta\,\dot{\varphi}, \qquad \frac{\partial T}{\partial \theta} = ml^2\sin\theta\cos\theta\,\dot{\varphi}^2.$$

The Euler-Lagrange equations then read

$$\begin{aligned} \frac{d}{dt}(ml^2\dot{\theta}) - ml^2\sin\theta\cos\theta\,\dot{\varphi}^2 + mgl\sin\theta &= 0, \\ \frac{d}{dt}(ml^2\sin^2\theta\,\dot{\varphi}) &= 0. \end{aligned}$$

Example 3: (see C) For the pearl on a rotating wire, we obtain directly with (4.4) (note that $R^t\dot{R}\boldsymbol{x} = \boldsymbol{\omega}\times\boldsymbol{x}$ with $\boldsymbol{\omega} = \omega\boldsymbol{e}_3$)

$$\begin{aligned} R^t\dot{\boldsymbol{x}} &= \boldsymbol{\omega}\times\Big(r\boldsymbol{e}_1 + \frac{1}{2}\alpha r^2\boldsymbol{e}_3\Big) + (\boldsymbol{e}_1 + \alpha r\boldsymbol{e}_3)\dot{r} \\ &= \omega r\boldsymbol{e}_2 + (\boldsymbol{e}_1 + \alpha r\boldsymbol{e}_3)\dot{r}. \end{aligned}$$

Thus, the kinetic energy is given by

$$T = \frac{1}{2}m\dot{\boldsymbol{x}}^2 = \frac{1}{2}m(R^t\dot{\boldsymbol{x}})^2 = \frac{1}{2}m\omega^2r^2 + \frac{1}{2}m(1+\alpha^2r^2)\dot{r}^2.$$

The potential energy due to the weight force takes the form $V(r) = mgx_3 = \frac{1}{2}mg\alpha r^2$. Thus, we obtain

$$\frac{\partial L}{\partial \dot{r}} = m(1+\alpha^2r^2)\dot{r}, \qquad \frac{\partial L}{\partial r} = m(\omega^2 - \alpha g)r + m\alpha^2r\dot{r}^2.$$

The Euler-Lagrange equations take the form

$$\frac{d}{dt}\Big(m(1+\alpha^2r^2)\dot{r}\Big) - m(\omega^2-\alpha g)r - m\alpha^2r\dot{r}^2 = 0.$$

Example 4: *Charged Particle in an Electromagnetic Field*

From the homogeneous Maxwell equations[4]

$$\operatorname{div}\boldsymbol{B} = 0\,, \qquad \operatorname{rot}\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = 0$$

[4]We use Gaussian units.

it directly follows that the magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$ and the electric field $\boldsymbol{E}(\boldsymbol{x},t)$ can be represented by electromagnetic potentials $\varphi, \boldsymbol{A}$ with

$$\boldsymbol{B} = \mathrm{rot}\,\boldsymbol{A}\,, \qquad \boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\,. \tag{4.38}$$

The motion of a particle (in Cartesian coordinates $\boldsymbol{x} \in \mathbb{R}^3$) is determined by the Lagrangian function

$$L(\boldsymbol{x},\dot{\boldsymbol{x}},t) = \frac{1}{2}m\dot{\boldsymbol{x}}^2 - e\Big(\varphi(\boldsymbol{x},t) - \frac{\dot{\boldsymbol{x}}}{c}\cdot\boldsymbol{A}(\boldsymbol{x},t)\Big)\,, \tag{4.39}$$

where m is the mass and e is the charge of the particle. In fact,

$$\frac{\partial L}{\partial x_k} = -e\frac{\partial \varphi}{\partial x_k} + \frac{e}{c}\sum_{i=1}^{3}\dot{x}_i\frac{\partial A_i}{\partial x_k}\,, \qquad \frac{\partial L}{\partial \dot{x}_k} = m\dot{x}_k + \frac{e}{c}A_k\,,$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} = m\ddot{x}_k + \frac{e}{c}\Big(\frac{\partial A_k}{\partial t} + \sum_{i=1}^{3}\frac{\partial A_k}{\partial x_i}\,\dot{x}_i\Big)\,,$$

so that the Euler-Lagrange equations (4.31)

$$m\ddot{x}_k = \underbrace{e\left(-\frac{\partial\varphi}{\partial x_k} - \frac{1}{c}\frac{\partial A_k}{\partial t}\right)}_{E_k} + \underbrace{\frac{e}{c}\sum_{i=1}^{3}\dot{x}_i\Big(\frac{\partial A_i}{\partial x_k} - \frac{\partial A_k}{\partial x_i}\Big)}_{(\dot{\boldsymbol{x}}\times\boldsymbol{B})_k}\,,$$

coincide with (1.16).

Example 5: *Double Pendulum*

In the (planar) double pendulum, two mass points move under the influence of a uniform gravitational field in a vertical plane, the first with a fixed distance l_1 from a fixed point (which we choose as the origin), and the second with a fixed distance l_2 from the first mass point. We denote the displacements from the vertical of the respective suspension points by the position coordinates θ_1 and θ_2. The position coordinates (in the plane) are then

$$\boldsymbol{x}_1 = l_1(\cos\theta_1, \sin\theta_1)\,, \qquad \boldsymbol{x}_2 = \boldsymbol{x}_1 + l_2(\cos\theta_2, \sin\theta_2)$$

and the configuration space is $S^1 \times S^1$.

For the kinetic energy, one finds with $\dot{\boldsymbol{x}}_1 = l_1\dot{\theta}_1(-\sin\theta_1, \cos\theta_1)$

$$T = \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + \frac{1}{2}m_2l_2\dot{\theta}_2^2\,,$$

while the potential energy is given by

$$V = -m_1l_1g\cos\theta_1 - m_2g(l_1\cos\theta_1 + l_2\cos\theta_2).$$

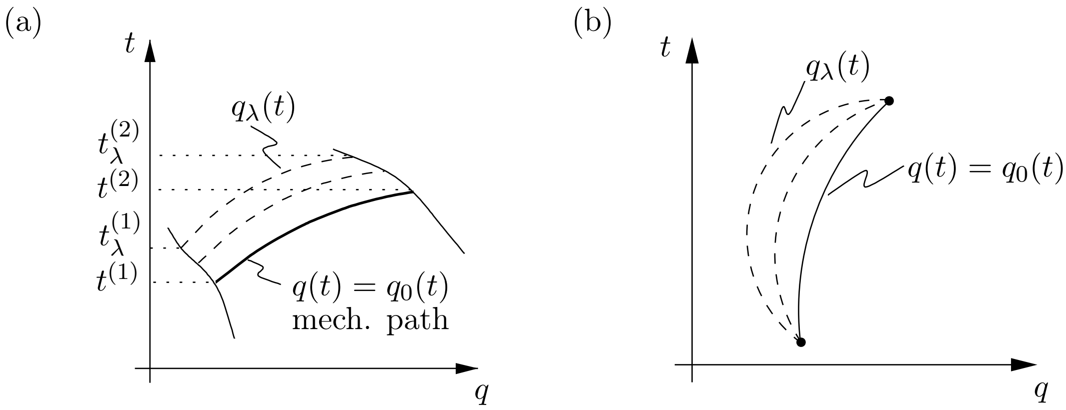


Figure 4.4: Variation of a mechanical path $q(t)$ with (a) or without (b) change of the endpoints $q^{(0)}$, $q^{(1)}$.

> The Lagrangian function is simply $L = T - V$. From this, one can easily derive the equations of motion. However, the double pendulum is *not integrable*: the equations of motion cannot be reduced to the computation of integrals. In fact, numerical calculations (and experiments) indicate a 'chaotic behavior'. The behavior of the double pendulum already strongly depends on minimal changes in the initial conditions after a finite time.

Remark: More generally than in (4.30), a system is called a *Lagrangian system* if there exists a Lagrangian function $L(q, \dot{q}, t)$ such that the equations of motion are given by the Euler-Lagrange equations (4.31). We have already examined such a system in example 4. With (4.31), one obtains in this case

$$\begin{aligned} \frac{d}{dt}\langle p, \delta q\rangle &= \frac{d}{dt}\sum_{\alpha=1}^{f} \frac{\partial L}{\partial \dot{q}^\alpha}\,\delta q^\alpha \overset{(4.31)}{=} \sum_{\alpha=1}^{f}\Big(\frac{\partial L}{\partial q^\alpha}\,\delta q^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha}\,\delta \dot{q}^\alpha\Big) \\ &= \delta L \end{aligned} \tag{4.40}$$

for a mechanical path with the *generalized momenta*

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}. \tag{4.41}$$

4.4 Hamilton's Principle

The laws of mechanics can be derived for Lagrangian systems from an extremal principle. This extremal principle is called the principle of the 'least' action or Hamilton's principle. To this end, we define the *action* of a path $q(t)$ ($t^{(1)} \leq t \leq t^{(2)}$)

as

$$S[q(t)] = \int_{t^{(1)}}^{t^{(2)}} dt\, L(q(t), \dot{q}(t), t). \tag{4.42}$$

We calculate the variation δS of the action around a mechanical path. We allow the possibility that the endpoints are also varied; that is, the endpoints $q_\lambda(t_\lambda^{(i)})$ of the comparison paths $q_\lambda(t)$ are generally also functions of the variation parameter λ, see Fig. 4.4. We write for the *variation of the endpoints*,

$$\begin{aligned} \Delta t^{(i)} &\equiv \frac{dt_\lambda^{(i)}}{d\lambda}\Big|_{\lambda=0}, \\ \Delta q^{(i)} &\equiv \frac{d}{d\lambda} q_\lambda(t_\lambda^{(i)})\Big|_{\lambda=0} = \delta q^{(i)} + \dot{q}\, \Delta t^{(i)} \end{aligned} \tag{4.43}$$

with $\delta q^{(i)} = \delta q(t^{(i)})$, where δq (as always) refers to the variation at fixed time. With these preliminary considerations, we obtain

$$\begin{aligned} \delta S &= \frac{d}{d\lambda} \int_{t_\lambda^{(1)}}^{t_\lambda^{(2)}} L(q_\lambda(t), \dot{q}_\lambda(t), t)\, dt\,\Big|_{\lambda=0} = \int_{(1)}^{(2)} \delta L\, dt + L\, \Delta t\Big|_{(1)}^{(2)} \\ &\overset{(4.40)}{=} \Big(\langle p, \delta q\rangle + L\, \Delta t\Big)\Big|_{(1)}^{(2)} \end{aligned}$$

With (4.43), we obtain the general formula

$$\delta S = \Big(\sum_{\alpha=1}^{f} p_\alpha\, \Delta q^\alpha - \Big(\sum_{\alpha=1}^{f} p_\alpha\, \dot{q}^\alpha - L\Big)\Delta t\Big)\Big|_{(1)}^{(2)} \tag{4.44}$$

for the variation of a mechanical path. In particular, it holds

$$\delta S = \delta \int_{(1)}^{(2)} L\, dt = 0 \tag{4.45}$$

for any variation of a mechanical path with *fixed endpoints*.

This is the content of *Hamilton's principle*:

> *The action of a mechanical path $q(t)$ is stationary (not necessarily minimal) with respect to any family $q_\lambda(t)$ of variation paths with fixed endpoints.*

From Hamilton's principle, the Euler-Lagrange equations immediately follow after performing the variation. We have

$$\begin{aligned} 0 = \delta S &= \int_{(1)}^{(2)} \delta L\, dt = \int_{(1)}^{(2)} \sum_{\alpha=1}^{f} \Big(\frac{\partial L}{\partial q^\alpha}\delta q^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha}\delta \dot{q}^\alpha\Big) dt \\ &= \sum_{\alpha=1}^{f} p_\alpha\, \delta q^\alpha\Big|_{(1)}^{(2)} - \int_{(1)}^{(2)} \sum_{\alpha=1}^{f} \Big(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha}\Big)\delta q^\alpha\, dt \end{aligned}$$

after partial integration. At fixed endpoints, $\delta q^\alpha(t) = 0$ for $t = t^{(1)}, t^{(2)}$, and thus (4.31) follows directly. We have thus shown that Hamilton's principle is equivalent to the Euler-Lagrange equations.

4.4.1 Equivalent Lagrangian Functions

Two Lagrangian functions L_1 and L_2, which differ only by the total time derivative of a function $F(q,t)$,

$$L_1 - L_2 = \frac{d}{dt}F(q,t) \equiv \sum_\alpha \frac{\partial F}{\partial q^\alpha}\dot{q}^\alpha + \frac{\partial F}{\partial t}\,, \tag{4.46}$$

describe the same dynamics, i.e., lead to the same Euler-Lagrange equations. This follows directly from Hamilton's principle. For fixed endpoints, indeed, for any otherwise arbitrary variation

$$\delta \int_{(1)}^{(2)} \frac{d}{dt}F(q,t)dt = \delta F(q,t)\Big|_{(1)}^{(2)} = 0 \tag{4.47}$$

and thus $\delta S_1 = \delta S_2$. Of course, one can also directly show that L_1 and L_2 lead to the same Euler-Lagrange equations.

Example 4 (Continuation): If one changes the potentials $(\varphi, \boldsymbol{A})$ by a *gauge transformation*

$$\varphi \to \varphi - \frac{1}{c}\frac{\partial \chi}{\partial t}\,, \qquad \boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\chi\,, \tag{4.48}$$

where χ is an arbitrary scalar function, the electromagnetic field (4.38) does not change. Therefore, the equations of motion (1.16) should also be invariant. In fact, L, given in Eq. (4.39), transforms under a gauge transformation (4.48) into the equivalent Lagrangian function

$$L + \frac{e}{c}\Big(\frac{\partial \chi}{\partial t} + \dot{\boldsymbol{x}}\cdot\boldsymbol{\nabla}\chi\Big) = L + \frac{e}{c}\frac{d\chi}{dt}$$

4.5 Conservation Quantities and Noether's Theorem

4.5.1 Conservation Quantities

A function $f(q,\dot{q},t)$ is called a *conservation quantity* or *constant of motion* if for *all* mechanical paths $q(t)$ it holds

$$\frac{d}{dt}f(q,\dot{q},t) = 0,$$

so that f remains constant during the motion. In Chapter 1.4, we have already encountered the conservation quantities momentum, angular momentum, and energy.

The Euler-Lagrange equation (4.31) is a second-order differential equation in the f degrees of freedom of the position coordinates. Therefore, in general, $2f$ integrations are needed to solve the problem. Conservation quantities are extraordinarily important, as each (independent) conservation quantity reduces the number of integrations required to solve the equations of motion by one.

Example: For the two-body problem from Chapter 2, we have $f = 3N = 6$ degrees of freedom (with $N = 2$). The Galilean principle of relativity provides the 10 conservation laws from Chapter 1.4. Thus, in general, the $2f - 10 = 2$ integrations of (2.12) and (2.14) remain. For the Kepler problem, we additionally found the LRL vector, which provides an additional independent conservation quantity. Thus, we could find the path without integration, and the only remaining integration is the determination of the time $t(r)$ from (2.12).

4.5.2 Cyclic Variables

Let L be the Lagrangian function of a Lagrangian system. If L is independent of a position coordinate q^α, i.e., if $\partial L/\partial q^\alpha = 0$, then the coordinate q^α is called *cyclic*. For each cyclic variable q^α, it then holds, due to the Euler-Lagrange equation

$$\frac{d}{dt}p_\alpha = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} = \frac{\partial L}{\partial q^\alpha} = 0, \tag{4.49}$$

i.e., the generalized momentum $p_\alpha = \partial L/\partial \dot{q}^\alpha$ is a conservation quantity.

Example 2 (Continuation): Let us consider the spherical pendulum again. The Lagrangian function L does not depend on φ according to Eq. (4.37). The corresponding conservation quantity is the generalized momentum

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\, l^2 \sin^2\theta\, \dot{\varphi}\,,$$

which corresponds to the angular momentum about the vertical axis. This is conserved since no torque acts in the vertical direction.

The fact that the Lagrangian function L does not depend on a cyclic variable q^α is equivalent to the Lagrangian function remaining invariant under the shift $\phi_\lambda(q)$ of the position coordinates, with

$$[\phi_\lambda(q)]^\beta = q^\beta + \delta^{\beta\alpha}\lambda \tag{4.50}$$

i.e., with $q_\lambda(t) = \phi_\lambda(q(t))$, it follows that $L(q_\lambda, \dot{q}_\lambda, t) = L(q, \dot{q}, t)$ independent of λ.[5] This reformulation illustrates the idea of Noether's theorem.

4.5.3 Noether's Theorem

Many of the conservation quantities we have found so far can be derived from Noether's theorem. We call a family $\phi_\lambda\colon \mathbb{R}^f \to \mathbb{R}^f$, $(\lambda \in \mathbb{R})$, of mappings of the

[5] The Kronecker symbol $\delta^{\alpha\beta}$ is defined by $\delta^{\alpha\beta} = 1$ for $\alpha = \beta$ and $\delta^{\alpha\beta} = 0$ otherwise.

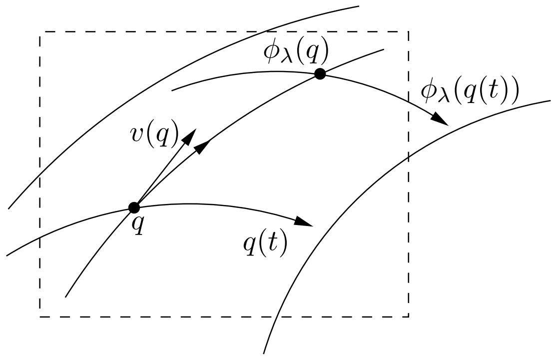


Figure 4.5: The vector field $v(q)$ at point q generates the flow $\phi_\lambda(q)$, i.e., the point q is (locally) shifted along $v(q)$ by the flow. Starting from a path $q(t)$, one obtains the family $q_\lambda(t) = \phi_\lambda(q(t))$. If the chart K only covers a subspace of $\mathbb{R}^f$, only a portion of the flow (dashed rectangle) is needed.

position coordinates[6] onto itself a *flow* if it satisfies the group property[7]

$$\phi_0 = \mathrm{id}\,, \qquad \phi_\lambda \circ \phi_\mu = \phi_{\lambda+\mu} \tag{4.51}$$

Every flow has a *generating vector field* $v(q)$ on $\mathbb{R}^f$, which is defined by

$$v(q) = \frac{\partial}{\partial\lambda}\phi_\lambda(q)\Big|_{\lambda=0} \in \mathbb{R}^f \tag{4.52}$$

see Fig. 4.5. Due to the group property of the flow, it holds (4.52) indeed for all $(\lambda, q) \in \mathbb{R} \times \mathbb{R}^f$, i.e.,

$$\frac{\partial}{\partial\lambda}\phi_\lambda(q) = \frac{\partial}{\partial\varepsilon}\phi_{\lambda+\varepsilon}(q)\Big|_{\varepsilon=0} = \frac{\partial}{\partial\varepsilon}\phi_\varepsilon(\phi_\lambda(q))\Big|_{\varepsilon=0} = v(\phi_\lambda(q))\,. \tag{4.53}$$

The function $f(\lambda) = \phi_\lambda(q)$ is thus the solution of the first-order differential equation

$$\frac{df(\lambda)}{d\lambda} = v(f(\lambda)), \qquad \text{with the initial condition } f(0) = q\,. \tag{4.54}$$

Thus, conversely, the vector field determines the flow.[8] The concept of flow can also be transferred to a general configuration space: In a chart, only a portion of the flow ϕ_λ can be represented; in general, multiple charts (an atlas) are needed to cover the entire configuration space.

Let now $q(t)$ be any curve in the configuration space. The flow ϕ_λ maps the curve $q(t)$ to the family $q_\lambda(t) = \phi_\lambda(q(t))$. We call the flow ϕ_λ a *continuous symmetry* of the Lagrangian function $L(q, \dot{q}, t)$ if

$$L(q_\lambda(t), \dot{q}_\lambda(t), t) = L(q(t), \dot{q}(t), t) + \frac{d}{dt}F_\lambda(q(t), t) \tag{4.55}$$

[6]For simplicity, we first treat the case where the chart covers all of $\mathbb{R}^f$.
[7]Here, id is the identity mapping with $\mathrm{id}(x) = x$.
[8]Provided that (4.54) has a global solution $f(q)$ for every initial value q.

holds for all $\lambda \in \mathbb{R}$ and for any curve $t \mapsto q(t) \in \mathbb{R}^f$; that is, the Lagrangian function is mapped to an equivalent Lagrangian function under the flow. With $F \equiv 0$, the Lagrangian function is called *invariant* under the continuous symmetry.

With this, we can formulate the *Noether's theorem*:

> *Let ϕ_λ be a continuous symmetry of L, then is*
>
> $$\langle p, v(q)\rangle - \delta F \equiv \sum_{\alpha=1}^{f} p_\alpha(q,\dot q,t)\, v^\alpha(q) - \delta F(q,t)\,, \tag{4.56}$$
>
> *with $p_\alpha = \partial L/\partial \dot q^\alpha$ obtained, i.e., on every mechanical path $q(t)$ it holds*[9]
>
> $$\frac{d}{dt}\Big(\langle p, v(q)\rangle - \delta F\Big) = 0\,.$$

Remarks:

- For every continuous symmetry, there is a conservation law.
- Noether's theorem is helpful because it is often easier to find continuous symmetries than to determine conserved quantities.
- The converse of Noether's theorem generally does not hold. Not every conserved quantity has a continuous symmetry; for example, the LRL vector $\boldsymbol{A}$ is not explained by any foot. This deficiency of the Lagrangian formulation is eliminated with the Hamiltonian formalism, where every conserved quantity also has a (canonical) flow, see Chap. (7.6).
- Noether's theorem also applies to flows $q_\lambda(t) = \phi_\lambda(q(t),t)$, which depend on time t. [10] The vector field is then given by $v\colon (q,t) \mapsto \partial_\lambda \phi_\lambda\big|_{\lambda=0} \in \mathbb{R}^f$. The proof remains unchanged and we obtain (4.56) with the time-dependent vector field $v(q,t)$ instead of $v(q)$.

Proof: For the variation family $q_\lambda(t)$ around any mechanical path $q(t)$, it holds by definition

$$\delta q(t) = \frac{\partial}{\partial\lambda} q_\lambda(t)\Big|_{\lambda=0} = v(q(t)).$$

According to the assumption (4.55), it also holds

$$\delta L = \frac{d}{d\lambda} L(q_\lambda, \dot q_\lambda, t)\Big|_{\lambda=0} = \frac{d}{dt}\delta F(q(t),t).$$

With (4.40), the claim follows directly.

[9] The conserved quantity depends only on δF in the expansion of $F_\lambda(q,t) = \delta F(q,t)\lambda + O(\lambda^2)$. In fact, it is sufficient that in Eq. (4.55) the terms of linear order in λ are a total time derivative, so that the function F_λ exists to every order of λ. This result follows from the integral representation $F_\lambda(q,t) = \int_0^\lambda \delta F(q_\mu,t)\, d\mu$ of F_λ.

[10] To maintain the group property, the flow is defined formally on the extended configuration space (q,t) by $\psi_\lambda(q,t) = (\phi_\lambda(q),t)$. The vector field is then given by $w(q,t) = \partial_\lambda \psi_\lambda(q,t)|_{\lambda=0} = (v,0) \in \mathbb{R}^f \times \mathbb{R}$.

Example 6: One can easily obtain the conservation of the generalized momentum, which corresponds to a cyclic coordinate, from Noether's theorem. If q^α is cyclic, the flow

$$[\phi_\lambda(q)]^\beta = q^\beta + \delta^{\beta\alpha}\lambda$$

defines a continuous symmetry of the Lagrangian with $F \equiv 0$. The corresponding vector field

$$v^\beta = \delta^{\alpha\beta}$$

then has only one non-trivial component. The conserved quantity provided by (4.56) is simply the conjugate momentum p_α.

Example 7: The Lagrangian

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{m}{2}\dot{\boldsymbol{x}}^2 - V(|\boldsymbol{x}|)$$

of a particle in 2 dimensions with the position coordinates[11] $\boldsymbol{x} \in \mathbb{R}^2$ is invariant under all rotations $R \in \mathrm{SO}(2)$, i.e., $L(\boldsymbol{x}_\lambda, \dot{\boldsymbol{x}}_\lambda) = L(\boldsymbol{x}, \dot{\boldsymbol{x}})$ with $x_\lambda = \phi_\lambda(\boldsymbol{x}) = R(\lambda)\boldsymbol{x}$ being the coordinates rotated by the angle λ. We determine the corresponding vector field

$$\boldsymbol{v}(\boldsymbol{x}) = R'(\lambda)\boldsymbol{x}\Big|_{\lambda=0} = \boldsymbol{e}_3 \times \boldsymbol{x} = -x_2\boldsymbol{e}_1 + x_1\boldsymbol{e}_2$$

and the impulses

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = m\dot{\boldsymbol{x}}.$$

Thus, we obtain with Noether's theorem the conserved quantity

$$L_3 = \boldsymbol{p} \cdot \boldsymbol{v}(\boldsymbol{x}) = m(x_1\dot{x}_2 - x_2\dot{x}_1).$$

This corresponds exactly to the angular momentum about the 3-axis. If we had assumed the polar coordinates (r, φ) as position coordinates instead of $\boldsymbol{x}$, we would have found that φ is cyclic and thus

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}^2$$

obtained. One can easily verify that $L_3 = p_\varphi$.

Example 8: A mass particle at position $x \in \mathbb{R}$ in a homogeneous gravitational field has the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - mgx.$$

The flow (*pure Galilean transformation*)

$$x_\lambda(t) = \phi_\lambda(x(t), t) = x(t) + \lambda t$$

[11] One may also use the Cartesian coordinates $\boldsymbol{x}$ as generalized coordinates.

is a symmetry of the problem, since

$$L(x_\lambda, \dot{x}_\lambda, t) = \frac{1}{2}m(\dot{x}+\lambda)^2 - mg(x+\lambda t) = L(x,\dot{x},t) + \frac{d}{dt}F_\lambda(x,t)$$

with

$$F_\lambda(x,t) = m\lambda\Big(x - \frac{1}{2}gt^2\Big) + \frac{1}{2}m\lambda^2 t.$$

The corresponding vector field is $v(x,t) = t$ and

$$\delta F = m\Big(x - \frac{1}{2}gt^2\Big).$$

Noether's theorem thus provides the conserved quantity

$$\beta = m\dot{x}v(x,t) - \delta F = m\Big(\dot{x}t + \frac{1}{2}gt^2 - x\Big). \tag{4.57}$$

Conservation of Energy: The (generalized) conservation of energy cannot be derived directly (from our form) of Noether's theorem. The conservation of energy is based on the fact that the Lagrangian $L = L(q,\dot{q})$ does not explicitly depend on time (autonomous system). The symmetry underlying the conservation of energy is therefore the time translation.

For the proof of conservation of energy, we consider the variation $q_\lambda(t) = q(t+\lambda)$ of a mechanical path $q(t)$ with $\delta q(t) = \dot{q}(t)$. Now, Eq. (4.40) requires that

$$\frac{d}{dt}\langle p, \dot{q}\rangle = \delta L = \frac{d}{dt}L - \frac{\partial L}{\partial t} \tag{4.58}$$

Since L does not explicitly depend on t, the *generalized energy*

$$E = \langle p, \dot{q}\rangle - L = \sum_{\alpha=1}^{f} p_\alpha(q,\dot{q},t)\dot{q}^\alpha - L(q,\dot{q},t) \tag{4.59}$$

is conserved along every mechanical path.

Example 9: If the Lagrangian is given by $L = T(q,\dot{q}) - V(q)$, with

$$T(q,\dot{q}) = \frac{1}{2}\sum_{\alpha,\beta=1}^{f} g_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta \qquad (g_{\alpha\beta} = g_{\beta\alpha}) \tag{4.60}$$

it holds

$$\sum_{\alpha=1}^{f}\dot{q}^\alpha\frac{\partial L}{\partial\dot{q}^\alpha} = \sum_{\alpha,\beta=1}^{f} g_{\alpha\beta}(q)\dot{q}^\alpha\,\dot{q}^\beta = 2T\,. \tag{4.61}$$

Such systems occur, for example, when the constraints (4.5) do not depend on time t, see (4.8) and (4.9). The constraint forces do no work, since $\dot{x}$ is a virtual displacement. Since in this case L does not explicitly depend on time, the conservation of energy holds in the form

$$E = T + V\,.$$

Example 2 (Continuation): Another example is the spherical pendulum. Since the Lagrangian (4.37) does not explicitly depend on time, the energy

$$E = \dot\theta\,\frac{\partial L}{\partial\dot\theta} + \dot\varphi\,\frac{\partial L}{\partial\dot\varphi} - L = \frac{l^2}{2}\dot\theta^2 + U(\theta)$$

is conserved with

$$U(\theta) = \frac{p_\varphi^2}{2l^2\sin^2\theta} - g\,l\,\cos\theta.$$

The effective potential U consists of the gravitational potential and the centrifugal potential, see (2.11).

Example 4 (Continuation): As an example, we consider a charged particle in a *static* (time-independent) field. In this case, the Maxwell equations are

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi(\boldsymbol{x})\,, \qquad \boldsymbol{B} = \mathrm{rot}\,\boldsymbol{A}(\boldsymbol{x})\,.$$

The Lagrangian is time-independent and thus the energy is conserved. The generalized momentum is given by

$$\boldsymbol{p} = \frac{\partial L}{\partial\dot{\boldsymbol{x}}} = m\dot{\boldsymbol{x}} + \frac{e}{c}\boldsymbol{A}.$$

Note that in this case the generalized momentum $\boldsymbol{p}$ does not coincide with the (kinematic) momentum $m\dot{\boldsymbol{x}}$ of Newtonian mechanics. Thus, we obtain the energy

$$E = \boldsymbol{p}\cdot\dot{\boldsymbol{x}} - L = \frac{1}{2}m\dot{\boldsymbol{x}}^2 + e\varphi(\boldsymbol{x}). \tag{4.62}$$

The first term is the kinetic energy and the second term is the potential energy in the electric potential φ.

Example 8 (Continuation): For the mass particle in the homogeneous gravitational field, the energy is conserved. With Example 9 we obtain

$$E = T + V = \frac{1}{2}m\dot x^2 + mgx.$$

We have previously found the additional conserved quantity β. With these two conserved quantities, the problem is completely integrated.

In fact, we can solve the equation for β for $\dot x$ with the result

$$\dot x = \frac{1}{t}\Big(x - \frac{1}{2}gt^2 + \frac{\beta}{m}\Big).$$

Substituting into the equation for E then yields after a short calculation

$$\gamma^2 \equiv \frac{2}{m}(E + g\beta) = \left(\frac{\frac{1}{2}gt^2 + x + \beta/m}{t}\right)^2.$$

This equation can be solved for x with the projectile parabola

$$x = -\frac{1}{2}gt^2 + \gamma t - \frac{\beta}{m}$$

as a result.

4.6 The 10 Classical Conservation Laws

For a (closed, Lagrangian) system $L = T - V$, whose potential V is invariant under Galilean transformations, we have already derived in Chapter 1.4 ten conserved quantities. We now want to trace these back to the 10 continuous parameters of the Galilean group (1.6).

Due to Galilean invariance, the Lagrangian of such a system is of the form

$$L(\underbrace{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N}_{=x}; \dot{\boldsymbol{x}}_1, \ldots, \dot{\boldsymbol{x}}_N) = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i^2 - V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \equiv T - V\,, \tag{4.63}$$

with

$$V(R\boldsymbol{x}_1 + \boldsymbol{b}, \ldots, R\boldsymbol{x}_N + \boldsymbol{b}) = V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)\,, \qquad (R \in \mathrm{SO}(3); \boldsymbol{b} \in \mathbb{R}^3),$$

see (1.36). The impulses are then given by

$$\boldsymbol{p}_i = \frac{\partial L}{\partial \dot{\boldsymbol{x}}_i} = m_i \dot{\boldsymbol{x}}_i \tag{4.64}$$

We will now discuss the parameters of the Galilean group individually:

(i) **Time translations:** As we have already shown in Example 9, the conserved quantity is the energy $E = T + V$.

(ii) **Spatial translations:** L is invariant under the joint translation

$$\phi_\lambda(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = (\boldsymbol{x}_1 + \lambda\boldsymbol{b}, \ldots, \boldsymbol{x}_N + \lambda\boldsymbol{b})$$

by the vector $\lambda\boldsymbol{b}$. The generating vector field is given by $v(x,t) = (\boldsymbol{b}, \ldots, \boldsymbol{b})$. The corresponding conserved quantity is then

$$\sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \boldsymbol{b} \equiv \boldsymbol{P} \cdot \boldsymbol{b}\,,$$

i.e., the **total momentum $\boldsymbol{P}$** is conserved, since $\boldsymbol{b}$ is arbitrary.

(iii) **Rotations:** L is invariant under the rotations

$$\phi_\lambda(\boldsymbol{x}_1, \ldots \boldsymbol{x}_N) = (R(\boldsymbol{e}, \lambda)\boldsymbol{x}_1, \ldots, R(\boldsymbol{e}, \lambda)\boldsymbol{x}_N)\,,$$

where $R(\boldsymbol{e}, \lambda)$ is the rotation about the axis $\boldsymbol{e}$ with angle λ. The corresponding vector field $v(x) = (\boldsymbol{e} \times \boldsymbol{x}_1, \ldots, \boldsymbol{e} \times \boldsymbol{x}_N)$, see (1.39), yields the conserved quantity

$$\sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot (\boldsymbol{e} \times \boldsymbol{x}_i) = \boldsymbol{e} \cdot \sum_{i=1}^{N} \boldsymbol{x}_i \times m_i \dot{\boldsymbol{x}}_i \equiv \boldsymbol{e} \cdot \boldsymbol{L}\,,$$

i.e., the **total angular momentum $\boldsymbol{L}$** is conserved.

(iv) **Special Galilean transformations:** A *special Galilean transformation* is a transformation into a uniformly moving coordinate system with velocity $\boldsymbol{v}$. This corresponds to the flow

$$\phi_\lambda(\boldsymbol{x}_1, \ldots \boldsymbol{x}_N, t) = (\boldsymbol{x}_1 + \lambda \boldsymbol{v}t, \ldots, \boldsymbol{x}_N + \lambda \boldsymbol{v}t).$$

with the generating vector field $v(x,t) = (\boldsymbol{v}t, \ldots, \boldsymbol{v}t)$. The Lagrangian transforms as

$$\begin{aligned} L(x_\lambda, \dot{x}_\lambda) &= \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{\boldsymbol{x}}_i + \lambda \boldsymbol{v})^2 - V(\boldsymbol{x}_1 + \lambda \boldsymbol{v}t, \ldots, \boldsymbol{x}_N + \lambda \boldsymbol{v}t) \\ &= L(x, \dot{x}) + \sum_{i=1}^{N} m_i\Big(\lambda \dot{\boldsymbol{x}}_i \cdot \boldsymbol{v} + \frac{1}{2}\lambda^2 \boldsymbol{v}^2\Big). \end{aligned}$$

Thus, the special Galilean transformations are a continuous symmetry (4.55) with

$$F_\lambda(x,t) = \sum_{i=1}^{N} m_i\Big(\lambda \boldsymbol{x}_i \cdot \boldsymbol{v} + \frac{1}{2}\lambda^2 \boldsymbol{v}^2 t\Big), \qquad \delta F(x,t) = \sum_{i=1}^{N} m_i \boldsymbol{x}_i \cdot \boldsymbol{v}\,.$$

The corresponding conserved quantity (4.56) is

$$\sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i \cdot \boldsymbol{v}t - \sum_{i=1}^{N} m_i \boldsymbol{x}_i \cdot \boldsymbol{v} = (\boldsymbol{P}t - M\boldsymbol{X}) \cdot \boldsymbol{v}\,;$$

i.e., the **center of mass integral** $M\boldsymbol{X} - \boldsymbol{P}t$ is conserved.

4.7 The Principle of Euler-Maupertuis

We restrict ourselves to autonomous systems, so that the constraint (4.5) does not explicitly depend on time and the potential depends only on the coordinates, $V \equiv V(q)$, see Example 9. With (4.8) we obtain the kinetic energy in the form

$$T(q,\dot{q}) = \frac{1}{2}\sum_{\alpha,\beta=1}^{f} g_{\alpha\beta}(q)\, \dot{q}^\alpha \dot{q}^\beta\,, \qquad g_{\alpha\beta}(q) = \frac{1}{2}\sum_{i=1}^{N} m_i \frac{\partial \boldsymbol{x}_i}{\partial q^\alpha} \cdot \frac{\partial \boldsymbol{x}_i}{\partial q^\beta}. \tag{4.65}$$

The kinetic energy thus defines a (Riemannian) *metric* on the configuration space with the *mechanical line element*

$$ds^2 = \frac{1}{2}\sum_{i=1}^{N} m_i (d\boldsymbol{x}_i)^2 = \frac{1}{2}\sum_{\alpha,\beta=1}^{f} g_{\alpha\beta}(q)\, dq^\alpha dq^\beta. \tag{4.66}$$

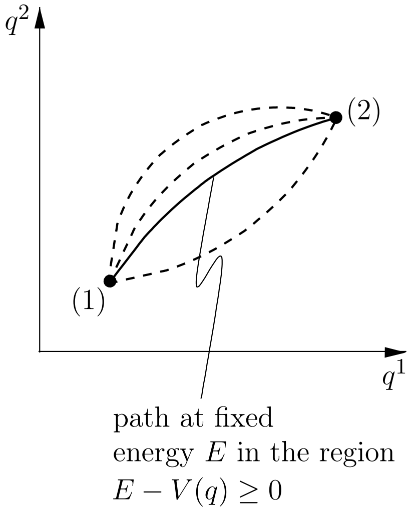


Figure 4.6: Variations of the path curve in the principle of Euler-Maupertuis. The variations are at fixed energy and fixed *position* of the starting and endpoint. Note that at fixed energy the *time* of the starting and endpoint will generally change under a variation of the path curve.

Thus, it follows directly that the kinetic energy

$$T(q,\dot{q}) = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\boldsymbol{x}}_i^2 = \left(\frac{ds}{dt}\right)^2 \tag{4.67}$$

corresponds simply to the square of ds/dt.

We are interested in a variational principle for the *path curve* (geometric shape of the curve) without specifying the temporal course. We use the conservation of energy $E = T + V$ and want to allow only variations at *fixed energy*. When we restrict the variations to a fixed energy, we can only hold the endpoints of the path curve but not the starting and ending times, see Fig. 4.6. In fact, the general result (4.44)

$$\delta S = -E\,\Delta t\Big|_{(1)}^{(2)} \tag{4.68}$$

holds for any variation of a mechanical path at fixed energy $E = \langle p, \dot{q}\rangle - L$ and fixed endpoints ($\Delta q^{(i)} = 0$). This fact motivates the introduction of the *reduced action*[12]

$$S_0[q(t)] = S + E(t^{(2)} - t^{(1)}) = \int_{t^{(1)}}^{t^{(2)}} dt\,(L(q,\dot{q}) + E), \tag{4.69}$$

with which we obtain the new extremal principle $\delta S_0 = 0$, since

$$\Delta t\Big|_{(1)}^{(2)} = \delta \int_{(1)}^{(2)} dt.$$

It remains to find a method to perform the variations of S_0 at fixed energy without reference to time (which will now change). The idea is to switch from integration

[12] The transition from S to S_0 corresponds to a Legendre transformation from t to E, see Chap. 8.

over time to integration over the line element ds. In fact, we find with $L + E = 2T$ and (4.67), that

$$(L+E)dt = 2T\frac{dt}{ds}ds = 2\sqrt{T}ds = 2\sqrt{E-V(q)}\,ds. \tag{4.70}$$

The last form is independent of time and we obtain the *principle of Euler-Maupertuis*[13]

$$0 = \delta S_0 = \delta \int_{(1)}^{(2)} \sqrt{E-V(q)}\,ds \tag{4.71}$$

for any variation of the path curve at fixed endpoints $q^{(i)}$, $i = 1, 2$.

Remark: To compute the integral in (4.71), one must parameterize the path curve $q(\tau)$ with τ being an *arbitrary* parameter. The principle of Euler-Maupertuis then explicitly states that

$$0 = \delta \int_{\tau^{(1)}}^{\tau^{(2)}} \underbrace{\sqrt{E-V(q(\tau))}\sqrt{T(q(\tau),q'(\tau))}}_{\equiv \tilde{L}(q,q')}\,d\tau, \qquad \text{with} \quad q' = \frac{dq}{d\tau}\,.$$

The integrand is then a (new) Lagrangian $\tilde{L}(q, q')$ and the extremal principle leads to the Euler-Lagrange equations

$$\frac{d}{d\tau}\frac{\partial \tilde{L}}{\partial q'^{\alpha}} - \frac{\partial \tilde{L}}{\partial q^{\alpha}} = 0$$

see Eq. (4.31).

Example: As an example, we consider the case of a free particle, $V(q) \equiv 0$. Then the variational principle (4.71)

$$\delta \int_{(1)}^{(2)} ds = 0\,,$$

i.e., the path curves for any energy $E > 0$ are *geodesics* (with respect to the mechanical metric (4.66)), i.e., curves of extremal length.

An illustration of this is a particle that slides frictionlessly on a 2-dimensional surface in $\mathbb{R}^3$. In this case, the kinetic energy is given by

$$T = \frac{m}{2}\dot{\boldsymbol{x}}^2 = \frac{m}{2}\left(\frac{ds}{dt}\right)^2,$$

where $ds^2 = (d\boldsymbol{x})^2$ refers here to the *Euclidean line element* of $\mathbb{R}^3$. A comparison with (4.67) shows that the Euclidean line element differs from the mechanical

[13] The constant factor 2 between S_0 and $\int \sqrt{E-V}\,ds$ is irrelevant for the extremal principle.

one only by a constant factor. Since this only corresponds to a different unit of length, this difference is insignificant for the extremal principle (4.71). We can therefore replace the mechanical length measure with the Euclidean one for a particle. Thus, the path curves of the particle are simply geodesics in the Euclidean sense, e.g., the great circles on the sphere.

Relation to Fermat's Principle: Fermat's principle states that light (in geometric optics) takes the path curve for which the travel time is an extremum. In a medium with spatially dependent *refractive index* $n(\boldsymbol{x})$, the effective light speed $c/n(\boldsymbol{x})$ also becomes spatially dependent. The travel time along a path curve is thus given as

$$\frac{\mathcal{L}[\boldsymbol{x}(\tau)]}{c} = \int_{(1)}^{(2)} \frac{n(\boldsymbol{x})}{c}\, ds \tag{4.72}$$

with $\mathcal{L}[\boldsymbol{x}(\tau)]$ being the optical path length of the path curve $\boldsymbol{x}(\tau)$.[14] Fermat's principle thus states that

$$\delta\mathcal{L}[\boldsymbol{x}(\tau)] = \delta \int_{(1)}^{(2)} n(\boldsymbol{x})\, ds = 0 \tag{4.73}$$

at fixed endpoints. One can now nicely recognize the analogy to the principle of Euler-Maupertuis (4.71) with the identification

$$n(\boldsymbol{x}) \propto \sqrt{E - V(\boldsymbol{x})}. \tag{4.74}$$

To perform the variation, we parameterize the path curve by an arbitrary parameter τ. Then (4.73) reads

$$\delta \int_{\tau^{(1)}}^{\tau^{(2)}} \underbrace{n(\boldsymbol{x})|\boldsymbol{x}'|}_{L(\boldsymbol{x},\boldsymbol{x}')}\, d\tau = 0,$$

with $'$ denoting the derivative with respect to τ.

Example: In a stratified medium with $n(\boldsymbol{x}) = n(x_3)$, x_1, x_2 are cyclic coordinates. Thus, we obtain the conserved quantities

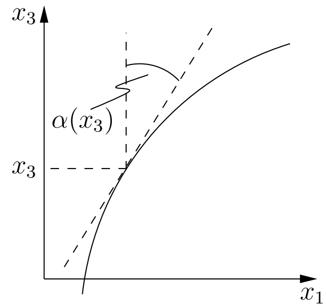
$$\frac{\partial L}{\partial x_i'} = n(x_3)\frac{x_i'}{|\boldsymbol{x}'|} = n(x_3)x_i', \qquad (i = 1, 2),$$

where in the last step we chose a parameterization by the arc length with $d\tau = ds$ so that $|\boldsymbol{x}'| = 1$.

[14] The parameter τ here denotes any parameterization.

Thus, we first find that the ray is straight ($x_2'/x_1' =$ const., and thus w.l.o.g. $x_2' = 0$). Introducing the angle $\alpha(x_3)$ between the 3-axis and the path, with $\sin\alpha(x_3) = x_1'/|\boldsymbol{x}'| = x_1'$, we then obtain the *Snell's law*

$$n(x_3)\,\sin\big(\alpha(x_3)\big) = \text{const.}$$



Chapter 5

Vibrational Problems

Another important class of mechanical motions are vibrations. In this chapter, we want to investigate the concept of the natural vibration of a conservatively vibrating system and a selected example (parametric resonance) of a forced vibration. Finally, we will also address the problem of a vibrating string.

5.1 Conservative Vibrating Systems

We consider a system whose position is determined by f coordinates $(q^1, \ldots, q^f) \equiv q \in \mathbb{R}^f$. The system is supposed to exhibit energy conservation in the form

$$E = T(\dot{q}) + V(q) \tag{5.1}$$

with

$$T(q) = \frac{1}{2} \sum_{\alpha,\beta=1}^{f} g_{\alpha\beta}\, q^\alpha q^\beta \qquad \text{and} \quad V(q) = \frac{1}{2} \sum_{\alpha,\beta=1}^{f} k_{\alpha\beta} q^\alpha q^\beta \tag{5.2}$$

exhibiting quadratic forms on $\mathbb{R}^f$ ($g_{\alpha\beta} = g_{\beta\alpha}$, $k_{\alpha\beta} = k_{\beta\alpha}$), where T is assumed to be positive definite. We can use the kinetic energy to introduce a *mechanical scalar product*

$$(x, y) = \frac{1}{2} \sum_{\alpha,\beta=1}^{f} g_{\alpha\beta}\, x^\alpha y^\beta = \frac{1}{2}\big(T(x+y) - T(x) - T(y)\big) \tag{5.3}$$

for two vectors $x, y \in \mathbb{R}^f$. It is easy to verify that (x, y) satisfies all the properties of a scalar product. In particular, $(q, q) = T(q)$. Thus, the space of position coordinates $\mathbb{R}^f$ becomes a Euclidean vector space.[1]

[1] A Euclidean vector space is a vector space with a scalar product.

The mapping $\tilde{V} = g^{-1}k$ is then a (with respect to the mechanical scalar product) symmetric[2] linear mapping $\tilde{V}: \mathbb{R}^f \to \mathbb{R}^f$, such that the *potential energy* is given by

$$V(q) = (q, \tilde{V}q) = (\tilde{V}q, q). \tag{5.4}$$

Energy conservation now requires that

$$0 = \frac{d}{dt}E = \frac{d}{dt}[(\dot{q}, \dot{q}) + (q, \tilde{V}q)] = 2(\dot{q}, \ddot{q} + \tilde{V}q). \tag{5.5}$$

For these relationships to hold for all $\dot{q}$, the *equations of motion* follow[3]

$$\ddot{q} = -\tilde{V}q. \tag{5.6}$$

As a symmetric mapping, $\tilde{V}$ has a complete orthonormal system of eigenvectors $e_1, \ldots, e_f \in \mathbb{R}^f$

$$\tilde{V}e_k = \lambda_k e_k, \qquad (e_k, e_l) = \delta_{kl} \tag{5.7}$$

with $\lambda_k \in \mathbb{R}$. In this basis, we have

$$q = \sum_{i=1}^{f} \xi_i e_i \tag{5.8}$$

with ξ_i being the *normal coordinates*. In the normal coordinates, the equation of motion (5.6) reduces to (*eigenvibrations*)

$$\ddot{\xi}_i = -\omega_i^2 \xi_i, \qquad \omega_i^2 = \lambda_i. \tag{5.9}$$

The system is thus equivalent to a system of f *uncoupled harmonic oscillators* with frequencies $\omega_1, \ldots, \omega_f$. The general solution of (5.6) is therefore a linear superposition (superposition)

$$q(t) = \sum_{i=1}^{f} e_i [\underbrace{(e_i, q(0))}_{=\xi_i(0)} \cos(\omega_i t) + \underbrace{(e_i, \dot{q}(0))}_{=\dot{\xi}_i(0)} \frac{1}{\omega_i} \sin(\omega_i t)] \tag{5.10}$$

of the f eigenvibrations. However, if the eigenfrequencies are not in a rational ratio to each other, generally no periodic motion results.

A vibrating system is called *stable* if *no* solution $q(t)$ grows unbounded as $t \to \infty$. We can immediately see that the system (5.1) is stable if $\omega_i^2 > 0$ for all i, i.e., if both T and V are positive definite. If V is only positive semidefinite, then at least one $\omega_i = 0$. In this case, we obtain

$$\frac{1}{\omega_i} \sin(\omega_i t) \xrightarrow{(\omega_i \to 0)} t, \tag{5.11}$$

i.e., in this case, the system is unstable.

---

[2]A symmetric mapping $\tilde{V}$ satisfies $(\tilde{V}x, y) = (x, \tilde{V}y)$ for all $x, y \in \mathbb{R}^f$.

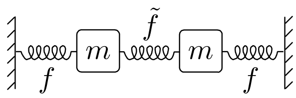
[3]The equations of motion can also be obtained as Euler-Lagrange equations for $L = T(\dot{q}) - V(q)$.

Remark: The problem (5.1) is important because it describes a general autonomous mechanical system near an equilibrium position q^*; without loss of generality, we set $q^* = 0$ with $V(q^*) = 0$. For the kinetic energy, we have found the general form (4.60) in this case. Since we are only interested in small displacements, we can simply identify $g_{\alpha\beta}$ from (5.2) with $g_{\alpha\beta}(q^*)$ from (4.61). Furthermore, we can expand the potential $V(q)$ around the equilibrium point with the result

$$V(q) \approx \sum_{\alpha=1}^{f} \overbrace{\frac{\partial V}{\partial q^\alpha}(q^*)}^{=0,\text{ since equilibrium}} q^\alpha + \frac{1}{2} \sum_{\alpha,\beta=1}^{f} \frac{\partial^2 V}{\partial q^\alpha \partial q^\beta}(q^*)\, q^\alpha q^\beta, \tag{5.12}$$

i.e., $V(q)$ is as required a quadratic form.

Example: As an example, we consider the simplest coupled oscillator from two identical harmonic oscillators with angular frequency $\Omega = \sqrt{f/m}$, which are connected by a spring with a spring constant $\tilde{f}$. The energy of the system is (with x_1, x_2 being the displacements from the unstressed equilibrium position)



$$E = T(\dot{x}) + V(x) = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2}f(x_1^2 + x_2^2) + \frac{1}{2}\tilde{f}(x_2 - x_1)^2.$$

It is easy to verify that $V(x) = (x, \tilde{V}x)$ holds with the symmetric matrix

$$\tilde{V} = \begin{pmatrix} \Omega^2 + \tilde{\Omega}^2 & -\tilde{\Omega}^2 \\ -\tilde{\Omega}^2 & \Omega^2 + \tilde{\Omega}^2 \end{pmatrix}$$

and $\tilde{\Omega} = \left(\tilde{f}/m\right)^{1/2}$. The eigenvalue problem (5.7) has the solution

$$e_1 = \frac{1}{\sqrt{m}}(1,1), \qquad \omega_1 = \Omega,$$
$$e_2 = \frac{1}{\sqrt{m}}(1,-1), \qquad \omega_2 = \sqrt{\Omega^2 + 2\tilde{\Omega}^2}.$$

In the first eigenvibration, the two oscillators simply oscillate in phase with a constant distance, so that the coupling spring is never stretched. The frequency of this vibration is, of course, simply the original frequency Ω. In the second eigenvibration, the oscillators oscillate out of phase and oscillate with the same amplitude. The corresponding frequency ω_2 is therefore increased.

We see that the eigenvibrations are intuitive, so we could have guessed them. This approach is facilitated by introducing the concept of symmetry.

Symmetries: A (discrete) symmetry is a linear mapping $S\colon \mathbb{R}^f \to \mathbb{R}^f$, which leaves T and V invariant; i.e., we require that $T(q) = T(Sq)$ and $V(q) = V(Sq)$ or (written with the mechanical scalar product)

$$(q, q) = (Sq, Sq) \qquad \text{and} \qquad (q, \tilde{V}q) = (Sq, \tilde{V}Sq). \tag{5.13}$$

The first relationship means that S is orthogonal (with respect to the mechanical scalar product).[4] The second condition thus requires,[5] that $S^{-1}\tilde{V}S = \tilde{V}$; i.e., S *commutes* with $\tilde{V}$

$$[S, \tilde{V}] \equiv S\tilde{V} - \tilde{V}S = 0. \tag{5.14}$$

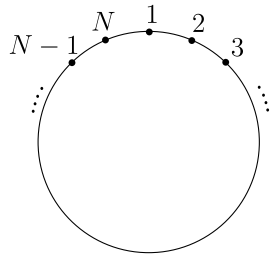
A symmetry is useful when one can determine its eigenspaces more easily than those of $\tilde{V}$. The reason is that due to (5.14), the eigenspaces of S are invariant under $\tilde{V}$; i.e.,

$$\text{with} \quad Se_i = \lambda_i e_i \qquad \Longrightarrow \qquad S\tilde{V}e_i = \tilde{V}Se_i = \lambda_i \tilde{V}e_i. \tag{5.15}$$

It is therefore sufficient to solve the eigenvalue problem of $\tilde{V}$ in each eigenspace of S ('*symmetry sector*') separately. This then leads to a reduction in the dimension of the problem.[6]

Example: *Cyclic Chain*

We consider a system of N identical mass points, which are arranged on a circle and connected by identical springs (spring constant f, fundamental frequency $\Omega = \sqrt{f/m}$). The coordinates $(x_1, \ldots, x_N)$ describe the displacements from the equilibrium position.



The equations of motion are $m\ddot{x}_i = -f[(x_i - x_{i-1}) + (x_i - x_{i+1})]$, and therefore from (5.6)

$$(\tilde{V}x)_i = \Omega^2(2x_i - x_{i-1} - x_{i+1}), \qquad (N + 1 \equiv 1).$$

Since all mass points are equal, the system has the symmetry

$$S\colon (x_1, \ldots, x_N) \mapsto (x_2, \ldots, x_{N+1} \equiv x_1).$$

The eigenvalue problem $Sx = \lambda x$ for S is easy to solve. We find $x_{l+1} = \lambda x_l = \lambda^2 x_{l-1} = \cdots = \lambda^l x_1$. Specifically, it must hold $x_1 = x_{N+1} = \lambda^N x_1$. The mapping S thus has N eigenvalues

$$\lambda_k = e^{ik\delta}, \qquad \left(k = 0, \ldots, N-1; \delta = \frac{2\pi}{N}\right)$$

[4] Due to (5.3), one can conclude from $(q, q) = (Sq, Sq)$ that $(x, y) = (Sx, Sy)$.
[5] Note that $(q, \tilde{V}q) = (Sq, SS^{-1}\tilde{V}Sq) = (q, S^{-1}\tilde{V}Sq)$ holds for all q.
[6] The problem can even be completely solved if all eigenspaces of S are one-dimensional.

with the corresponding eigenvectors (not normalized)

$$\tilde{e}_k = \big(1, e^{ik\delta}, e^{2ik\delta}, \ldots, e^{(N-1)ik\delta}\big).$$

Since the eigenspaces of S are one-dimensional, the reduced problem in a symmetry sector becomes trivial. We immediately obtain $\tilde{V}\tilde{e}_k = \omega_k^2 \tilde{e}_k$ with

$$\omega_k^2 = \Omega^2(2 - e^{-ik\delta} - e^{ik\delta}) = 2\Omega^2(1 - \cos k\delta) = 4\Omega^2 \sin^2 \frac{k\delta}{2},$$

i.e., the eigenfrequencies of the cyclic chain are

$$\omega_k = 2\Omega \sin \frac{k\pi}{N}.$$

We see that the modes k and $k' = N - k$ are degenerate. The general motion of the chain is given by (5.10).[7]

Special eigenmodes are:

- $k = 0$: with $\omega_0 = 0, \tilde{e}_0 = (1, \ldots, 1)$. This mode is called the *zero mode* and its solution $\xi_0(t) = \xi_0(0) + \dot{\xi}(0)t$ corresponds to a uniform rotation of the chain.
- $k = N/2$ (for N even): with $\omega_{N/2} = 2\Omega$, $\tilde{e}_{N/2} = (1, -1, \ldots, -1)$. Thus, neighboring mass points oscillate out of phase.

The other eigenvectors are not real. Here, $\tilde{e}_k$ and $\tilde{e}_{-k} \equiv \tilde{e}_{N-k}$ are each degenerate. They correspond to the oppositely traveling waves $\mathrm{Re}(\tilde{e}_{\pm k} e^{-i\omega_k t})_j = \cos(kj\delta \mp \omega_k t)$ at the same frequency ω_k. The real eigenvectors of this frequency are the standing waves

$$\frac{1}{2}(\tilde{e}_k + \tilde{e}_{-k}) = (1, \cos k\delta, \ldots, \cos(N-1)k\delta),$$
$$\frac{1}{2i}(\tilde{e}_k - \tilde{e}_{-k}) = (0, \sin k\delta, \ldots, \sin(N-1)k\delta).$$

5.2 Parametric Resonance

In addition to resonance in a forced vibration, when the frequency of the driver matches the natural frequency, there is another type of resonance phenomenon called *parametric resonance*, which we will address in this chapter. A generic example is the problem of a harmonic oscillator

$$\ddot{x} = -\omega^2(t)x, \qquad \omega(t+T) = \omega(t); \tag{5.16}$$

whose frequency is periodically modulated. Note that at no point does a force act on the oscillator. In fact, $x \equiv 0$ is always a solution of (5.16) and the oscillation is therefore not directly driven from the outside.

[7] Of course, one should first form real and normalized eigenvectors e_k for $\tilde{V}$ from the degenerate $\tilde{e}_k$, $\tilde{e}_{N-k}$, see also below.

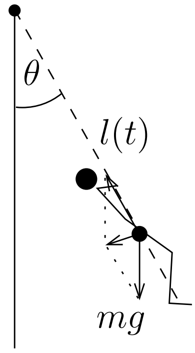


Figure 5.1: A person stands on a swing (mathematical pendulum). They drive the motion by moving the center of mass up and down, thereby periodically changing the distance $l(t)$ from the suspension point to the center of mass.

Example: Such systems often arise from linearization around a periodic solution of a nonlinear problem (see e.g. (3.62)). Equation (5.16) also describes the swinging while standing, see Fig. 5.1. The distance from the suspension point to the center of mass is periodically changed along the rope with $l(t) = l(t+T)$. The angular momentum equation (4.32) requires that

$$\frac{d}{dt}(ml^2\dot{\theta}) = -mgl\sin\theta\,.$$

For small displacements ($\theta \ll 1$), this leads to

$$\frac{d}{dt}(l^2\dot{\theta}) = l\frac{d^2}{dt^2}(l\theta) - l\ddot{l}\theta = -gl\theta$$

and thus

$$\frac{d^2}{dt^2}(l\theta) = -\frac{g-\ddot{l}}{l}(l\theta)\,.$$

This corresponds to $\omega^2(t) = (g-\ddot{l})/l$ and $x = l\theta$ but is exactly the equation of motion (5.16) of parametric resonance.

In a first step, we rewrite the equation (5.16) into the system

$$\dot{z} = A(t)z, \qquad z = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \qquad A(t) = \begin{pmatrix} 0 & 1 \\ -\omega^2(t) & 0 \end{pmatrix}. \tag{5.17}$$

of first-order differential equations; it holds, of course, $A(t+T) = A(t)$. Let $P(t)$ be the propagator of (5.17) for the time interval from 0 to t defined by

$$z(t) = P(t)z(0). \tag{5.18}$$

Example: We seek the propagator of (5.17) for $\omega(t) \equiv \omega$ constant. The general solution of the harmonic oscillator problem is then given by (cf. (5.10))

$$x(t) = x(0)\cos(\omega t) + \dot{x}(0)\frac{1}{\omega}\sin(\omega t).$$

We obtain

$$z(t) = \begin{pmatrix} x(0)\cos(\omega t) + \dot{x}(0)\omega^{-1}\sin(\omega t) \\ -x(0)\omega\sin(\omega t) + \dot{x}(0)\cos(\omega t) \end{pmatrix}$$

and thus the propagator

$$P_\omega(t) = \begin{pmatrix} \cos(\omega t) & \omega^{-1}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix} \tag{5.19}$$

of an oscillator with constant angular frequency ω.

The periodicity of A requires that

$$P(nT) = P(T)^n\,, \qquad (n = 1, 2, 3, \ldots)\,. \tag{5.20}$$

Exponentially growing solutions occur when $P(T)$ has an eigenvalue $\lambda \in \mathbb{C}$ with $|\lambda| > 1$; conversely, all solutions are bounded and the system is stable if all eigenvalues satisfy $|\lambda_k| \leq 1$.

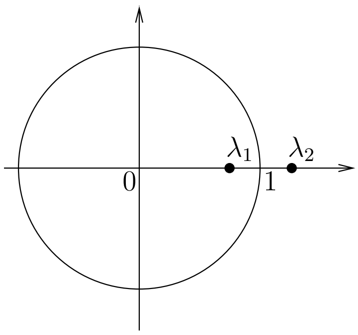
Since the trace of A vanishes ($\mathrm{Sp}\,A(t) = 0$), it follows[8]

$$\frac{d}{dt}\,\mathrm{Det}\,P(t) = \mathrm{Sp}\Big(\underbrace{\frac{dP}{dt}P(t)^{-1}}_{A(t)}\Big)\;\mathrm{Det}\,P(t) = 0 \tag{5.21}$$

and therefore $\mathrm{Det}\,P(t) = \mathrm{Det}\,P(0) = 1$.

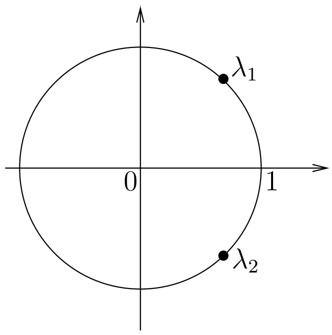
The two eigenvalues λ_1, λ_2 of $P(T)$ thus satisfy $\lambda_1\lambda_2 = 1$. Furthermore, $P(T)$ is real, and if λ_i is an eigenvalue, then λ_i^* must also be an eigenvalue. This gives rise to two possibilities:

1. $\lambda_i = \lambda_i^*$: λ_i real



$|\,\mathrm{Sp}\,P(T)| = |\lambda_1 + \lambda_2| > 2$

2. $\lambda_1 = \lambda_2^*$: $\lambda_i\lambda_i^* = 1$



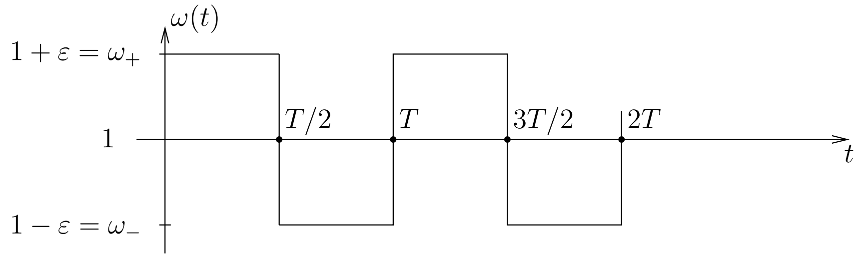
$|\,\mathrm{Sp}\,P(T)| = |\lambda_1 + \lambda_2| \leq 2$

In the first case, the system is unstable ($|\lambda_2| > 1$), in the second case stable ($|\lambda_1| = |\lambda_2| = 1$).[9] The stability of the system is therefore determined by the trace of $P(T)$.

[8]It holds $\mathrm{Det}(1+\varepsilon A) = 1+\varepsilon\,\mathrm{Sp}(A)+O(\varepsilon^2)$. Thus, it follows (to first order in ΔP) $\mathrm{Det}(P+\Delta P) = \mathrm{Det}(P)\;\mathrm{Det}(1+\Delta P\,P^{-1}) = \mathrm{Det}(P) + \mathrm{Sp}(\Delta P\,P^{-1})\;\mathrm{Det}(P)$.

[9]The limiting case $\lambda_1 = \lambda_2 = 1$ is stable.

Example: We investigate the system for the simple case where $\omega(t)$ is simply given by the step function



between the values $\omega_+ = 1+\varepsilon$ and $\omega_- = 1-\varepsilon$. The value $\omega = 1$ for the undisturbed oscillation with $\varepsilon = 0$ corresponds to the choice of time unit: i.e., we have set the period of the undisturbed oscillation to $T_0 = 2\pi$. The parameter ε characterizes the strength of the disturbance.

Due to the step form of $\omega(t)$, we directly obtain

$$P(T) = P_{\omega_-}(T/2)P_{\omega_+}(T/2)$$

for our problem with $P_\omega(t)$ from (5.19). For the stability analysis, we determine[10]

$$\begin{aligned}
\mathrm{Sp}\,P(T) &= \cos\frac{\omega_+T}{2}\cos\frac{\omega_-T}{2} - \frac{\omega_+}{\omega_-}\sin\frac{\omega_+T}{2}\sin\frac{\omega_-T}{2} \\
&\quad + \cos\frac{\omega_+T}{2}\cos\frac{\omega_-T}{2} - \frac{\omega_-}{\omega_+}\sin\frac{\omega_+T}{2}\sin\frac{\omega_-T}{2} \\
&= (\cos T + \cos\varepsilon T) + \underbrace{\frac{1}{2}\left(\frac{\omega_+}{\omega_-} + \frac{\omega_-}{\omega_+}\right)}_{=(1+\varepsilon^2)(1-\varepsilon^2)}(\cos T - \cos\varepsilon T) \\
&= \frac{2}{1-\varepsilon^2}\left(\cos T - \varepsilon^2\cos\varepsilon T\right) \\
&= 2 - \frac{4}{1-\varepsilon^2}\left(\sin^2\frac{T}{2} - \varepsilon^2\sin^2\frac{\varepsilon T}{2}\right) \qquad (5.22)\\
\text{or:}\quad &= -2 + \frac{4}{1-\varepsilon^2}\left(\cos^2\frac{T}{2} - \varepsilon^2\cos^2\frac{\varepsilon T}{2}\right).
\end{aligned}$$

We now consider T (and thus the modulation frequency) as a variable and seek the T ranges in which the system is unstable (resonance ranges). There are two

[10] In the second equation we use the identities $\cos(a+b)+\cos(a-b) = 2\cos a\,\cos b$ and $\cos(a+b)-\cos(a-b) = -2\sin a\,\sin b$; in the fourth or fifth the identities $\cos(2a) = 1-2\sin^2 a$ and $\cos(2a) = 2\cos^2 a - 1$.

possibilities for instability:

$$\text{(a)} \qquad \mathrm{Sp}\, P(T) > 2: \qquad \sin^2 \frac{T}{2} < \varepsilon^2 \sin^2 \frac{\varepsilon T}{2}\,, \tag{5.23}$$

$$\text{(b)} \qquad \mathrm{Sp}\, P(T) < -2: \qquad \cos^2 \frac{T}{2} < \varepsilon^2 \cos^2 \frac{\varepsilon T}{2}\,. \tag{5.24}$$

We discuss the two cases for $\varepsilon \ll 1$:

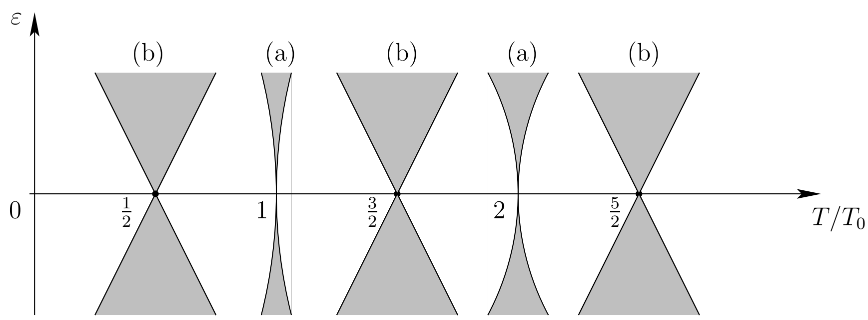
(a) For $\varepsilon = 0$, $\mathrm{Sp}\, P(T) = 2$ if $T = 2\pi n$, $(n = 1, 2, 3, \ldots)$. For small ε, we seek solutions of (5.23) in the form $T = 2\pi n + x$ with $|x| \ll 1$. Approximately, the resonance ranges are given by

$$|x| < 2\pi n \varepsilon^2\,. \tag{5.25}$$

(b) For $\varepsilon = 0$, $\mathrm{Sp}\, P(T) = -2$ if $T = (2n+1)\,\pi$, $(n = 0, 1, 2, \ldots)$. For small ε, we set $T = (2n+1)\pi + x$, and (5.24) then reads

$$|x| < 2\varepsilon\,. \tag{5.26}$$

In case (b), the width of the resonance intervals is independent of n and vanishes like $O(\varepsilon)$ as $\varepsilon \to 0$. In case (a), this width increases proportionally to n at fixed ε and vanishes like $O(\varepsilon^2)$. In arbitrary units, $T/2\pi = T/T_0$ is the ratio of the periods of the disturbance and the free oscillation. Thus, the stability diagram is approximately given by:



Note that the area of resonance at $T = \frac{1}{2}T_0$ is larger than when driving with the natural frequency $(T = T_0)$. This means it is better to drive a parametric resonator with double its natural frequency than with its natural frequency.

5.3 Vibrating String

We model a string by a periodic arrangement of identical mass points of mass m at position $x_i = ai$ along the 1-axis $(i \in \mathbb{Z})$, see Fig. 5.2. Each pair of neighboring mass points interacts with each other via a spring force with spring constant f. We assume

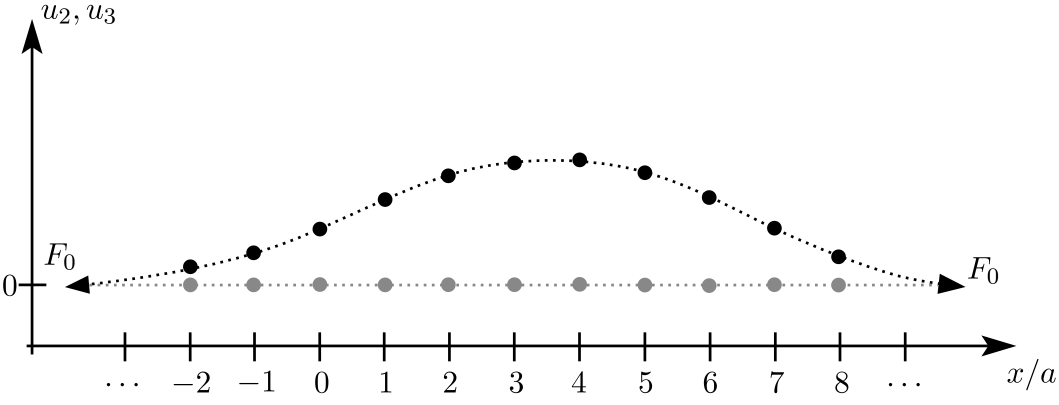


Figure 5.2: Modeling a string as an arrangement of mass points at each position $x_i = ai$ along the 1-axis. Neighboring mass points are connected with a spring with spring constant f. The string is stretched with an external force $F_0 = af$, which guarantees the correct equilibrium position (gray). The vector $\boldsymbol{u}_i$ then describes the displacement of the i-th mass point from the equilibrium position.

that the string is stretched by the external force F_0 such that we achieve the desired displacement $a = F_0/f$ in equilibrium.

The potential energy of the springs takes the form

$$V = \frac{1}{2} f \sum_{i \in \mathbb{Z}} (\boldsymbol{u}_{i+1} - \boldsymbol{u}_i + a\boldsymbol{e}_1)^2 \tag{5.27}$$

with the displacements $\boldsymbol{u}_i$ of the i-th mass point from the equilibrium position.

We are interested in the description of small displacements. In particular, we want to consider transverse displacements with $\boldsymbol{u}_i \cdot \boldsymbol{e}_1 = 0$. Thus, we can rewrite the potential energy as

$$V = V_0 + \frac{1}{2} f \sum_{i \in \mathbb{Z}} (\boldsymbol{u}_{i+1} - \boldsymbol{u}_i)^2 \tag{5.28}$$

with V_0 being a constant (independent of $\boldsymbol{u}_i$). We now directly obtain the equations of motion

$$m\ddot{\boldsymbol{u}}_i = -\frac{\partial V}{\partial \boldsymbol{u}_i} = f(\boldsymbol{u}_{i+1} + \boldsymbol{u}_{i-1} - 2\boldsymbol{u}_i) \tag{5.29}$$

of the i-th mass point.

To realistically describe a string, we now want to gradually increase the density of mass points; i.e., we take the *continuum limit*

$$a \to 0, \quad m \to 0, \qquad \text{with } \rho = m/a \text{ fixed.} \tag{5.30}$$

We replace $x_i = ai$ with the continuous coordinate $x \in \mathbb{R}$ and $\boldsymbol{u}_i$ becomes the differentiable function $\boldsymbol{u}(x)$. In particular, it holds (by Taylor expansion)

$$\boldsymbol{u}_{i\pm1} \approx \boldsymbol{u}(x) \pm a\boldsymbol{u}'(x) + \frac{1}{2}a^2\boldsymbol{u}''(x). \tag{5.31}$$

Thus, we obtain the continuum limit

$$\frac{f}{m}(\boldsymbol{u}_{i+1} + \boldsymbol{u}_{i-1} - 2\boldsymbol{u}_i) \xrightarrow{(a\to0)} \frac{F_0}{\rho}\boldsymbol{u}''(x). \tag{5.32}$$

The equations of motion (5.29) transition in the continuum limit to the *wave equation*

$$\frac{\partial^2}{\partial t^2}\boldsymbol{u}(x,t) = c^2\frac{\partial^2}{\partial x^2}\boldsymbol{u}(x,t) \tag{5.33}$$

with $\boldsymbol{u}(x,t)$ being the displacement of the string at point x at time t and

$$c = \sqrt{\frac{F_0}{\rho}} \tag{5.34}$$

being the propagation speed. The wave equation is a partial differential equation, as the parameter i, which initially numbered the particles, has now become another parameter $x = ai$ of the function $\boldsymbol{u}$.

We will restrict ourselves to the case where the string only performs motion in the 13-plane with $\boldsymbol{u}(x,t) = u(x,t)\boldsymbol{e}_3$. The general solution of the wave equation has the form (*d'Alembert*)

$$u(x,t) = f(x-ct) + g(x+ct) \tag{5.35}$$

with $f, g\colon \mathbb{R} \to \mathbb{R}$ being two arbitrary (twice differentiable) functions. One can easily verify the solution by substitution. Here, f describes a disturbance that propagates to the right, maintaining its shape. Similarly, g propagates to the left while remaining shape-invariant. The general solution is a superposition of a right-moving and a left-moving wave.

Example: *Guitar*

In the case where the string is fixed at the points $x = 0$ and $x = L$, i.e., $u(0,t) = 0$ and $u(L,t) = 0$, neither of the two functions provides a solution to the problem. The reason is that a right-moving wave is reflected at the point $x = L$ (and transforms into a left-moving wave). The solution to the problem is therefore necessarily a superposition of a right-moving and a left-moving wave, i.e., a *standing wave* of the form

$$\begin{aligned} u(x,t) &= A\sin(k(x-ct)) + A\sin(k(x+ct)) \\ &= 2A\sin(kx)\cos(kct). \end{aligned} \tag{5.36}$$

For the standing wave to satisfy the boundary condition at $x = L$, it must hold

$$k = k_n = n\frac{\pi}{L}, \qquad (n = 1, 2, \dots) \tag{5.37}$$

Thus, with (5.36), the string oscillates with the frequency

$$\nu_n = \frac{k_n c}{2\pi} = n\frac{c}{2L}, \tag{5.38}$$

i.e., with the frequency $\nu_1 = c/(2L)$ (and its overtones).

Chapter 6

Rigid Bodies

A rigid body is a system of mass points whose relative distances between *all* pairs of points remain constant during motion. Although this represents an idealization, this concept is very useful, and the mechanics of the motion of rigid bodies deserves a detailed presentation. The example we will primarily focus on is the gyroscope, where a point of the rigid body is held fixed.

6.1 Mass Distribution and Inertia Tensor

Mass Distribution: A rigid body is described by a *mass distribution*

$$dm = \rho(\boldsymbol{y})\, d^3y \tag{6.1}$$

with the mass density $\rho(\boldsymbol{y})$ with respect to a *body-fixed* coordinate system $\boldsymbol{y} \in \mathbb{R}^3$. The general configuration in the *space-fixed* inertial system $\boldsymbol{x}$ is then given by

$$\boldsymbol{x}(t) = R(t)\boldsymbol{y} + \boldsymbol{b}(t), \qquad t \mapsto R(t) \in \mathrm{SO}(3) \text{ and } t \mapsto \boldsymbol{b}(t) \in \mathbb{R}^3, \tag{6.2}$$

cf. Fig. 6.1. Unlike a mass point, the configuration space is 6-dimensional: three coordinates describe the position $\boldsymbol{b}(t)$ of the reference point, while the additional three coordinates $R(t)$ describe the orientation of the rigid body relative to a standard configuration. Often we are interested in a situation where a point of the rigid body (not necessarily the center of mass) is held fixed. In this case, there is only a rotational motion about this fixed point, which we set at $\boldsymbol{x} = \boldsymbol{y} = 0$, and no translation. A rigid body with a fixed point is called a *gyroscope*.

With the mass distribution dm, one can compactly write the various quantities of mechanics, e.g.

- the total mass: $M = \int dm$,
- the center of mass: $\boldsymbol{X} = M^{-1} \int dm\, \boldsymbol{x}$,

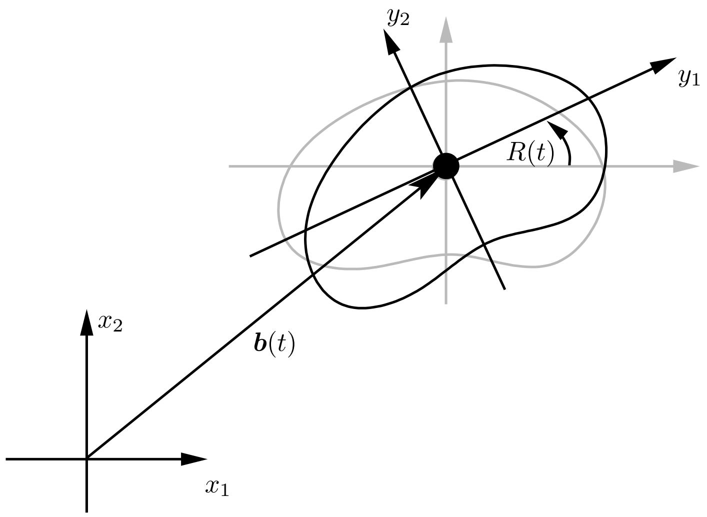


Figure 6.1: The general position of a rigid body is described by a translation by $\boldsymbol{b}(t)$ and a rotation $R \in \mathrm{SO}(3)$. The configuration space is therefore 6-dimensional.

- the angular momentum: $\boldsymbol{L} = \int dm\, \boldsymbol{x} \times \dot{\boldsymbol{x}}$,
- the kinetic energy: $T = \frac{1}{2} \int dm\, \dot{\boldsymbol{x}}^2$.

Example: (*concentrated mass points*) One can see the concept of a mass point as a special case of the general mass distribution dm. For this, one introduces the concept of a concentrated mass point at the location $\boldsymbol{y}_0$ with mass 1 as $dm = \delta(\boldsymbol{y} - \boldsymbol{y}_0) d^3y$ (Dirac delta function) with

$$\int dm\, f(\boldsymbol{y}) = \int d^3y\, f(\boldsymbol{y})\delta(\boldsymbol{y} - \boldsymbol{y}_0) = f(\boldsymbol{y}_0)$$

for any function $f\colon \mathbb{R}^3 \to \mathbb{R}$. Thus, for the mass distribution, one obtains

$$dm = \sum_{i=1}^{N} m_i\, \delta(\boldsymbol{y} - \boldsymbol{y}_i)$$

the results from Chap. 1 for N mass points. As an example, we consider the kinetic energy

$$T = \frac{1}{2} \int dm\, \dot{\boldsymbol{x}}^2 = \sum_{i=1}^{N} \frac{1}{2} \int d^3y\, \delta(\boldsymbol{y} - \boldsymbol{y}_i)\, \dot{\boldsymbol{x}}^2 = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\boldsymbol{x}}_i{}^2$$

with $\boldsymbol{x}(\boldsymbol{y}_i, t) = \boldsymbol{x}_i(t)$ the position of the i-th particle at time t. The expression for T agrees with (1.33).

Kinetic Energy: We can use the relationship (6.2) to express the kinetic energy in body-fixed coordinates. We obtain

$$\begin{aligned} T &= \frac{1}{2}\int dm\,\dot{\boldsymbol{x}}^2 = \frac{1}{2}\int dm(\dot{R}\boldsymbol{y}+\dot{\boldsymbol{b}})^2 = \frac{1}{2}\int dm(\dot{\boldsymbol{b}}^2 + \overbrace{\dot{R}\boldsymbol{y}\cdot\dot{R}\boldsymbol{y}}^{=R^t\dot{R}\boldsymbol{y}\cdot R^t\dot{R}\boldsymbol{y}} + 2\dot{R}\boldsymbol{y}\cdot\dot{\boldsymbol{b}}) \\ &= \frac{1}{2}M\dot{\boldsymbol{b}}^2 + \frac{1}{2}\int dm\,(\boldsymbol{\omega}\times\boldsymbol{y})^2 + M\dot{R}\boldsymbol{Y}\cdot\dot{\boldsymbol{b}} \end{aligned} \tag{6.3}$$

with the center of mass $\boldsymbol{Y} = R^t(\boldsymbol{X}-\boldsymbol{b})$ and the angular velocity $\boldsymbol{\omega}$ in y-coordinates.[1]

In the following, we will either place the body-fixed coordinate system on the center of mass (so that $\boldsymbol{Y} = 0$) or (in the case of a gyroscope) on the fixed point (so that $\boldsymbol{b} = 0$). In both cases, the last term of (6.3) vanishes. Thus, the kinetic energy $T = \frac{1}{2}M\dot{\boldsymbol{b}}^2 + T_{\text{rot}}$ separates into the kinetic energy of the reference point (as if the entire mass were concentrated at the reference point) and the *rotational energy*

$$\begin{aligned} T_{\text{rot}} &= \frac{1}{2}\int dm\,(\boldsymbol{\omega}\times\boldsymbol{y})^2 = \frac{1}{2}\int dm\,\left(\boldsymbol{y}^2\boldsymbol{\omega}^2 - (\boldsymbol{y}\cdot\boldsymbol{\omega})^2\right) \\ &= \frac{1}{2}\sum_{i,k=1}^{3}\Theta_{ik}\omega_i\omega_k \end{aligned} \tag{6.4}$$

where we have defined the coefficients of the *inertia tensor* Θ by

$$\Theta_{ik} = \int dm\,(\boldsymbol{y}^2\delta_{ik} - y_i y_k). \tag{6.5}$$

Angular Momentum: For the calculation of the angular momentum, we again consider the two cases with $\boldsymbol{Y} = 0$ or $\boldsymbol{b} = 0$. From (6.2), we obtain the result

$$\boldsymbol{L} = \int dm\,\boldsymbol{x}\times\dot{\boldsymbol{x}} = M\boldsymbol{b}\times\dot{\boldsymbol{b}} + \underbrace{\int dm\,Ry\times\dot{R}y}_{\equiv R\boldsymbol{S}}, \tag{6.6}$$

where the first term describes the angular momentum of the reference point. The second term is the angular momentum with respect to this point (with $\boldsymbol{y} = 0$). In y-coordinates, it has the form[2]

$$\begin{aligned} \boldsymbol{S} &= \int dm\,\boldsymbol{y}\times R^t\dot{R}\boldsymbol{y} = \int dm\,\overbrace{\boldsymbol{y}\times(\boldsymbol{\omega}\times\boldsymbol{y})}^{=\boldsymbol{y}^2\boldsymbol{\omega}-(\boldsymbol{y}\cdot\boldsymbol{\omega})\boldsymbol{y}} \\ &\equiv \Theta\,\boldsymbol{\omega}. \end{aligned} \tag{6.7}$$

With the angular momentum vector $\boldsymbol{S}$, we can express the rotational energy as

$$T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{S}. \tag{6.8}$$

[1] As already mentioned in Chapter 1.5, we write $R^t\dot{R}\boldsymbol{y} = \boldsymbol{\omega}\times\boldsymbol{y}$ with $\boldsymbol{\omega}$ being the y-coordinates of the angular velocity.

[2] Note that $R(\boldsymbol{a}\times\boldsymbol{b}) = R\boldsymbol{a}\times R\boldsymbol{b}$ for arbitrary $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$.

Inertia Tensor: The mapping given by (6.7) $\Theta\colon \boldsymbol{\omega} \mapsto \boldsymbol{S}$ is the *inertia tensor* of the rigid body with respect to $\boldsymbol{y} = 0$. In components, it has the form

$$S_i = \sum_{k=1}^{3} \Theta_{ik}\omega_k \tag{6.9}$$

with Θ_{ik} from (6.5). Since the mass distribution in the y-system is fixed, the components of Θ_{ik} do not depend on time; they rather characterize an intrinsic property of the rigid body. Furthermore, the Θ-matrix is symmetric, $\Theta_{ik} = \Theta_{ki}$. The inertia tensor Θ therefore has three orthonormal eigenvectors $\boldsymbol{f}_i$ (*principal axes of inertia*) with

$$\Theta\, \boldsymbol{f}_i = \theta_i \boldsymbol{f}_i\,, \qquad (i = 1, 2, 3)\,. \tag{6.10}$$

As can be seen from (6.4), the quadratic form associated with Θ is $2T_{\text{rot}} \geq 0$ and is therefore positive semi-definite. In fact, it is positive definite if the mass distribution is not degenerate (i.e., not concentrated on a line through $\boldsymbol{y} = 0$). In the following, we will mostly describe the system in the principal axis system, in which

$$\Theta = \begin{pmatrix} \theta_1 & & \\ & \theta_2 & \\ & & \theta_3 \end{pmatrix}, \qquad S_i = \theta_i\omega_i, \qquad T = \frac{1}{2}\sum_{i=1}^{3} \theta_i\omega_i^2 \tag{6.11}$$

holds with the *principal moments of inertia* $\theta_i > 0$.

6.2 Equations of Motion

The 6 equations of motion that determine the motion of a rigid body are given by

$$M\ddot{\boldsymbol{X}} = \boldsymbol{F} \qquad \text{(Impulse Law)}, \tag{6.12}$$

$$\dot{\boldsymbol{L}} = \boldsymbol{M} \qquad \text{(Angular Momentum Law)}, \tag{6.13}$$

with $\boldsymbol{F}$ being the total force and $\boldsymbol{M}$ the total torque (with respect to $\boldsymbol{x} = 0$).

Due to the simplicity of the relationship between $\boldsymbol{\omega}$ and $\boldsymbol{S}$ via the inertia tensor in the body-fixed system, it is useful to transform the angular momentum law into the body-fixed system. After applying R^t to the angular momentum law (6.13) using $\dot{\boldsymbol{L}} = M\boldsymbol{b} \times \ddot{\boldsymbol{b}} + \dot{R}\boldsymbol{S} + R\dot{\boldsymbol{S}}$ (from (6.6)), we obtain

$$\dot{\boldsymbol{S}} + \boldsymbol{\omega} \times \boldsymbol{S} + R^t(\boldsymbol{b} \times M\ddot{\boldsymbol{b}}) = R^t\boldsymbol{M}. \tag{6.14}$$

These equations of motion, together with the impulse law (6.12) and the relationship $\boldsymbol{S} = \Theta\,\boldsymbol{\omega}$, determine the trajectory. We want to simplify the equations of motion a bit further in the two cases:

Center of Mass as Reference Point ($\boldsymbol{Y} = 0$): In this case, $\boldsymbol{b} = \boldsymbol{X}$. Then, from the impulse law, we obtain the relationship

$$\dot{\boldsymbol{S}} + \boldsymbol{\omega} \times \boldsymbol{S} = \boldsymbol{N} \tag{6.15}$$

with $\boldsymbol{N} = R^t(\boldsymbol{M} - \boldsymbol{X} \times \boldsymbol{F})$ being the torque with respect to the center of mass in y-components.

Gyroscope ($\boldsymbol{b} = 0$): In the case of a gyroscope held at the suspension point $\boldsymbol{x} = \boldsymbol{y} = 0$, only the angular momentum law is needed. The relationship between the space-fixed and the body-fixed coordinate systems in this case is given by $\boldsymbol{x} = R\boldsymbol{y}$. The equations of motion reduce again to (6.15), this time with $\boldsymbol{N} = R^t\boldsymbol{M}$, the torque with respect to the suspension point.

In the principal axis system (see (6.11)), the *Euler equations* arise from (6.15) in components

$$\begin{aligned} \theta_1\,\dot{\omega}_1 + (\theta_3 - \theta_2)\,\omega_2\,\omega_3 &= N_1, \\ \theta_2\,\dot{\omega}_2 + (\theta_1 - \theta_3)\,\omega_3\,\omega_1 &= N_2, \\ \theta_3\,\dot{\omega}_3 + (\theta_2 - \theta_1)\,\omega_1\,\omega_2 &= N_3. \end{aligned} \tag{6.16}$$

These equations define a system of nonlinear first-order differential equations. Through $\boldsymbol{\omega}(t)$, $\Omega(t) = R^t\dot{R}$ is also given,[3] and to determine $R(t)$, the linear differential equation

$$\dot{R} = R\,\Omega(t) \tag{6.17}$$

needs to be solved.

In general, the explicit solution of the equations of motion of a rigid body is difficult. In the remainder of the chapter, we will restrict ourselves to the treatment of gyroscopes, as for these only $R(t)$ needs to be determined, not additionally $\boldsymbol{b}(t)$. Depending on the principal moments of inertia, we distinguish the following types of gyroscopes:

all θ_i different	(asymmetric gyroscope),
$\theta_1 = \theta_2 \neq \theta_3$	(symmetric gyroscope),
$\theta_1 = \theta_2 = \theta_3$	(spherical gyroscope).

6.3 The Free Gyroscope

For the *free gyroscope*, the Euler equations hold with $\boldsymbol{N} = 0$. A gyroscope is free if either no driving forces act or only the weight force in the case that the

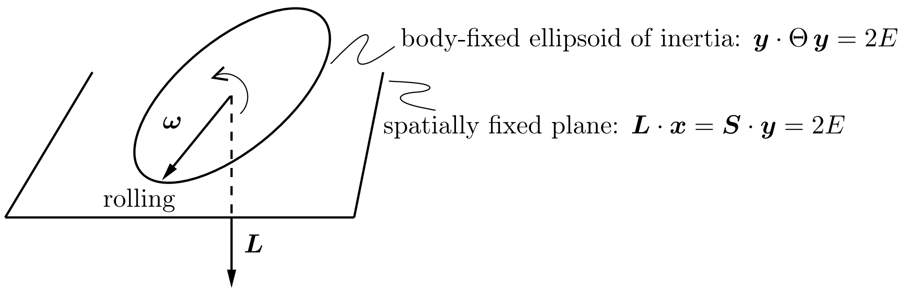
[3] We remind again that $\boldsymbol{\omega} \times \boldsymbol{x} = \Omega\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^3$.

suspension point coincides with the center of mass. Since then $\boldsymbol{b} = 0$, it follows that $T = T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{S}$. With the equation of motion $\dot{\boldsymbol{S}} = -\boldsymbol{\omega} \times \boldsymbol{S}$, one obtains

$$\dot{T} = \boldsymbol{\omega} \cdot \dot{\boldsymbol{S}} = 0, \tag{6.18}$$

i.e., the (kinetic) energy is conserved with $T = E$. Moreover, $\boldsymbol{L}$ is also conserved.

The motions of the free gyroscope with energy E can thus be geometrically represented with the *construction of Poincaré*:



The equations are satisfied by construction for $\boldsymbol{y} = \boldsymbol{\omega}$. In fact, they uniquely determine (for given R) $\boldsymbol{\omega}$ and thus $\dot{R}$. For let $\boldsymbol{y}$ be any point in the space-fixed plane. Then we decompose $\boldsymbol{y} = \boldsymbol{\omega} + (\boldsymbol{y} - \boldsymbol{\omega})$, and find

$$\boldsymbol{y} \cdot \Theta\, \boldsymbol{y} = \underbrace{\boldsymbol{\omega} \cdot \Theta\, \boldsymbol{\omega}}_{=2E} + 2\, \underbrace{(\boldsymbol{y} - \boldsymbol{\omega}) \cdot \overbrace{\Theta\, \boldsymbol{\omega}}^{=\boldsymbol{S}}}_{=2E-2E=0} + \underbrace{(\boldsymbol{y} - \boldsymbol{\omega}) \cdot \Theta(\boldsymbol{y} - \boldsymbol{\omega})}_{\geq 0}\ . \tag{6.19}$$

Thus, $\boldsymbol{y} \cdot \Theta\, \boldsymbol{y} \geq 2E$ and '=' holds only for $\boldsymbol{y} = \boldsymbol{\omega}$.

The space-fixed plane (perpendicular to $\boldsymbol{L}$) is therefore the tangent plane of the inertia ellipsoid at the point $\boldsymbol{\omega}$. Since $\boldsymbol{\omega}$ is the instantaneous axis of rotation, the contact point of the ellipsoid with the tangent plane has zero velocity, i.e., the inertia ellipsoid associated with the principal axes of a gyroscope *rolls* on the plane with a fixed center (torque $\boldsymbol{L}$ fixed) without slipping. Meanwhile, the angular velocity $\boldsymbol{\omega}$ (contact point of the ellipsoid) moves around $\boldsymbol{L}$.

A simple analytical treatment of the free gyroscope is only possible in special cases. We will describe two of them below.

Permanent Rotations: A simple solution exists if the gyroscope rotates around one of its principal axes. As an example, we choose

$$\boldsymbol{\omega} = (\omega_0, 0, 0)\,, \tag{6.20}$$

where ω_0 is constant. This is obviously a solution of the Euler equations. To investigate the stability of this solution, we consider a disturbance $\boldsymbol{\omega} = (\omega_0 + \omega_1, \omega_2, \omega_3)$ by retaining only linear terms in the small quantities $\omega_1, \omega_2, \omega_3$. In this approximation, the Euler equations are given by

$$\begin{aligned} \theta_1 \dot{\omega}_1 &= 0\,, \\ \theta_2 \dot{\omega}_2 &= (\theta_3 - \theta_1)\omega_0 \omega_3\,, \\ \theta_3 \dot{\omega}_3 &= (\theta_1 - \theta_2)\omega_0 \omega_2\,. \end{aligned} \tag{6.21}$$

The first equation states that ω_1 remains constant, and the other two can be written in matrix form

$$\begin{pmatrix} \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\theta_3 - \theta_1}{\theta_2}\omega_0 \\ \frac{\theta_1 - \theta_2}{\theta_3}\omega_0 & 0 \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_3 \end{pmatrix}. \tag{6.22}$$

The eigenvalues $\pm\lambda$ of the 2×2 matrix are given by

$$\lambda^2 = \frac{(\theta_3 - \theta_1)(\theta_1 - \theta_2)}{\theta_2 \theta_3}\omega_0^2 \tag{6.23}$$

and are therefore either both real or both purely imaginary. The first case (with $\lambda \neq 0$) occurs if θ_1 is *between* θ_2 and θ_3, i.e., if $\theta_2 < \theta_1 < \theta_3$, or if $\theta_3 < \theta_1 < \theta_2$. Then, (6.22) has exponentially growing solutions, i.e., the rotation is *unstable*. In the remaining cases, $\lambda^2 < 0$, the eigenvalues are purely imaginary and the solutions are bounded. This means that of the three principal axes, only two exhibit stable permanent rotations, namely those with the smallest and largest principal moments of inertia.

The same result follows even outside the linear approximation: As we have seen above, the kinetic energy T and the angular momentum $\boldsymbol{L}$ are conserved, and therefore we have

$$T = \frac{1}{2}\sum_{i=1}^{3} \theta_i \omega_i^2 = \frac{1}{2}\sum_{i=1}^{3} \frac{S_i^2}{\theta_i} \equiv E\,, \tag{6.24}$$

$$\boldsymbol{L} \cdot \boldsymbol{L} = \boldsymbol{S} \cdot \boldsymbol{S} = \sum_{i=1}^{3} S_i^2 \equiv l^2\,. \tag{6.25}$$

Thus, $\boldsymbol{S}$ always lies on the intersection of an ellipsoid (with the semi-axes $\sqrt{2E\theta_i}$) and a sphere of radius l, see Fig. 6.2. The intersection curves for $\theta_1 < \theta_2 < \theta_3$ near the principal axes $\boldsymbol{f}_1$ and $\boldsymbol{f}_3$ are small closed curves, i.e., $\boldsymbol{S}$ (and thus $\boldsymbol{\omega}$) remains near them. Not near $\boldsymbol{f}_2$.

The Symmetric Free Gyroscope: In the special case of a symmetric body, the inertia ellipsoid is a rotational ellipsoid, i.e., two of the three principal moments of inertia are equal. Without loss of generality, let $\theta_1 = \theta_2$, and thus $\boldsymbol{f}_3$ is the *figure*

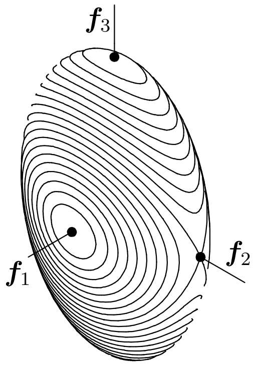


Figure 6.2: In the diagram, the intersection curves for the case $\theta_1 < \theta_2 < \theta_3$ are shown.

axis. The Euler equations then simplify to

$$\begin{aligned} \dot{\omega}_1 &= -\alpha\,\omega_2\,, \\ \dot{\omega}_2 &= \alpha\,\omega_1\,, \\ \dot{\omega}_3 &= 0\,. \end{aligned} \qquad \left(\alpha = \frac{\theta_3 - \theta_1}{\theta_1}\omega_3\right), \tag{6.26}$$

The last equation requires that ω_3 is constant. Thus, α is also constant. The solution of the differential equation for ω_1 and ω_2 is then given by

$$\omega_1(t) + i\omega_2(t) = (\omega_1(0) + i\omega_2(0))e^{i\alpha t}\,. \tag{6.27}$$

The vector $\boldsymbol{\omega}$ thus rotates around the figure axis with constant angular velocity α. The motion $R(t)$ results as a special case of the next section or with the construction of Poincaré: It is simply seen that in the symmetric gyroscope, the angular momentum $\boldsymbol{S}$, the axis of rotation $\boldsymbol{\omega}$, and the figure axis $\boldsymbol{f}_3$ always lie in one plane.[4]

Now, if the axis of rotation moves around the figure axis with angular velocity α, this implies in space-fixed coordinates that both the figure axis and the axis of rotation move around the space-fixed angular momentum $\boldsymbol{L}$ with angular velocity α. It is said that the figure axis *precesses*.

Example (Euler's Theory of Pole Oscillations)

> The kinematic North Pole (direction of the instantaneous axis of rotation $\boldsymbol{\omega}$) of the Earth rotates around the geometric North Pole (figure axis). The external torques acting on the Earth are so weak that the rotational motion can be considered as force-free. The Earth is symmetric around the pole axis and slightly flattened at the poles, so that θ_1 is less than θ_3. Specifically, the numerical ratio of the moments of inertia is given by
>
> $$\frac{\theta_1 - \theta_3}{\theta_1} = -0.0033\,. \tag{6.28}$$

[4]In body-fixed coordinates, it holds that $\boldsymbol{\omega} \times \boldsymbol{f}_3 = (\omega_2, -\omega_1, 0)$, $\boldsymbol{S} = (\theta_1\omega_1, \theta_1\omega_2, \theta_3\omega_3)$ and thus $\boldsymbol{S} \cdot (\boldsymbol{\omega} \times \boldsymbol{f}_3) = 0$.

The period of the precession movement is therefore

$$T = \frac{2\pi}{\alpha} = \underbrace{\frac{2\pi}{\omega_3}}_{=1\text{day}} \underbrace{\frac{\theta_1}{\theta_3 - \theta_1}}_{\approx 300} \approx 300 \text{ days}\,. \tag{6.29}$$

It follows that the rotational axis of the Earth describes a circle around the North Pole over the course of 10 months. A (though irregular) motion of this kind has been observed. The amplitude of the precession is very small, the axis of rotation never wanders more than about 4.5 m from the North Pole. The path is, however, completely irregular, and the fundamental period seems to be approximately 427 days (and not 300 days). The fluctuations are attributed to small displacements of the mass distribution on the Earth, such as those caused by atmospheric movements, while the difference in the period arises from the fact that the Earth is not completely rigid, but has the elastic properties of a material like steel.[5]

6.4 The Heavy Symmetric Gyroscope

The other important example is the gyroscope in a homogeneous gravitational field. In the following, we consider the case of the symmetric gyroscope, i.e., we again assume that $\theta_1 = \theta_2$. For symmetry reasons, the center of mass then lies on the figure axis. In the body-fixed coordinate system, it therefore has the components

$$\boldsymbol{y}_S = (0, 0, l)\,, \qquad l > 0\,. \tag{6.30}$$

We choose the space-fixed x-system such that the x_3-axis points vertically upward. As position coordinates, we use *the Euler angles* (φ, θ, ψ), cf. Fig. 6.3. The body-fixed y-system (with the basis vectors $\boldsymbol{f}_i$) arises from the space-fixed x-system (with the basis vectors $\boldsymbol{e}_i$) through the following sequence of 3 rotations[6]

$$R = R_1(\varphi) R_2(\theta) R_3(\psi)$$

with:

Rotation	Rotation Axis	Rotation Angle	
1.	x_3	φ	($\boldsymbol{e}_1$ goes over to the node line)
2.	K	θ	($\boldsymbol{e}_3$ goes to $\boldsymbol{f}_3$)
3.	y_3	ψ	

[5]The force-free precession of the Earth's axis should not be confused with its slow precession around the normal to the ecliptic. This astronomical precession arises from gravitational forces of the Sun and the Moon, which were considered negligible in the above discussion. This assumption is justified since the period of astronomical precession is very large (26,000 years).

[6]We describe the rotations as a transformation from the x- to the y-system. The order of rotation is thus exactly reversed, as in the definition $\boldsymbol{x} = R\boldsymbol{y}$. The order of the Euler angles is chosen such that typically $\dot\psi > \dot\theta > \dot\varphi$ is.

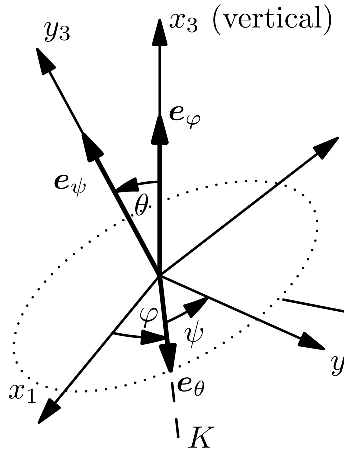


Figure 6.3: Coordinate system for describing the heavy symmetric gyroscope: The angle θ indicates the inclination of the gyroscope. The angle θ is also called the 'nutation angle'. The angle φ is called the 'precession angle'. It indicates the orientation of the gyroscope axis projected onto the 'ground' (the node line K). The angle ψ is the 'self-rotation angle'.

where the 'node line' K is the image of the x_1-axis under the 1st rotation. The corresponding chart of SO(3) is the region

$$\{0 < \theta < \pi; 0 < \varphi, \psi < 2\pi\} \subset \mathbb{R}^3 . \tag{6.31}$$

The three Euler angles are also called precession angle (φ), nutation angle (θ), and self-rotation angle (ψ).

The rotation axes are given by the unit vectors $\boldsymbol{e}_\varphi$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\psi$. In the body-fixed coordinate system, $\boldsymbol{e}_\psi = \boldsymbol{f}_3$ is precisely the unit vector in the y_3 direction. The other two rotation axes are somewhat more complicated. However, one can easily find geometrically that

$$\begin{aligned} \boldsymbol{e}_\theta &= \cos\psi\, \boldsymbol{f}_1 - \sin\psi\, \boldsymbol{f}_2 , \\ \boldsymbol{e}_\varphi &= \sin\theta(\sin\psi\, \boldsymbol{f}_1 + \cos\psi\, \boldsymbol{f}_2) + \cos\theta\, \boldsymbol{f}_3 \end{aligned} \tag{6.32}$$

with $\boldsymbol{f}_i$ being the unit vectors along the y_i-axes.

The angular velocity (in body-fixed components) is thus given by

$$\begin{aligned} \boldsymbol{\omega} &= \dot\varphi\, \boldsymbol{e}_\varphi + \dot\theta\, \boldsymbol{e}_\theta + \dot\psi\, \boldsymbol{e}_\psi , \\ &= \left(\dot\varphi \sin\theta \sin\psi + \dot\theta \cos\psi, \dot\varphi \sin\theta\cos\psi - \dot\theta\sin\psi, \dot\varphi\cos\theta + \dot\psi\right) . \end{aligned} \tag{6.33}$$

The kinetic energy

$$T = \frac{\theta_1}{2}(\omega_1^2 + \omega_2^2) + \frac{\theta_3}{2}\omega_3^2 \tag{6.34}$$

thus calculates to

$$T = \frac{\theta_1}{2}\left(\dot\varphi^2 \sin^2\theta + \dot\theta^2\right) + \frac{\theta_3}{2}\left(\dot\psi + \dot\varphi\cos\theta\right)^2 . \tag{6.35}$$

The center of mass has the form in the space-fixed coordinate system

$$\boldsymbol{x}_S = (*, *, l\cos\theta) . \tag{6.36}$$

This leads to the Lagrangian

$$L = T - V\,, \tag{6.37}$$

with

$$V = mgl\cos\theta\,. \tag{6.38}$$

Conservation Laws: Since L does not depend on t, φ, and ψ, the system has the three conserved quantities:

$$T + V = E\,, \tag{6.39}$$

$$p_\varphi = \frac{\partial L}{\partial\dot\varphi} = \dot\varphi\,\theta_1\sin^2\theta + (\dot\psi + \dot\varphi\cos\theta)\,\theta_3\cos\theta \equiv L_3\,, \tag{6.40}$$

$$p_\psi = \frac{\partial L}{\partial\dot\psi} = (\dot\psi + \dot\varphi\cos\theta)\,\theta_3 \equiv S_3\,. \tag{6.41}$$

We denote the corresponding constant values by E, L_3, and S_3. Here, L_3 and S_3 are the projections of the angular momentum onto $\boldsymbol{e}_\varphi = \boldsymbol{e}_3$ (vertical) and onto $\boldsymbol{e}_\psi = \boldsymbol{f}_3$ (figure axis), respectively. From $\boldsymbol{S} = \Theta\,\boldsymbol{\omega}$, it directly follows

$$p_\varphi = \boldsymbol{S}\cdot\boldsymbol{e}_\varphi\,, \qquad p_\psi = \boldsymbol{S}\cdot\boldsymbol{e}_\psi\,. \tag{6.42}$$

The conservation of $T + V$ and p_φ also holds more generally if the gyroscope is not symmetric; however, p_ψ is only conserved for symmetric gyroscopes.

Integration of the Equations of Motion: From (6.40) and (6.41), we obtain

$$\dot\psi + \dot\varphi\cos\theta = \frac{S_3}{\theta_3}\,, \qquad \dot\varphi = \frac{L_3 - S_3\cos\theta}{\theta_1\sin^2\theta}\,. \tag{6.43}$$

Substituting into (6.39) yields

$$E' \equiv E - \frac{S_3^2}{2\theta_3} = \frac{\theta_1}{2}\dot\theta^2 + \frac{(L_3 - S_3\cos\theta)^2}{2\theta_1\sin^2\theta} + mgl\cos\theta\,. \tag{6.44}$$

This is a first-order differential equation for the *nutation motion* $\theta(t)$. To solve it, we introduce the variable $u = \cos\theta$. With $\dot u = -\sin\theta\,\dot\theta$, one finds

$$\dot u^2 = (\alpha - \beta u)(1 - u^2) - (a - bu)^2 \equiv f(u)\,, \tag{6.45}$$

where the constants α, β, a, and b are given by

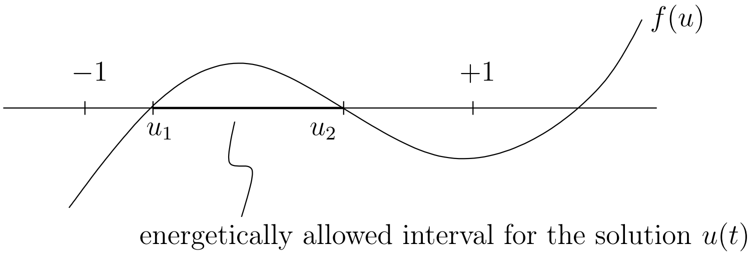
$$\alpha = \frac{2E'}{\theta_1}\,, \qquad \beta = \frac{2mgl}{\theta_1} > 0\,, \qquad a = \frac{L_3}{\theta_1}\,, \qquad b = \frac{S_3}{\theta_1} \tag{6.46}$$

Note that u is only defined for $-1 \le u \le 1$. Since the left side is non-negative, $\dot u^2 \ge 0$, the general solution is restricted to the region where $f(u) \ge 0$; there it reads

$$t(u) - t(v) = \int_v^u \frac{dx}{\sqrt{f(x)}}\,. \tag{6.47}$$

The function $f(u)$ is a cubic polynomial. The roots of this cubic polynomial correspond to the angles for which $\dot{\theta}$ changes its sign, i.e., the turning angles of θ. For large u, βu^3 is the dominating term; since $\beta > 0$, $f(u)$ is therefore positive for large positive u and negative for large negative u.

At the points $u = \pm 1$, $f(u) = -(a \mp b)^2$, and is therefore always negative (we treat the general case $b \neq \pm a$)[7]. Thus, at least one of the roots must lie in the region $u > 1$, which corresponds to no physical angle. Due to (6.45), $f(u) \geq 0$ must hold somewhere in the physical range ($|u| \leq 1$). Therefore, for a gyroscope, $f(u)$ must have two roots $u_1 \leq u_2$ that lie within the interval $[-1, 1]$ with the general shape



of $f(u)$. The points u_1 and u_2 are the turning points of the nutation motion. This motion is periodic with the period

$$T = 2 \int_{u_1}^{u_2} \frac{dx}{\sqrt{f(x)}} \,. \tag{6.48}$$

Using (6.43), one can then also determine $\varphi(t)$ and $\psi(t)$ from a solution for $u(t) = \cos\theta(t)$. In particular, we have

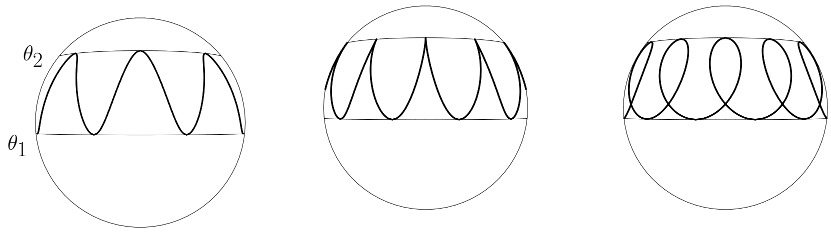
$$\dot{\varphi} = \frac{a - bu}{1 - u^2} \,, \tag{6.49}$$

and thus $\dot{\varphi}(t)$ has *the same* period T. On the other hand, $\varphi(t)$ describes the precession of the figure axis around the vertical. By integrating (6.49), we find that this precession is the sum of a linear function in t (mean precession) and a periodic function of period T.

It is common to describe the motion of the gyroscope by plotting the intersection curve of the figure axis on a sphere with unit radius around the suspension point. This curve is called the *locus* of the figure axis. The spherical coordinates of a point on the locus in the space-fixed system are identical to the Euler angles θ and φ.[8] The third angle, ψ, only describes the rotation of the gyroscope around its own symmetry axis. As we have seen above, the locus runs between the two limiting circles $\theta_1 = \arccos u_1$ and $\theta_2 = \arccos u_2$, where $\dot{\theta}$ vanishes on both circles. The shape of the locus curve is largely determined by the value of the root $u_0 \equiv a/b$ of $a - bu$. It can be shown that $u_0 > u_1$. From (6.49), we obtain the result that the precession motion changes direction for $u > u_0 = a/b$. Thus, we obtain the following locus curves:

[7] The limiting cases correspond to the standing gyroscope ($b = a$, $S_3 = L_3$) and the hanging gyroscope ($b = -a$, $S_3 = -L_3$).

[8] The angle φ is conventionally measured from the $(-x_2)$-axis.



$a/b > u_2$ $\qquad$ $a/b = u_2$ $\qquad$ $u_1 < a/b < u_2$

In the first case, $\dot{\varphi} > 0$ for all times. The figure axis thus performs a *precession* around the vertical while θ performs a *nutation* between the limiting circles. The second case with $a/b = u_2$ is not as extraordinary as one might initially think. It corresponds to the initial conditions $\dot{\theta}(0) = \dot{\varphi}(0) = 0$ (releasing the gyroscope without initial precession of the figure axis). Then, $u(0) = u_0 = a/b$ holds (due to $\dot{\varphi}(0) = 0$) and $f(u_0) = 0$ (due to $\dot{\theta}(0) = 0$). In this case, the gyroscope 'falls' from θ_2 down to θ_1 and then reorients itself. In the last case, $\dot{\varphi}$ changes sign at the limiting circle $\theta_0 = \arccos u_0$. For $\theta_1 \leq \theta < \theta_0$, normal precession occurs, while for a more upright gyroscope (with $\theta_0 < \theta \leq \theta_2$), the precession runs backward.

6.4.1 Examples

Pure Precession: We obtain a *pure precession* as a special case if $u_1 = u_2 \equiv \bar{u}$. Then $\dot{\varphi}$ is constant and $f(u)$ has a double root at $\bar{u}$, i.e., $f(\bar{u}) = f'(\bar{u}) = 0$.

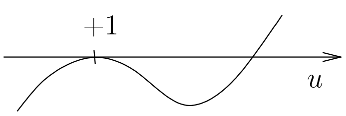
Standing Gyroscope: We consider the stability of the *standing gyroscope* with angular velocity ω. Initially, $\theta = \dot{\theta} = 0$ and thus $L_3 = S_3 = \theta_3\omega$, $E' = mgl$. We obtain

$$\alpha = \beta = \frac{2mgl}{\theta_1}, \qquad a = b = \frac{\theta_3\omega}{\theta_1}.$$

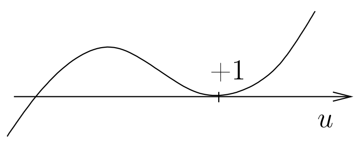
Thus, we have

$$f(u) = (1-u)^2[\alpha(1+u) - a^2]$$

which has a double root at $u = 1$ and thus $\theta = 0$ is an equilibrium point. For the stability of this motion, we consider the behavior of f. Depending on the sign of $[\cdots]_{u=1} = 2\alpha - a^2$, we obtain



$2\alpha < a^2$
d.h. $\omega^2 > 4\theta_1 mgl/\theta_3^2$
(stable)



$2\alpha > a^2$
d.h. $\omega^2 < 4\theta_1 mgl/\theta_3^2$
(unstable)

as a possible behavior.

The first case corresponds to a *fast gyroscope* with $|\omega| > \omega_c \equiv 2\sqrt{\theta_1 mgl}/\theta_3$ and $u = 1$, $\theta = 0$ as the only physical solution. The standing position is thus stable and the gyroscope *sleeps*. In the second case ($|\omega| < \omega_c$), the equilibrium solution $\theta = 0$ is unstable, as an energetically allowed interval follows to the left of $u = 1$. The upright spinning top begins to wobble as soon as $|\omega|$ drops below the stability threshold ω_c due to friction losses. Thus, the *sleepless* gyroscope awakens after a certain time and begins to tumble.

Chapter 7

Hamiltonian Systems

For various aspects of mechanics (and especially for quantum mechanics and statistical mechanics), the Hamiltonian description of mechanics is important. The basic idea is that, instead of using the position coordinates and their velocities to describe the system, one replaces the velocities with the conjugate momenta. This approach will now be explained in detail.

7.1 Hamiltonian Function

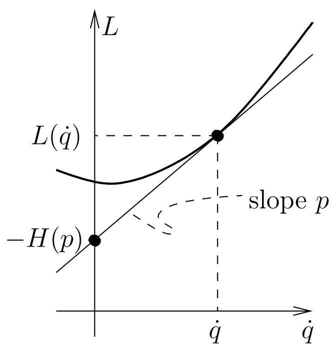
Consider a Lagrangian system for which the Lagrangian function L depends on the position coordinates q^α, their velocities $\dot{q}^\alpha$, and possibly time t, $L \equiv L(q, \dot{q}, t)$. The (canonical) conjugate momenta are defined by

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} \equiv p_\alpha(q, \dot{q}, t) \ , \qquad (\alpha = 1, \ldots, f) \tag{7.1}$$

As we will see later, the motion of a mechanical system can often be described more naturally in the *phase space*, whose coordinates are the position coordinates q^α and the corresponding momenta p_α. To achieve this description, we need to express the velocities $\dot{q}^\alpha$ in terms of p_α.

The *Hamiltonian function* $H(q, p, t)$ arises from the Lagrangian function $L(q, \dot{q}, t)$ through a contact (or Legendre) transformation at fixed (q, t). In this process, the variable $\dot{q}$ is replaced by p via:

Geometrically:



Analytically:

One solves (7.1) for

$$\dot q = \dot q(q,p,t) \tag{7.2}$$

and substitutes this into

$$\langle p, \dot q\rangle - L = \sum_{\alpha=1}^{f} p_\alpha \, \dot q^\alpha - L(q,\dot q,t) \equiv H(q,p,t) \tag{7.3}$$

(cf. (4.59)).

We want to determine the relationship between the partial derivatives of H and those of L. We obtain directly

$$\begin{aligned}
\frac{\partial H}{\partial p_\alpha} &= \dot q^\alpha + \sum_{\beta=1}^{f}\left(p_\beta - \frac{\partial L}{\partial \dot q^\beta}\right)\frac{\partial \dot q^\beta}{\partial p_\alpha} = \dot q^\alpha \\
\frac{\partial H}{\partial q^\alpha} &= \sum_{\beta=1}^{f}\left(p_\beta - \frac{\partial L}{\partial \dot q^\beta}\right)\frac{\partial \dot q^\beta}{\partial q^\alpha} - \frac{\partial L}{\partial q^\alpha} = -\frac{\partial L}{\partial q^\alpha} \\
\frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t}
\end{aligned} \tag{7.4}$$

since $\partial L/\partial \dot q^\beta = p_\beta$.

7.2 The Canonical Equations of Motion

The equations of motion for the phase space coordinates (q,p) can now be easily derived from the Euler-Lagrange equations $\dot p_\alpha = \partial L/\partial q^\alpha$. The resulting Hamiltonian (*canonical*) equations of motion are

$$\dot q^\alpha = \frac{\partial H}{\partial p_\alpha}\,, \qquad \dot p_\alpha = -\frac{\partial H}{\partial q^\alpha}\,, \qquad (\alpha = 1,\dots f)\,. \tag{7.5}$$

These are $2f$ first-order differential equations for the $2f$ unknown functions $q^1(t),\dots q^f(t)$, $p_1(t),\dots p_f(t)$.

Examples:

(i) Point mass in an external potential $V(\boldsymbol{x})$:

$$\begin{aligned}
L &= \frac{m}{2}\dot{\boldsymbol{x}}^2 - V(\boldsymbol{x})\,, \qquad \boldsymbol{p} = m\dot{\boldsymbol{x}}\,, \\
H &= \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{x})\,.
\end{aligned}$$

(ii) The same system in polar coordinates

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r)\,, \qquad p_r = m\dot{r}\,, \qquad p_\varphi = mr^2\dot{\varphi}\,,$$
$$H = \frac{1}{2m}(p_r^2 + r^{-2}p_\varphi^2) + V(r)\,.$$

(iii) The general autonomous case (cf. Example 9 from Chapter 4.5.3)

$$L = \frac{1}{2}\sum_{\alpha,\beta=1}^{f} g_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta - V(q)\,, \qquad p_\alpha = \sum_{\beta=1}^{f} g_{\alpha\beta}(q)\dot{q}^\beta\,,$$
$$H = \frac{1}{2}\sum_{\alpha,\beta=1}^{f} g^{\alpha\beta}(q)p_\alpha p_\beta + V(q)\,,$$

where $(g^{\alpha\beta}(q))_{\alpha,\beta=1,\dots,f}$ is the inverse of the matrix $(g_{\alpha\beta}(q))_{\alpha,\beta=1,\dots,f}$.

(iv) Charged particle in an external electromagnetic field:

$$L = \frac{m}{2}\dot{\boldsymbol{x}}^2 - e\Big(\varphi(\boldsymbol{x},t) - \frac{\dot{\boldsymbol{x}}}{c}\cdot\boldsymbol{A}(\boldsymbol{x},t)\Big)\,, \qquad \boldsymbol{p} = m\dot{\boldsymbol{x}} + \frac{e}{c}\boldsymbol{A}\,,$$
$$H = \frac{1}{2m}\Big(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{x},t)\Big)^2 + e\varphi(\boldsymbol{x},t)\,.$$

Since the Lagrangian function does not explicitly depend on time, in these cases the Hamiltonian function is simply the generalized energy, cf. (4.59). However, it should be noted that H is a function of (q, p, t), i.e., $\dot{q}$ must be expressed in terms of p.

7.3 Phase Space and Poisson Brackets

We denote the $2f$ variables

$$x \equiv (x_1, \dots, x_{2f}) \equiv (q^1, p_1, \dots, q^f, p_f) \tag{7.6}$$

as the *phase coordinates* of the system, and the corresponding space as the *phase space*. If the configuration space is $\mathbb{R}^f$, then the phase space is simply $\mathbb{R}^{2f}$. The canonical equations (7.5) in this notation read

$$\dot{x}_i = \sum_{k=1}^{2f} \varepsilon_{ik}\frac{\partial H}{\partial x_k}\,, \qquad \varepsilon = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}. \tag{7.7}$$

The matrix ε has the important properties $\varepsilon^t = \varepsilon^{-1} = -\varepsilon$.

Now let $F(q,p) = F(x)$ be any function on the phase space. Along a mechanical path $x(t) = (q(t), p(t))$, the change in $F(x(t))$ is given by

$$\begin{aligned} \frac{d}{dt}F(q(t),p(t)) &= \sum_{k=1}^{2f} \frac{\partial F}{\partial x_k}\dot{x}_k = \sum_{i,k=1}^{2f} \frac{\partial F}{\partial x_k}\varepsilon_{ki}\frac{\partial H}{\partial x_i} \\ &= \sum_{\alpha=1}^{f}\left(\frac{\partial F}{\partial q^\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial H}{\partial q^\alpha}\right). \end{aligned} \tag{7.8}$$

For two arbitrary functions of the phase space $F_1(q,p)$ and $F_2(q,p)$, the *Poisson bracket* is defined as

$$\{F_1, F_2\} \equiv \sum_{i,k=1}^{2f} \frac{\partial F_1}{\partial x_i}\varepsilon_{ik}\frac{\partial F_2}{\partial x_k} = \sum_{\alpha=1}^{f}\left(\frac{\partial F_1}{\partial q^\alpha}\frac{\partial F_2}{\partial p_\alpha} - \frac{\partial F_1}{\partial p_\alpha}\frac{\partial F_2}{\partial q^\alpha}\right) \tag{7.9}$$

of F_1 and F_2. With this notation, the time derivative of F (along a mechanical path) can be rewritten as

$$\frac{d}{dt}F(q(t),p(t)) = \{F, H\}(q(t),p(t)), \tag{7.10}$$

where we have made explicit that the Poisson bracket must be evaluated at the point $x(t)$. For $F(x) = x_i$, we find the canonical equations of motion in the form

$$\dot{x}_i = \{x_i, H\}. \tag{7.11}$$

Example: Particularly simple Poisson brackets exist between momentum and position coordinates. In fact, it is easy to calculate that the "canonical Poisson brackets" hold

$$\{q^\alpha, q^\beta\} = 0\,, \qquad \{q^\alpha, p_\beta\} = \delta^\alpha{}_\beta\,, \qquad \{p_\alpha, p_\beta\} = 0\,,$$

where $\delta^\alpha{}_\beta$ is the Kronecker symbol.

The Poisson brackets are linear

$$\{aF_1 + bF_2, F_3\} = a\{F_1, F_3\} + b\{F_2, F_3\}, \qquad (a, b \in \mathbb{R}) \tag{7.12}$$

and antisymmetric (since $\varepsilon^t = -\varepsilon$)

$$\{F_1, F_2\} = -\{F_2, F_1\}. \tag{7.13}$$

They also satisfy the *Jacobi identity*

$$\{\{F_1, F_2\}, F_3\} + \{\{F_2, F_3\}, F_1\} + \{\{F_3, F_1\}, F_2\} = 0\,. \tag{7.14}$$

We will provide the proof of this at the end of Chapter 7.5.

7.4 Canonical Transformations

As we saw at the beginning of Chapter 7.3, the Hamiltonian equations of motion in phase space take the simple form $\dot{x}_i = \{x_i, H\}$ or also (with $\varepsilon^{-1} = \varepsilon^t$)

$$\sum_{i=1}^{2f} \varepsilon_{ik}\, \dot{x}_i = \frac{\partial H}{\partial x_k}\,. \tag{7.15}$$

We now want to study the structure of bijective coordinate transformations

$$\bar{x}_i = \phi_i(x_1, \dots, x_{2f}) \tag{7.16}$$

that leave the canonical equations (7.15) form-invariant. This means that for *every* Hamiltonian function H, the equations of motion

$$\sum_{i=1}^{2f} \varepsilon_{ik} \dot{\bar{x}}_i = \frac{\partial \bar{H}}{\partial \bar{x}_k}\,, \tag{7.17}$$

with $\bar{H}(\bar{x}) = (\bar{H} \circ \phi)(x) = H(x)$ are equivalent to (7.15). Transformations with this property are called *canonical transformations.*

From (7.16) and (7.17), we directly obtain

$$\sum_{i,j=1}^{2f} \varepsilon_{ik} \frac{\partial \bar{x}_i}{\partial x_j} \dot{x}_j = \sum_{l=1}^{2f} \frac{\partial H}{\partial x_l} \frac{\partial x_l}{\partial \bar{x}_k}\,. \tag{7.18}$$

Now we define the *Jacobian matrix*

$$A_{ij}(x) \equiv \frac{\partial \phi_i}{\partial x_j} = \frac{\partial \bar{x}_i}{\partial x_j}, \tag{7.19}$$

and thus obtain the condition[1]

$$\sum_{j=1}^{2f} \underbrace{\left(\sum_{i,k=1}^{2f} A_{ij} \varepsilon_{ik} A_{kl} \right)}_{=\varepsilon_{jl},\ \text{since invariant to (7.15)}} \dot{x}_j = \frac{\partial H}{\partial x_l}\,.$$

We conclude that the canonical transformations are characterized by

$$A^t(x) \varepsilon A(x) = \varepsilon \qquad (\text{for all } x)\,. \tag{7.20}$$

The linear mappings $A\colon \mathbb{R}^{2f} \to \mathbb{R}^{2f}$ with the property $A^t \varepsilon A = \varepsilon$ are called *symplectic* and form the group $\mathrm{Sp}(2f)$. In particular, it holds

$$(\mathrm{Det}\, A)^2 = 1 \tag{7.21}$$

and even $\mathrm{Det}\, A = +1$, which we will not prove here. Thus, the inverse of a canonical transformation exists. If $A \in \mathrm{Sp}(2f)$, then the following properties hold:

[1] Note that $\partial x_j / \partial \bar{x}_i = (A^{-1})_{ji}$.

- $A^{-1} \in \mathrm{Sp}(2f)$, since $(A^{-1})^t \varepsilon A^{-1} = (A^{-1})^t (A^t \varepsilon A) A^{-1} = \varepsilon$,
- $A^t \in \mathrm{Sp}(2f)$, since $\varepsilon = -\varepsilon^{-1} = -((A^{-1})^t \varepsilon A^{-1})^{-1} = -A\varepsilon^{-1}A^t = A\varepsilon A^t$.

A canonical transformation preserves *all* Poisson brackets, i.e.

$$\{F_1, F_2\}(x) = \{\bar{F}_1, \bar{F}_2\}(\bar{x})\,, \quad \text{where } F_1(x) = \bar{F}_1(\bar{x}),\ F_2(x) = \bar{F}_2(\bar{x}). \tag{7.22}$$

This follows simply from the fact that[2]

$$\begin{aligned}\{F_1, F_2\} &= \sum_{j,k=1}^{2f} \frac{\partial F_1}{\partial x_j}\,\varepsilon_{jk}\,\frac{\partial F_2}{\partial x_k} = \sum_{i,l=1}^{2f} \frac{\partial \bar{F}_1}{\partial \bar{x}_i} \underbrace{\left(\sum_{j,k=1}^{2f} A_{ij}\,\varepsilon_{jk}\,A_{lk}\right)}_{=(A\varepsilon A^t)_{il}=\varepsilon_{il},\ \text{since } A^t\in\mathrm{Sp}(2f)} \frac{\partial \bar{F}_2}{\partial \bar{x}_l} \\ &= \{\bar{F}_1, \bar{F}_2\}\,,\end{aligned} \tag{7.23}$$

With the form (7.11) of the canonical equations of motion, it is also clear in the reverse direction that any transformation that preserves all Poisson brackets (7.22) defines a canonical transformation.

Example 1: (*Point transformations*) In the Lagrangian formalism, any new position coordinates $\bar{q}^\alpha = \bar{q}^\alpha(q^1, \ldots, q^f)$ can be introduced. The Lagrangian function defined by $T - V$ remains invariant with $\bar{L}(\bar{q}, \dot{\bar{q}}) = L(q, \dot{q})$. The conjugate momenta are therefore given by

$$\bar{p}_\alpha = \frac{\partial \bar{L}}{\partial \dot{\bar{q}}^\alpha} = \sum_{\beta=1}^{f} \frac{\partial L}{\partial \dot{q}^\beta}\frac{\partial \dot{q}^\beta}{\partial \dot{\bar{q}}^\alpha} \overset{(4.10)}{=} \sum_{\beta=1}^{f} p_\beta \frac{\partial q^\beta}{\partial \bar{q}^\alpha}\,. \tag{7.24}$$

As we will see in Chapter 7.7.1, this transformation is of course also canonical. This result can also be obtained by directly calculating the Jacobian matrix.

Example 2: The group of canonical transformations is however significantly larger than the group of point transformations. This fact is one of the advantages of the Hamiltonian compared to the Lagrangian formalism. For example, the transformation $(f = 1)$

$$\bar{q} = \sqrt{pq^3}\,, \qquad \bar{p} = \sqrt{\frac{p}{q}}\,.$$

is also a canonical transformation. To prove this, we calculate the Jacobian matrix

$$A = \frac{1}{2}\begin{pmatrix} 3\sqrt{pq} & \sqrt{p^{-1}q^3} \\ -\sqrt{pq^{-3}} & \sqrt{p^{-1}q^{-1}} \end{pmatrix}.$$

It can be easily verified that this is symplectic.

[2] In the literature, this relationship is often written as $\{F_1, F_2\}_{\bar{x}} = \{F_1, F_2\}_x$, where the index of the Poisson bracket indicates the "variables" of the functions F_1, F_2

7.5 Canonical Flows

Let $x = (x_1, \ldots, x_{2f})$ be phase coordinates in the phase space $\Gamma = \mathbb{R}^{2f}$. Let $\phi_\lambda \colon x \mapsto \bar{x} \equiv x(\lambda)$ be a *canonical flow*, i.e., a family of canonical transformations with $x(0) = \phi_0(x) = x$. By differentiating with respect to λ, we obtain the generating vector field $v(x)$, (cf. (4.53))

$$\frac{dx(\lambda)}{d\lambda} = \frac{\partial}{\partial\lambda}\phi_\lambda(x) = v(\phi_\lambda(x)). \tag{7.25}$$

The fact that the transformation ϕ_λ is to be canonical for every λ means that the Jacobian matrix

$$(A_\lambda)_{ik}(x) = \frac{\partial(\phi_\lambda)_i}{\partial x_k} = \frac{\partial \bar{x}_i}{\partial x_k} \tag{7.26}$$

is symplectic for all x and λ.

We now want to characterize the vector fields $v(x)$ that generate *canonical flows*. From (7.25), we obtain (at each point x)

$$\frac{\partial}{\partial\lambda}(A_\lambda)_{ik} = \frac{\partial}{\partial x_k} v_i(\phi_\lambda(x)) = \sum_{l=1}^{2f} \underbrace{\frac{\partial v_i}{\partial \bar{x}_l}(\bar{x})}_{\equiv V_{il}(\bar{x})}(A_\lambda)_{lk}\,, \tag{7.27}$$

or in matrix notation

$$\frac{\partial}{\partial\lambda}A_\lambda(x) = V(\bar{x})\, A_\lambda(x)\,, \qquad \text{with } \bar{x} = \phi_\lambda(x)\,. \tag{7.28}$$

For $\lambda = 0$, we trivially have a symplectic transformation with $A_0(x) = 1$. For the transformation to remain symplectic for all λ, we must therefore only require that

$$0 = \frac{\partial}{\partial\lambda}(A_\lambda^t \varepsilon A_\lambda) = A_\lambda^t(V^t\varepsilon + \varepsilon V)A_\lambda\,. \tag{7.29}$$

Since $\operatorname{Det} A_\lambda \neq 0$ and $\varepsilon^t = -\varepsilon$, this is equivalent to $(\varepsilon^t V)^t = \varepsilon^t V$ or (the kl-element) written out

$$\frac{\partial}{\partial \bar{x}_k} \underbrace{\sum_{i=1}^{2f} \varepsilon_{il} v_i(\bar{x})}_{\equiv g_l(\bar{x})} = \frac{\partial}{\partial \bar{x}_l} \underbrace{\sum_{i=1}^{2f} \varepsilon_{ik} v_i(\bar{x})}_{= g_k(\bar{x})}\,. \tag{7.30}$$

These conditions mean that $g(x)$ is the gradient of a function $F(x)$, i.e.

$$g_k(x) = \sum_{i=1}^{2f} \varepsilon_{ik} v_i(x) = \frac{\partial F}{\partial x_k}(x) \quad \text{or} \quad v_i(x) = \sum_{k=1}^{2f} \varepsilon_{ik} \frac{\partial F}{\partial x_k}(x)\,. \tag{7.31}$$

The vector fields of canonical flows are characterized by (7.31).

Thus, we obtain the important result that the differential equation (7.25), which characterizes canonical flows, has the form of canonical equations (cf. (7.11))

$$\frac{dx_i(\lambda)}{d\lambda} = \sum_{k=1}^{2f} \varepsilon_{ik} \frac{\partial F}{\partial x_k}(x(\lambda)) = \{x_i, F\}\,, \tag{7.32}$$

where the parameter λ plays the role of time and F that of the Hamiltonian function.

From the derivation, the reverse also follows directly: Canonical equations generate canonical flows in phase space. The function $F(x)$ is called the *generating function* of the canonical flow.

Example: The dynamics (time evolution) of an *autonomous* Hamiltonian system $H = H(x)$ generates a canonical flow in phase space with $\phi_t(q,p) = (q(t), p(t))$. Here, $q(t), p(t)$ satisfy the canonical equations of motion (7.15) with the initial conditions $q(0) = q$ and $p(0) = p$. The flow is canonical because the corresponding vector field $v = (\dot{q}, \dot{p})$ satisfies the condition (7.31) with $F = H$.

An important consequence is the fact that the Poisson bracket is compatible with time evolution. In fact, we obtain at a fixed time t with $\bar{q} = q(t)$, $\bar{p} = p(t)$ and $F_i = \bar{F}_i \circ \phi_t$ from (7.23) immediately that

$$\{\bar{F}_1, \bar{F}_2\}(\bar{q}, \bar{p}) = \{F_1, F_2\}(q,p)$$

and thus

$$\{\bar{F}_1, \bar{F}_2\} \circ \phi_t = \{\bar{F}_1 \circ \phi_t, \bar{F}_2 \circ \phi_t\}; \tag{7.33}$$

i.e., the time evolution ϕ_t "commutes" with the Poisson bracket.

Proof of the Jacobi Identity: Let $F_i(x)$, $i = 1,2,3$ be functions on the phase space. We consider the flow $x(\lambda) = \phi_\lambda(x)$ generated by F_3. Using (7.10), we obtain for the changes of the quantities F_1, F_2 and $\{F_1, F_2\}$ under the flow the conditions

$$\frac{d}{d\lambda} F_i(x(\lambda))\Big|_{\lambda=0} = \{F_i, F_3\}, \quad \frac{d}{d\lambda}\{F_1, F_2\}(x(\lambda))\Big|_{\lambda=0} = \{\{F_1, F_2\}, F_3\}, \tag{7.34}$$

where the Poisson brackets are evaluated at $x = x(0)$. From (7.33), we know that the Poisson bracket commutes with the flow, i.e.

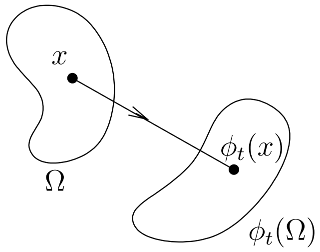
$$\{F_1, F_2\}(x(\lambda)) = \{F_1 \circ \phi_\lambda, F_2 \circ \phi_\lambda\}(x).$$

Differentiating this relationship with respect to λ gives (using the product rule)

$$\frac{d}{d\lambda}\{F_1, F_2\}(x(\lambda))\Big|_{\lambda=0} = \left\{ \frac{dF_1(x(\lambda))}{d\lambda}\Big|_{\lambda=0}, F_2 \right\} + \left\{ F_1, \frac{dF_2(x(\lambda))}{d\lambda}\Big|_{\lambda=0} \right\}. \tag{7.35}$$

Substituting (7.34) yields (utilizing the antisymmetry) the Jacobi identity (7.14).

7.5.1 The Liouville Theorem



The phase volume

$$\mu(\Omega) = \int_\Omega dx_1 \dots dx_{2f} \tag{7.36}$$

of any subset $\Omega \subset \Gamma$ of the phase space is invariant under time evolution, i.e.

$$\mu(\phi_t(\Omega)) = \mu(\Omega)\ . \tag{7.37}$$

To prove this, we only need the fact that ϕ_t is a canonical flow so that at any time t the mapping $\phi_t\colon x \mapsto \bar{x} = x(t)$ is a canonical transformation. Thus, we obtain

$$\mu(\phi_t(\Omega)) = \int_{\phi_t(\Omega)} d\bar{x}_1 \dots d\bar{x}_{2f} = \int_\Omega \underbrace{\left|\mathrm{Det}\,\frac{\partial \bar{x}_i}{\partial x_j}\right|}_{=\,1,\ \text{due to (7.21)}} dx_1 \dots dx_{2f} = \mu(\Omega)\,. \tag{7.38}$$

The equation (7.37) states that every canonical flow ϕ_t is volume-preserving.

7.6 Conservation Laws

As we saw in Chapter 7.3, the time change of a quantity $F(x)$ along a mechanical path is given by

$$\frac{d}{dt}F(x(t)) = \{F, H\}(x(t)) \tag{7.39}$$

The quantity F is conserved if and only if the Poisson bracket of F with H vanishes.

Now let F_1 and F_2 be two conserved quantities, i.e.

$$\{F_1, H\} = \{F_2, H\} = 0\,. \tag{7.40}$$

Then, from the Jacobi identity, it follows that $\{F_1, F_2\}$ also has a vanishing Poisson bracket with H. Thus, the Poisson bracket of two conserved quantities is again a conserved quantity (*Poisson's theorem*).

Example: The angular momentum $\boldsymbol{L}$ is defined by $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$. If two components of the angular momentum $\boldsymbol{e}_1 \cdot \boldsymbol{L}$ and $\boldsymbol{e}_2 \cdot \boldsymbol{L}$ are conserved, then so is the third component, since

$$\{\boldsymbol{e}_1 \cdot \boldsymbol{L}, \boldsymbol{e}_2 \cdot \boldsymbol{L}\} = (\boldsymbol{e}_1 \times \boldsymbol{e}_2) \cdot \boldsymbol{L}\,. \tag{7.41}$$

Moreover, it holds

$$\{\boldsymbol{e}_1 \cdot \boldsymbol{L}, \boldsymbol{e}_2 \cdot \boldsymbol{p}\} = (\boldsymbol{e}_1 \times \boldsymbol{e}_2) \cdot \boldsymbol{p}\,; \tag{7.42}$$

i.e., if one momentum component and the angular momentum are conserved, then in fact all momentum components are conserved.

We denote the canonical flows generated by F and H in the sense of Chapter 7.5 by ψ_λ and ϕ_t. It holds

$$\frac{d}{dt}F(\phi_t(x))\Big|_{t=0} = \{F,H\} = -\{H,F\} = -\frac{d}{d\lambda}H(\psi_\lambda(x))\Big|_{\lambda=0}. \tag{7.43}$$

Thus, F is a conserved quantity if and only if H is invariant under the canonical flow generated by F with

$$H(\psi_\lambda(x)) = H(x). \tag{7.44}$$

To every conserved quantity, there corresponds a continuous symmetry ψ_λ of H, and vice versa. In a certain sense, this statement is the converse of Noether's theorem.

Example: Let $\mathbb{R}^6$ be the phase space with coordinates $(\boldsymbol{x}, \boldsymbol{p})$ and

$$F(\boldsymbol{x},\boldsymbol{p}) = \boldsymbol{e}\cdot\boldsymbol{L} = \boldsymbol{e}\cdot(\boldsymbol{x}\times\boldsymbol{p}), \tag{7.45}$$

where $\boldsymbol{e}$ is a fixed unit vector. The differential equations (7.32) of the canonical flow generated by F are (in the form (7.5))

$$\frac{d\boldsymbol{x}}{d\lambda} = \frac{\partial F}{\partial \boldsymbol{p}} = \boldsymbol{e}\times\boldsymbol{x}, \qquad \frac{d\boldsymbol{p}}{d\lambda} = -\frac{\partial F}{\partial \boldsymbol{x}} = \boldsymbol{e}\times\boldsymbol{p}. \tag{7.46}$$

After integration, we obtain the corresponding canonical flow

$$\psi_\lambda\colon (\boldsymbol{x},\boldsymbol{p}) \mapsto (R(\lambda)\boldsymbol{x}, R(\lambda)\boldsymbol{p}), \tag{7.47}$$

where $R(\lambda)$ is the rotation in $\mathbb{R}^3$ about the axis $\boldsymbol{e}$ with angle λ. The components (7.45) of the angular momentum are thus the generating functions of the rotations. With (7.43), the most general form of the angular momentum theorem is given by

$$\frac{d}{dt}\boldsymbol{e}\cdot(\boldsymbol{x}\times\boldsymbol{p})\Big|_{t=0} = -\frac{d}{d\lambda}H(R(\lambda)\boldsymbol{x}, R(\lambda)\boldsymbol{p})\Big|_{\lambda=0}. \tag{7.48}$$

Thus, $\boldsymbol{e}\cdot(\boldsymbol{x}\times\boldsymbol{p})$ is particularly conserved if H is invariant under rotations about the axis $\boldsymbol{e}$. Similarly, the momentum component $\boldsymbol{e}\cdot\boldsymbol{p}$ is the generator of the translations $\psi_\lambda\colon (\boldsymbol{x},\boldsymbol{p}) \mapsto (\boldsymbol{x}+\lambda\boldsymbol{e}, \boldsymbol{p})$. This results in the momentum theorem

$$\frac{d}{dt}\boldsymbol{e}\cdot\boldsymbol{p}\Big|_{t=0} = -\frac{d}{d\lambda}H(\boldsymbol{x}+\lambda\boldsymbol{e},\boldsymbol{p})\Big|_{\lambda=0}. \tag{7.49}$$

Occasionally, one is interested in the time change of $F(x,t)$, i.e., a quantity F that explicitly depends on time. We obtain (instead of (7.39))

$$\frac{d}{dt}F(x(t),t) = \sum_{i=1}^{2f}\frac{\partial F}{\partial x_i}\dot{x}_i + \frac{\partial F}{\partial t} = \{F,H\} + \frac{\partial F}{\partial t} \tag{7.50}$$

In particular, for $H = H(x,t)$, the *energy theorem* holds

$$\frac{dH}{dt} = \underbrace{\{H,H\}}_{=0} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \tag{7.51}$$

due to (7.13). For autonomous systems with $H = H(x)$, the energy H is therefore conserved.

7.7 The Hamiltonian Principle in Phase Space

According to the definition of the Hamiltonian function (7.3), it holds

$$L(q,\dot{q},t) = \langle p, \dot{q}\rangle - H = \sum_{\alpha=1}^{f} p_\alpha \dot{q}^\alpha - H(q,p,t)\,. \tag{7.52}$$

We therefore expect that the equations of motion can be derived from the variational principle

$$\delta \int_{(1)}^{(2)} \Big(\langle p, \dot{q}\rangle - H\Big) dt = 0\,. \tag{7.53}$$

where the position $q^\alpha(t^{(i)})$ is held fixed at the endpoints $t^{(i)}$.

In phase space, however, we consider q^α and p_β as independent variables, while in the original formulation of the variational principle, the variation of $\dot{q}$ was determined by that of q. It is therefore *a priori* not obvious that (7.53) yields the correct equations of motion. However, the variational principle (7.53) is indeed equivalent to the canonical equations of motion. Under a variation of the independent variables q^α, p_β (with fixed endpoints $q^{(i)}$), we obtain

$$\begin{aligned}
&\delta \int_{(1)}^{(2)} \Big(\sum_{\alpha=1}^{f} p_\alpha \dot{q}^\alpha - H\Big) dt \\
&= \int_{(1)}^{(2)} \sum_{\alpha=1}^{f} \left(p_\alpha \delta\dot{q}^\alpha + \dot{q}^\alpha \delta p_\alpha - \frac{\partial H}{\partial q^\alpha}\delta q^\alpha - \frac{\partial H}{\partial p_\alpha}\delta p_\alpha \right) dt \\
&= \sum_{\alpha=1}^{f} p_\alpha \underbrace{\delta q^\alpha}_{=0} \Big|_{(1)}^{(2)} + \int_{(1)}^{(2)} \sum_{\alpha=1}^{f} \left[\left(\dot{q}^\alpha - \frac{\partial H}{\partial p_\alpha} \right) \delta p_\alpha - \left(\dot{p}_\alpha + \frac{\partial H}{\partial q^\alpha} \right) \delta q^\alpha \right] dt\,,
\end{aligned}$$

where we have partially integrated the first term on the left side of the second line. The boundary terms vanish, and we obtain exactly the Hamiltonian equations of motion.

7.7.1 Generating Canonical Transformations

As preparation for the discussion of the Hamilton-Jacobi equation in Chapter 8, we now want to discuss how to systematically construct canonical transformations. We consider a Hamiltonian system with phase space coordinates $x = (q^1, p_1, \ldots, q^f, p_f)$ and Hamiltonian function $H(x,t)$. We subject the system to a (possibly) time-dependent canonical transformation

$$\bar{x}_i = \phi_i(x_1, \ldots, x_{2f}, t). \tag{7.54}$$

Since the transformation is canonical, the equations of motion in the new coordinates $\bar{x} = (Q^1, P_1, \ldots, Q^f, P_f)$ are again the Hamiltonian equations. However, since the canonical transformation depends on time, we generally need to introduce a new Hamiltonian function $K(\bar{x}, t)$. Equivalently, the claim is that the two variational problems

$$\delta \int_{(1)}^{(2)} \Big(\langle p, \dot{q}\rangle - H\Big) dt = 0 \quad \text{and} \quad \delta \int_{(1)}^{(2)} \Big(\langle P, \dot{Q}\rangle - K\Big) dt = 0 \tag{7.55}$$

have solutions $x(t)$ and $\bar{x}(t)$, which correspond under (7.54). A sufficient condition for this is (cf. (4.46)), that the two integrands in (7.55) differ by a total derivative of a function $F(q, Q, t)$.[3]

This means that for *all* curves $x(t)$ it holds[4]

$$\langle p(t), \dot{q}(t)\rangle - H(x(t), t) = \langle P(t), \dot{Q}(t)\rangle - K(\bar{x}(t), t) + \frac{dF}{dt}. \tag{7.56}$$

Rearranging gives from (7.56) the expression

$$\frac{dF}{dt} = \Big(\langle p, \dot{q}\rangle - \langle P, \dot{Q}\rangle\Big) + (K - H). \tag{7.57}$$

It is useful to express the function F not in terms of the variables q and Q but rather in terms of the old coordinates q and the new momenta P. The transition from Q to P as an independent variable is achieved by a Legendre transformation. We define

$$S(q, P, t) = F(q, Q, t) + \langle P, Q\rangle \qquad \text{with} \quad P_\alpha = -\frac{\partial F}{\partial Q^\alpha}(q, Q, t). \tag{7.58}$$

[3] Since we hold the endpoints δq^α and δQ^α fixed in the variational calculus, F does not modify the Euler-Lagrange equations.

[4] Often, this relationship is also written compactly

$$\langle p, dq\rangle - H dt = \langle P, dQ\rangle - K dt + dF$$

as an identity of differentials. Thus, we have $dF = \langle p, dq\rangle - \langle P, dQ\rangle + (K - H)dt$. The Legendre transformation on S is then obtained with the product rule $d\langle P, Q\rangle = \langle dP, Q\rangle + \langle P, dQ\rangle$ and $dS = dF + d\langle P, Q\rangle$.

Analogous to (7.4), we obtain for the partial derivatives[5]

$$\frac{\partial S}{\partial P_\alpha} = Q^\alpha, \qquad \frac{\partial S}{\partial q^\alpha} = \frac{\partial F}{\partial q^\alpha}, \qquad \frac{\partial S}{\partial t} = \frac{\partial F}{\partial t}. \tag{7.59}$$

Thus, we directly obtain

$$\begin{aligned} \frac{dF}{dt} &\equiv \sum_{\alpha=1}^{f} \left(\frac{\partial F}{\partial q^\alpha} \dot{q}^\alpha + \frac{\partial F}{\partial Q^\alpha} \dot{Q}^\alpha \right) + \frac{\partial F}{\partial t} \\ &= \sum_{\alpha=1}^{f} \left(\frac{\partial S}{\partial q^\alpha} \dot{q}^\alpha - P_\alpha \dot{Q}^\alpha \right) + \frac{\partial S}{\partial t} \end{aligned} \tag{7.60}$$

for the total time derivative of the function $F(q, Q, t)$.

For a given function $F(q, Q, t)$, by comparing (7.57) with (7.60), we obtain the relationships

$$p_\alpha = \frac{\partial S}{\partial q^\alpha}(q, P, t)\,, \qquad Q^\alpha = \frac{\partial S}{\partial P_\alpha}(q, P, t)\,, \qquad K = H + \frac{\partial S}{\partial t}\,, \tag{7.61}$$

between the old (q, p) and the new (Q, P) canonical coordinates (and the new Hamiltonian function K). To explicitly obtain the canonical transformation (7.54), one must solve the first equation for P_α and substitute the result into the second equation. This gives

$$Q^\alpha = Q^\alpha(q, p, t), \qquad P_\alpha = P_\alpha(q, p, t). \tag{7.62}$$

The condition for solvability at a point (q_0, P_0, t_0) is that there

$$\mathrm{Det}\left(\frac{\partial^2 S}{\partial q^\alpha \partial P_\beta} \right) \neq 0\,. \tag{7.63}$$

One calls $S(q, P, t)$ the *generating function* of the (at any time t) canonical transformation (7.62). The transformation is canonical for every *fixed* time $t = t_0$, since it is generated there by $S_0(q, P) = S(q, P, t_0)$. For the generating function S_0 the time derivative vanishes, and thus the new variables $\bar{x} = \phi(x, t_0)$ satisfy the canonical equations of motion with the Hamiltonian $\bar{H}(\bar{x}) \equiv K_0(\bar{x}) = H(x)$. This corresponds exactly to the definition (7.17) of a canonical transformation.

By freely choosing the function S, one can elegantly construct canonical transformations. To show this, we consider two examples (analogous to the examples in Chapter 7.4).

[5]Replace $(L, H, \dot{q}, p)$ in (7.4) with $(-F, S, Q, P)$.

Example 1: (*Point transformations*) Let $S = \sum_{\beta=1}^{f} P_\beta Q^\beta(q^1, \dots, q^f)$, where the functions $Q^\alpha(q)$ are given. Then, from (7.61)

$$Q^\alpha = Q^\alpha(q^1, \dots, q^f)\,, \qquad p_\alpha = \sum_{\beta=1}^{f} P_\beta \frac{\partial Q^\beta}{\partial q^\alpha}\,. \tag{7.64}$$

After solving the second equations for P_α with $P_\alpha = \sum_{\beta=1}^{f} p_\beta(\partial q^\beta/\partial Q^\alpha)$, these are the point transformations discussed in connection with (7.24); where we have now proven that the transformation is canonical.

Example 2: For $f = 1$, we consider the generating function $S(q, P) = \frac{1}{2}q^2\,P^2$. Equation (7.61) yields

$$Q = q^2 P\,, \qquad p = qP^2. \tag{7.65}$$

Thus, we obtain the canonical transformation

$$Q = \sqrt{pq^3}\,, \qquad P = \sqrt{\frac{p}{q}}\,. \tag{7.66}$$

Remark: The generating function $S(q, P, t)$ of the canonical transformation is not the same as the generating function $F(q, p)$ of a canonical flow $(Q, P) = x(\lambda) = \phi_\lambda(q, p)$ from (7.32). However, the terms are closely related. For small λ, it holds

$$S(q, P, \lambda) = \sum_{\alpha=1}^{f} P_\alpha q^\alpha + \lambda F(q, P) + O(\lambda^2)\,. \tag{7.67}$$

Proof: From (7.61), we know

$$p_\alpha = P_\alpha + \lambda \frac{\partial F}{\partial q^\alpha}\bigg|_{(q,P)} + O(\lambda^2)\,, \quad Q^\alpha = q^\alpha + \lambda \frac{\partial F}{\partial p_\alpha}\bigg|_{(q,P)} + O(\lambda^2)\,. \tag{7.68}$$

Since $Q^\alpha - q^\alpha = O(\lambda)$ and $P_\alpha - p_\alpha = O(\lambda)$, we can also rewrite the result as

$$Q^\alpha = q^\alpha + \lambda \frac{\partial F}{\partial p_\alpha}\bigg|_{(Q,P)} + O(\lambda^2)\,, \quad P_\alpha = p_\alpha - \lambda \frac{\partial F}{\partial q^\alpha}\bigg|_{(Q,P)} + O(\lambda^2)\,. \tag{7.69}$$

Thus, $(Q, P) = (q(\lambda), p(\lambda))$ locally is the solution of the canonical equations $dq^\alpha(\lambda)/d\lambda = \partial F/\partial p_\alpha$, $dp_\alpha(\lambda)/d\lambda = -\partial F/\partial q^\alpha$ with the initial condition $(q(0), p(0)) = (q, p)$ with $F(q, p)$ being the generating function, cf. (7.32).

Chapter 8

The Hamilton-Jacobi Theory

The Hamilton-Jacobi theory is another reformulation of classical mechanics. It encompasses all results of the Lagrangian and Hamiltonian formulations, but also provides a deeper understanding of integrability. Furthermore, it allows, as presented in Chapter 8.5, a direct transition from classical mechanics to quantum mechanics.

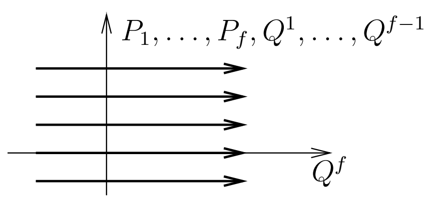
8.1 The Time-Independent Case

We now consider an autonomous system, i.e., a system where the Hamiltonian does not explicitly depend on t, $H = H(q^1, \ldots, q^f, p_1, \ldots, p_f)$. We attempt to find a time-independent canonical transformation such that K is just one of the new momentum coordinates, e.g., $K(Q, P) = P_f$, where P_f corresponds to the energy E. If successful, the equations of motion in the new coordinates are given by

$$\begin{aligned} \dot{P}_\alpha &= -\frac{\partial K}{\partial Q^\alpha} = 0\,, \qquad (\alpha = 1, \ldots, f)\,, \\ \dot{Q}^\alpha &= \frac{\partial K}{\partial P_\alpha} = 0\,, \qquad (\alpha = 1, \ldots, f-1)\,, \qquad \dot{Q}^f = \frac{\partial K}{\partial P_f} = 1\,. \end{aligned} \tag{8.1}$$

The equations of motion are then trivially solvable with

$$\begin{aligned} P_\alpha(t) &= P_\alpha(0)\,, \qquad (\alpha = 1, \ldots, f)\,, \\ Q^\alpha(t) &= Q^\alpha(0)\,, \qquad (\alpha = 1, \ldots, f-1)\,, \\ Q^f(t) &= Q^f(0) + t\,. \end{aligned}$$



Canonical flow to the Hamiltonian $K = P_f$.

As we have seen in Chapter 7.7.1, a generating function $S_0 = S_0(q, P)$ must be determined such that (see (7.61))

$$H\Big(q^1, \ldots, q^f, \frac{\partial S_0}{\partial q^1}, \ldots, \frac{\partial S_0}{\partial q^f}\Big) = P_f \equiv E\,. \tag{8.2}$$

This equation is called the *time-independent Hamilton-Jacobi equation.* It is a (parameterized by P_f) first-order partial differential equation for S_0. For S_0 to be solvable for q^α, a family of solutions is needed[1]

$$S_0(q^1, \ldots, q^f, P_1, \ldots, P_f) \qquad \text{with} \quad \text{Det}\left(\frac{\partial^2 S_0}{\partial q^\alpha \partial P_\beta}\right) \neq 0\,, \tag{8.3}$$

parameterized by $(f-1)$ parameters $P_1, \ldots, P_{f-1}$ and of course also P_f. Such a solution is called a *complete integral.*[2]

It is important that $Q(q, P, t)$ can be solved for q with the condition (cf. (7.63))

$$\text{Det}\left(\frac{\partial^2 S_0}{\partial q^\alpha \partial P_\beta}\right) \neq 0. \tag{8.4}$$

Necessary for solvability is therefore at least that

$$\text{Rang}\left(\frac{\partial^2 S_0}{\partial q^\alpha \partial P_\beta}\right)_{\substack{\alpha=1,\ldots,f\\ \beta=1,\ldots,f-1}} = f-1\,. \tag{8.5}$$

This condition is indeed also sufficient. From (8.2), by differentiating with respect to $P_1, \ldots,\, P_f$, we obtain

$$\sum_\alpha \frac{\partial H}{\partial p_\alpha}\frac{\partial^2 S_0}{\partial q^\alpha \partial P_\beta} = 0\,, \quad (\beta = 1, \ldots, f-1)\,, \qquad \sum_\alpha \frac{\partial H}{\partial p_\alpha}\frac{\partial^2 S_0}{\partial q^\alpha \partial P_f} = 1\,.$$

The last column $(\beta = f)$ of the matrix (8.4) cannot therefore be a linear combination of the first $f-1$. Moreover, a necessary condition for the solvability of (8.2) is that in the considered region of phase space nowhere

$$\frac{\partial H}{\partial p_1} = \ldots = \frac{\partial H}{\partial p_f} = 0 \tag{8.6}$$

holds.

From the complete solution of the Hamilton-Jacobi equation, the motion in the original coordinates results as follows. For given values of the conserved quantities $P_1, \ldots, P_f$, each of the equations

$$\frac{\partial S_0}{\partial P_\beta}(q, P) = Q^\beta\,, \qquad (\beta = 1, \ldots, f-1) \tag{8.7}$$

[1] In general, $S_0(q, P) + A$ with $A \in \mathbb{R}$ is a solution. However, the additional additive constant A is irrelevant in mechanics.

[2] As we have seen in Chapter 5.3, a *general solution* of a partial differential equation depends on arbitrary functions (and not just on parameters).

for $q = (q^1, \dots, q^f)$ corresponds to a surface in configuration space, which, according to (8.5), has linearly independent normals. Their average is the (1-dimensional) trajectory. The temporal traversal of this trajectory is given by

$$\frac{\partial S_0}{\partial E}(q, P) = Q^f(0) + t \equiv t_0 + t. \tag{8.8}$$

The $2f$ constants $Q^1, P_1, \dots, t_0, E$ arise from the initial conditions.

Example 1: The harmonic oscillator has the Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{1}{2}fq^2.$$

with the angular frequency $\omega = \sqrt{f/m}$. The Hamilton-Jacobi equation for $S_0(q)$ is the ordinary differential equation

$$\frac{1}{2m}\left(\frac{dS_0}{dq}\right)^2 + \frac{1}{2}fq^2 = E$$

with the solution

$$S_0(q) = \int_0^q dx\,\sqrt{m(2E - fx^2)}. \tag{8.9}$$

The motion is given by (8.8). With

$$\frac{\partial S_0}{\partial E} = \int_0^q dx\,\sqrt{\frac{m}{2E - fx^2}} = \frac{1}{\omega}\arcsin\!\left(q/\sqrt{2E/f}\right) = t_0 + t$$

we obtain the general solution

$$q(t) = \sqrt{2E/f}\,\sin\!\big(\omega(t_0 + t)\big).$$

8.2 Separable Problems

If one can write the Hamilton-Jacobi equation (8.2) as

$$f\left(q^1, \frac{\partial S_0}{\partial q^1}\right) = F\left(q^2, \dots, q^f, \frac{\partial S_0}{\partial q^2}, \dots, \frac{\partial S_0}{\partial q^f}\right), \tag{8.10}$$

the coordinate q^1 is called *separable*. The separation ansatz

$$S_0(q^1, \dots, q^f) = S_1(q^1) + \tilde{S}_0(q^2, \dots, q^f) \tag{8.11}$$

then leads to the two equations

$$f\left(q^1, \frac{dS_1}{dq^1}\right) = P_1\,, \qquad F\left(q^2, \dots, q^f, \frac{\partial \tilde{S}_0}{\partial q^2}, \dots, \frac{\partial \tilde{S}_0}{\partial q^f}\right) = P_1\,, \tag{8.12}$$

where P_1 is constant, since the left or right side of (8.10) does not depend on $q^2, \dots, q^f$ or q^1, respectively. From (8.12), by solving the first equation, one finds the function $S_1(q^1, P_1)$.

The problem is *completely separable* if one can proceed in the same way with the second equation, and so on. In this case, the result is a complete solution

$$S_0(q^1, \dots, q^f, P_1, \dots, P_{f-1}, P_f) = S_1(q^1, P_1) + S_2(q^2, P_1, P_2) + \dots \\ + S_f(q^f, P_1 \dots P_{f-1}) \quad (8.13)$$

of the Hamilton-Jacobi equation (8.2), since each term can still depend on $P_f = E$. The (complete) separability of a problem always refers to special coordinates $q = (q^1, \dots, q^f)$, whose existence is an exception rather than the rule. For a list of problems that are completely separable, see Landau & Lifschitz.

Example 2: (*Cyclic Coordinate*) If the coordinate q^1 is cyclic, it does not explicitly enter into the Hamilton-Jacobi equation. Thus, one can trivially separate the system with $f(q^1, dS_1/dq^1) = dS_1/dq^1$. We immediately obtain $S_1 = P_1 q^1$ and

$$S_0 = P_1 q^1 + \tilde{S}_0.$$

The constant P_1 is of course nothing other than the conjugate momentum $\partial S_0 / \partial q^1$.

Example 3: (*The Plane Central Force Problem*) We consider the plane central force problem in polar coordinates (see Example (ii) in Chapter 7.2), for which the Hamiltonian is given by

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r)$$

The Hamilton-Jacobi equation is

$$\frac{1}{2m}\left[\left(\frac{\partial S_0}{\partial r}\right)^2 + r^{-2}\left(\frac{\partial S_0}{\partial \varphi}\right)^2\right] + V(r) = E\,,$$

with $E \equiv P_2$. This equation is separable, as it can be rewritten as

$$\left(\frac{\partial S_0}{\partial \varphi}\right)^2 = 2mr^2(E - V(r)) - r^2\left(\frac{\partial S_0}{\partial r}\right)^2. \quad (8.14)$$

With the separation ansatz $S_0(r, \varphi) = S_r(r) + S_\varphi(\varphi)$, both sides of (8.14) must equal a separation constant $l^2 \equiv P_1^2$. It follows that

$$S_\varphi(\varphi) = l\varphi\,, \qquad S_r(r) = \int^r dx\,\sqrt{2m(E - V(x)) - l^2 x^{-2}}\,. \quad (8.15)$$

up to irrelevant additive constants. The trajectory $\varphi(r)$ and the temporal dependence $t(r)$ now result from (8.7) and (8.8)

$$\frac{\partial S_0}{\partial l} = \varphi - \int^r \frac{l}{x^2} \frac{dx}{\sqrt{2m(E - V(x)) - l^2 x^{-2}}} = \varphi_0 \,,$$
$$\frac{\partial S_0}{\partial E} = \int^r \frac{m\, dx}{\sqrt{2m(E - V(x)) - l^2 x^{-2}}} = t_0 + t \,,$$

where the constants t_0, φ_0 are determined by the initial conditions. This result is of course consistent with (2.14) and (2.12).

Remark: A complete solution of the Hamilton-Jacobi equation directly provides the f conserved quantities P_β. That P_β are conserved quantities follows from

$$\{H(x), P_\beta(x)\} = \{P_f, P_\beta\} = 0, \tag{8.16}$$

where we have used (7.22) and (8.2). Moreover, they are in *involution*, i.e.

$$\{P_\alpha(x), P_\beta(x)\} = \{P_\alpha, P_\beta\} = 0, \quad (\text{for all } \alpha, \beta). \tag{8.17}$$

The conserved quantities are also *independent*[3] of each other, since (from (8.4))

$$\text{Det}\left(\frac{\partial P_\alpha(q,p)}{\partial p_\beta}\right) = \text{Det}\left(\frac{\partial^2 S_0}{\partial q^\beta \partial P_\alpha}\right)^{-1} \neq 0 \,. \tag{8.18}$$

A mechanical system with f degrees of freedom is called *completely integrable* (in the Liouville sense) if there are f independent conserved quantities in involution. Thus, every system for which there is a complete solution of the Hamilton-Jacobi equation is completely integrable.[4]

Note that in Lagrangian mechanics, $2f$ integrals of motion are needed to completely integrate a problem. It turns out that with the Hamiltonian method, only f conserved quantities are needed, which must, however, be in involution. Each additional conserved quantity reduces the number of degrees of freedom by 2.

8.3 Angle-Action Variables

We would like to examine the structure of phase space for a completely separable problem with

$$S_0(q, P) = \sum_{\alpha=1}^{f} S_\alpha(q^\alpha, P), \tag{8.19}$$

[3] The quantities P_α are independent if the matrix $\partial P_\alpha / \partial x_k$ has full rank f.

[4] This result also holds in the reverse: a completely integrable system has a (global) solution of the Hamilton-Jacobi equation.

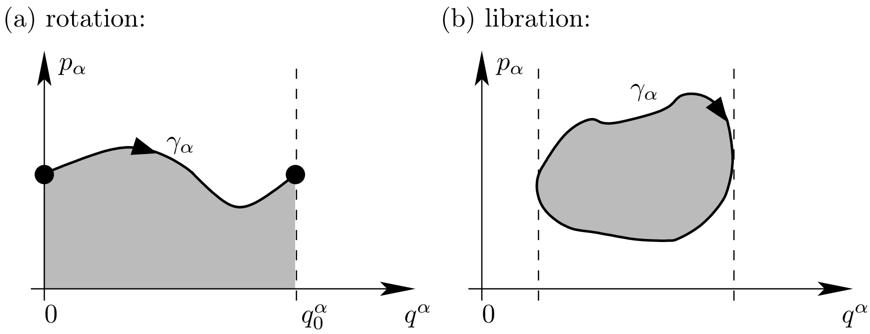


Figure 8.1: The action variable J_α is the (oriented) area (divided by 2π) under the curve $p_\alpha(q^\alpha)$. There are two classes of trajectories. (a) Rotation (only possible for periodic variables): The curve connects two equivalent points with $q^\alpha = 0$ and $q^\alpha = q_0^\alpha$; q_0^α denotes the period. (b) Libration: The curve encloses an area in phase space. The motion in q is bounded by two turning points (dashed lines) at fixed P.

which performs finite motion in all coordinates, in a bit more detail.[5] One can then make a special choice of canonical coordinates, the so-called *angle-action variables*, in which the configuration space becomes a torus and the motion becomes conditionally periodic. These variables are useful for perturbation theory, problems with slowly (adiabatically) varying parameters, and for the semiclassical description of quantum mechanical problems.

The canonical momentum $p_\alpha(q^\alpha, P) = \partial S_\alpha / \partial q^\alpha$ is (at fixed P) only a function of the coordinate q^α. Since the motion is finite, the trajectory γ_α in the subspace (q^α, p_α) is closed. We define the *action variables* J_α by the line integral along γ_α

$$J_\alpha(P) = \frac{1}{2\pi} \oint_{\gamma_\alpha} p_\alpha(q^\alpha)\, dq^\alpha = \frac{1}{2\pi} \oint_{\gamma_\alpha} \langle p, dq \rangle, \qquad (P \text{ fixed}). \tag{8.20}$$

There are two classes of trajectories, see Fig. 8.1:

(a) *Rotation* (e.g., angle φ in polar coordinates): q^α is a periodic variable with period q_0^α and the curve γ_α connects $q^\alpha = 0$ with $q^\alpha = q_0^\alpha$,

(b) *Libration* (e.g., position q in the harmonic oscillator): the curve γ_α is a closed curve in phase space (q^α, p_α).

From the definition, it follows that $J_\alpha = J_\alpha(P)$ depends only on the conserved quantities P and therefore J_α itself are conserved quantities. If

$$\mathrm{Det}\left(\frac{\partial J_\alpha}{\partial P_\beta}\right) \neq 0$$

[5] Although technically more difficult, most results also hold for a general completely integrable system with the necessary adjustments.

the relation $J(P)$ is invertible with $P = P(J)$ and one can define J as a new momentum. The transition from (q,p) to (ϕ, J) is then accomplished by the generating function $\tilde{S}_0(q,J) = S_0(q,P(J))$. The new coordinates

$$\phi^\alpha(q,J) = \frac{\partial \tilde{S}_0}{\partial J_\alpha}(q,J) \tag{8.21}$$

are called *angle variables*. In general, ϕ^α is a function of (all) $q^1, \ldots, q^f, J_1, \ldots, J_f$.

The new Hamiltonian is given by $K(J) = P_f(J)$ and does not depend on the angle variables. The angle variables thus satisfy the canonical equations of motion

$$\dot{\phi}^\alpha = \frac{\partial K(J)}{\partial J_\alpha} = \omega^\alpha(J). \tag{8.22}$$

Since J are conserved quantities, all angle variables depend only linearly on time with

$$\phi^\alpha = \omega^\alpha(J)t + \phi_0^\alpha; \tag{8.23}$$

ϕ_0^α are integration constants determined by the initial conditions.

We will now show that ϕ^α are actually periodic variables, so that ϕ^α and $\phi^\alpha + 2\pi$ correspond to the same state (q,p). Thus, the configuration space T with $\phi \in T$ is an f-dimensional torus ($T = S^1 \times \cdots \times S^1$) and the motion (8.23) corresponds to a *conditionally periodic* motion, where each angle variable ϕ^α performs a periodic motion with the angular frequency ω^α.

To show that ϕ^α is a periodic variable (with period 2π), we want to calculate the change $\Delta\phi^\alpha$ when the system transitions along the path γ_β to the same state.[6] This question concerns only the configuration space and thus the relation $\phi(q,P)$ and not whether the mechanical path is periodic. We therefore consider the (virtual) displacement (at fixed P or J)

$$\delta\phi^\alpha = \sum_{\beta=1}^{f} \frac{\partial \phi^\alpha}{\partial q^\beta}\,\delta q^\beta = \sum_{\beta=1}^{f} \frac{\partial^2 \tilde{S}_0}{\partial J_\alpha \partial q^\beta}\,\delta q^\beta = \frac{\partial}{\partial J_\alpha}\langle p(q,J), \delta q\rangle. \tag{8.24}$$

After integrating over the curve γ_β, we obtain

$$\Delta\phi^\alpha = \frac{\partial}{\partial J_\alpha} \underbrace{\oint_{\gamma_\beta} \langle p, dq\rangle}_{=2\pi J_\beta} = 2\pi\delta_{\alpha\beta}, \tag{8.25}$$

so that after traversing the curve γ_β, only the angle variable ϕ^β has changed by 2π. Since the system transitions to the same state under all curves γ_β ($\beta = 1, \ldots, f$), we obtain the desired result that all ϕ^α are periodic variables.

[6] Note that γ_β is a *closed* curve.

With this general result, we can make statements about the motion of an integrable system without explicitly solving the equations of motion. For simplicity, we assume that all motions are of the libration class.[7] The angle-action variables uniquely define the position $q(\phi, J)$, and it must hold that $q(\phi + 2\pi m, J) = q(\phi, J)$ with $m \in \mathbb{Z}^f$. We can therefore write the (periodic) function $q(\phi)$ as a Fourier series. This gives us the temporal dependence of $q^\alpha(t)$ as

$$q^\alpha(t) = \sum_{m\in\mathbb{Z}^f} A^\alpha_m e^{i\langle m,\phi\rangle} \equiv \sum_{m\in\mathbb{Z}^f} A^\alpha_{m_1,\ldots,m_f} \exp\left(i \sum_{\beta=1}^{f} m_\beta(\omega^\beta t + \phi_0^\beta)\right). \tag{8.26}$$

A similar development can be found for $p(t)$ and thus for any function $F(q, p)$.

Since the frequencies are generally not commensurable, the motion $q(t)$ of the system is not periodic but only *conditionally periodic*. In some cases, two (or more) of the angular frequencies ω^α may be commensurable. For example, if $\omega^1/\omega^2 \in \mathbb{Q}$ holds, one speaks of a *degeneration* of the motion of ϕ^1 and ϕ^2. In the extreme case, where all f angular frequencies are commensurable, the motion of the system is called *completely degenerate*. In this case, the motion is of course strictly periodic, and all trajectories are closed.

Example 1 (Continuation): The trajectory of the harmonic oscillator (at fixed E) is a libration (between the turning points $\pm q^*$ with $q^* = \sqrt{2E/f}$). We obtain the action variable

$$J = \frac{1}{2\pi}\oint p(q)\,dq = \frac{2}{2\pi}\int_{-q^*}^{q^*} \sqrt{m(2E - fq^2)}\,dq = \frac{E}{\omega}.$$

The action $\tilde{S}_0(J)$ is given by $S_0(E)$ from (8.9) with the relation $E = \omega J$. We thus obtain the corresponding angle variable

$$\phi = \frac{\partial \tilde{S}_0}{\partial J} = \omega\frac{\partial S_0}{\partial E} = \arcsin\left(\frac{q}{q^*}\right).$$

The motion is completely degenerate (as always with only one degree of freedom) with a period of $2\pi/\omega$.

Remark: We have seen that the configuration space of a completely integrable system is an f-dimensional torus. The KAM theorem (Kolmogorov (1956), Arnold (1965), Moser (1965)) describes the stability of this result against (small) perturbations with

$$H = \underbrace{H_0}_{\text{integrable}} + \varepsilon H_1.$$

[7] For rotations, the result (8.26) holds for $\tilde{q}^\alpha = q^\alpha - q_0^\alpha \phi^\alpha/2\pi$.

The result is: If H_0 has angular frequencies ω^α that are 'sufficiently incommensurable,' then the perturbed system H for $\varepsilon \ll 1$ predominantly has solutions that are also conditionally periodic and differ only slightly from those of H_0. Sufficiently incommensurable means in this case that $\beta > 0$ and $\tau > f - 1$ exist such that

$$|\langle k, \omega \rangle| = \left| \sum_{\alpha=1}^{f} k_\alpha \omega^\alpha \right| \geq \frac{\beta}{\left(\sum_{\alpha=1}^{f} |k_\alpha| \right)^\tau}, \quad \text{(for all } k \in \mathbb{Z}^n, k \neq 0).$$

Quas classical Quantization: With the angle-action variables, one can (quasically) quantize integrable systems (*Einstein-Keller quantization*). It is required that the action variables are not continuous quantities, but multiples of the *Planck action quantum*

$$\hbar = 1.055 \cdot 10^{-34}\,\mathrm{J\,s} \tag{8.27}$$

Specifically, it is required that

$$J_\alpha = \hbar(n_\alpha + \nu_\alpha), \qquad (n_\alpha \in \mathbb{Z}), \tag{8.28}$$

where $\nu_\alpha = 0$ for rotation and $\nu_\alpha = \frac{1}{2}$ for libration.[8] For a libration, the area in Fig. 8.1 is always positive, and thus only $n_\alpha = 0, 1, 2, \ldots$ make sense.

Example 1 (Continuation): For the harmonic oscillator, we found $J = E/\omega$. Since the motion is of the libration type, the quas classical quantization requires that

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \qquad (n = 0, 1, \ldots),$$

which exactly corresponds to the spectrum of the harmonic oscillator in quantum mechanics.

8.3.1 Adiabatic Invariants

We consider an (autonomous) integrable system determined by the Hamiltonian $H \equiv H(q, p; \lambda)$, which depends on an external parameter λ. The motion of the system is characterized by the time scale

$$T = \frac{2\pi}{\omega_{\min}}$$

with the smallest angular frequency $\omega_{\min} = \min_\alpha \omega^\alpha$. We want to understand the mechanical system when the parameter $\lambda = \lambda(t)$ in the Hamiltonian is changed *adiabatically* (i.e., slowly) with[9]

$$T \left| \frac{d\lambda}{dt} \right| \ll |\lambda|. \tag{8.29}$$

[8] This rule is somewhat simplified; in general, $4\nu_\alpha$ is given by the Keller-Maslov index.

[9] It is difficult to formulate an exact condition for the applicability of the adiabatic result. In the derivation, we will implicitly take the limit $T|\dot{\lambda}| \to 0$.

Since the (autonomous) system is integrable, there exists a generating function $S_0(q, J; \lambda)$ with $E = K(J; \lambda)$, so that J are the action variables (at constant λ). The associated angle variables are denoted (as always) by ϕ^α.

We now transform the (full) time-dependent problem $H(q, p; \lambda(t))$ with the time-dependent generating function $S(q, J, t) = S_0(q, J; \lambda(t))$. This gives us the new (time-dependent) Hamiltonian

$$\tilde{K} = K(J; \lambda) + \frac{\partial S}{\partial t} = K(J; \lambda) + \Lambda(\phi, J; \lambda)\dot{\lambda} \tag{8.30}$$

with

$$\Lambda = \frac{\partial S_0(q, J; \lambda)}{\partial \lambda}. \tag{8.31}$$

To interpret $\tilde{K}$ as a new Hamiltonian in the canonical variables (ϕ, J), q in S_0 must of course be expressed in terms of ϕ and J using (8.21).

The canonical equations for the angle-action variables are then

$$\dot{J}_\alpha = -\frac{\partial \tilde{K}}{\partial \phi^\alpha} = -\frac{\partial \Lambda}{\partial \phi^\alpha}\dot{\lambda}, \tag{8.32}$$

$$\dot{\phi}^\alpha = \frac{\partial \tilde{K}}{\partial J} = \omega^\alpha(J; \lambda) + \frac{\partial \Lambda}{\partial J_\alpha}\dot{\lambda} \tag{8.33}$$

with the angular frequencies $\omega^\alpha = \partial K(J; \lambda)/\partial J_\alpha$. It is thus directly evident that the variables J are no longer conserved quantities. The equations (8.32) and (8.33) are still exact. We now want to average the first equation (8.32) over a time τ, with $\tau|\dot{\lambda}| \ll |\lambda|$ and $\tau \gg T$.

Due to the first condition, we have $\dot{\overline{\lambda}} = \overline{\dot{\lambda}} \approx \dot{\lambda}$ and we can pull $\dot{\lambda}$ as a constant from the average on the right side of (8.32). Thus, we obtain

$$\overline{\dot{J}_\alpha} = -\overline{\frac{\partial \Lambda}{\partial \phi^\alpha}\dot{\lambda}} \approx -\overline{\frac{\partial \Lambda}{\partial \phi^\alpha}}\dot{\lambda} \tag{8.34}$$

with $\overline{f}(t) = \tau^{-1}\int_t^{t+\tau} dt'\, f(t')$.

The second condition yields

$$\overline{f(\phi, J)} = \frac{1}{\tau}\int_t^{t+\tau} dt'\, f(\omega t' + \phi_0, J) \approx \frac{1}{(2\pi)^f}\int_{[0,2\pi]^f} d^f\phi\, f(\phi, J) \tag{8.35}$$

for any function $f(\phi, J)$ in phase space, since the motion runs over many periods of ϕ_α and thus covers the entire phase space (at fixed J); here we have assumed that the motion is not degenerate.[10] Thus, we immediately obtain that

$$\overline{\frac{\partial \Lambda}{\partial \phi^\alpha}} \approx \frac{1}{(2\pi)^f}\int_{[0,2\pi]^f} d^f\phi\, \frac{\partial \Lambda}{\partial \phi^\alpha} = 0 \tag{8.36}$$

[10] The adiabatic theorem also holds for a degenerate motion. One simply replaces the average over the torus in (8.35) with the corresponding subset of reachable coordinates.

since $\Lambda(\phi, J; \lambda)$ (like any physical quantity) is a 2π-periodic function in ϕ^α. The equation (8.34) yields the central result

$$\overline{\dot{J}_\alpha} = \dot{\overline{J_\alpha}} \overset{(8.35)}{\approx} \dot{J}_\alpha \approx 0, \tag{8.37}$$

i.e., the action variables J_α remain constant during a slow change of an external parameter λ and are thus *adiabatic invariants.*

Example 1 (Continuation): We now consider the situation where the angular frequency of the oscillator is slowly changed. According to the considerations in this chapter, the angle variable $J = E/\omega$ remains approximately constant. We thus obtain the result that

$$E(t) = \omega(t)J,$$

i.e., the energy of the oscillator changes proportionally to the angular frequency.

8.4 Time-Dependent Hamilton-Jacobi Equation

We want to consider a (non-autonomous) system whose Hamiltonian $H \equiv H(q^1, \ldots, q^f, p_1, \ldots, p_f, t)$ now also depends on time. We are looking for a *time-dependent* canonical transformation such that in the new coordinates the Hamiltonian $K(Q^1, \ldots, Q^f, P_1, \ldots, P_f, t) \equiv 0$ is. Then $Q^1, \ldots, P_f$ are constant and the motion is transformed to rest. According to (7.61), the corresponding generating function $S(q, P, t)$ is a solution of the *time-dependent Hamilton-Jacobi equation*

$$\frac{\partial S}{\partial t} + H\Big(q, \frac{\partial S}{\partial q}, t\Big) = 0\,, \tag{8.38}$$

with the condition

$$\operatorname{Det}\left(\frac{\partial^2 S}{\partial q^\alpha \partial P_\beta}\right) \neq 0\,. \tag{8.39}$$

The motion in the original coordinates results from the f equations

$$\frac{\partial S}{\partial P_\alpha}(q, P, t) = Q^\alpha\,, \qquad \text{thus} \quad q^\alpha = q^\alpha(Q, P, t)\,, \tag{8.40}$$

where the constants $(Q, P) = (Q^1, \ldots, Q^f, P_1, \ldots, P_f)$ are determined by the initial conditions. Substituting into $p_\alpha = (\partial S/\partial q^\alpha)(q, P, t)$ then yields $p_\alpha = p_\alpha(Q, P, t)$.

If the system is autonomous, then (8.38) is equivalent to (8.2). In this case, we can choose the separation ansatz

$$S(q, t) = S_0(q) + S_t(t) \tag{8.41}$$

and obtain

$$H\Big(q, \frac{\partial S_0}{\partial q}\Big) = -\dot{S}_t \equiv P_f\,, \tag{8.42}$$

where $P_f \equiv E$ is the separation constant with $S_t(t) = -P_f t$. Since

$$\frac{\partial S}{\partial P_\beta} = \frac{\partial S_0}{\partial P_\beta} - \delta_{\beta f} t\,, \tag{8.43}$$

the equations (8.40) then just reduce to the time-independent Hamilton-Jacobi equations (8.7) and (8.8).

Remark: The generating function $S(q, P, t)$ of the Hamilton-Jacobi theory is related to the action $S[q(t)]$ from Hamilton's principle. With (7.61), we obtain for the derivative along the mechanical path

$$\frac{d}{dt} S(q, P, t) = \sum_{\alpha=1}^{f} (p_\alpha \dot{q}^\alpha + Q^\alpha \dot{P}^\alpha) + (K - H) = \sum_{\alpha=1}^{f} p_\alpha \dot{q}^\alpha - H = L, \tag{8.44}$$

where we have used $K = 0$ and $\dot{P}_\alpha = 0$. Integrating over time leads to the result

$$S(q^{(1)}, P, t^{(1)}) - S(q^{(0)}, P, t^{(0)}) = \int_{(0)}^{(1)} dt\, L(q, \dot{q}, t) = S[q(t)]. \tag{8.45}$$

The statement of the equation is that $S(q, P, t)$ is given by the action $S[q(t)]$ calculated along the mechanical path $q\colon t \mapsto q(t)$: the difference of the generating *function* $S(q, P, t)$ is the action *functional* evaluated on the mechanical path connecting $q^{(0)}$ with $q^{(1)}$. For this reason, the generating function of the canonical transformation is also referred to as action. An analogous relationship exists between the reduced action $S_0[q(t)]$ of the Euler-Maupertuis principle and the generating function $S_0(q, P)$ in the time-independent case.

It is very clear from the above construction how, in the Hamilton-Jacobi method, the difficulty of the problem has shifted from solving the equations of motion (in the new coordinates trivially) to finding the coordinate transformation (non-trivial). The generating function that mediates the coordinate transformation is given by integration over the solution of the problem. Knowledge of the generating function is thus equivalent to solving the problem.

8.5 Outlook on Wave Mechanics

The starting point is the analogy between the mechanics of a particle and ray optics, particularly evident in the variational principles

$$\delta \int_{(1)}^{(2)} ds\, \sqrt{E - V(\boldsymbol{x})} = 0 \qquad \text{(Euler-Maupertuis)}, \tag{8.46}$$

$$\delta \int_{(1)}^{(2)} ds\, n(\boldsymbol{x}) = 0 \qquad \text{(Fermat)}. \tag{8.47}$$

Schrödinger (1926) built upon this to create a wave mechanics that relates to classical mechanics as wave optics relates to ray optics; i.e., mechanics is based on a wave mechanics according to the scheme

$$\begin{array}{ccc} \text{geometric optics} & \longleftrightarrow & \text{mechanics} \\ \uparrow & & \uparrow \\ \text{wave optics} & \longleftrightarrow & \text{wave mechanics}\,. \end{array}$$

As a wavelength, de Broglie (1923) postulated the quantity

$$\lambda = \frac{2\pi\hbar}{p}\,,$$

(or also $p = \hbar k$) depending on the momentum p of the particle.

Mechanics

We initially restrict ourselves to a particle in $\mathbb{R}^3$ with the Hamiltonian

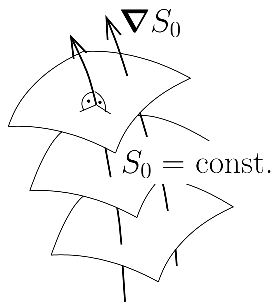
$$H = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{x}). \tag{8.48}$$

Then the Hamilton-Jacobi equation reads

$$(\boldsymbol{\nabla} S_0)^2 = 2m\big(E - V(\boldsymbol{x})\big). \tag{8.49}$$

Every solution of this equation describes a bundle of mechanical trajectories at energy E. The momentum is determined by

$$\boldsymbol{p} = m\dot{\boldsymbol{x}} = \boldsymbol{\nabla} S_0. \tag{8.50}$$



The trajectories are thus orthogonal trajectories of the surfaces $S_0 = \text{const}$. Along each of these trajectories, we have $d\boldsymbol{x}/ds = \dot{\boldsymbol{x}}/(ds/dt) = \boldsymbol{\nabla} S_0/|\boldsymbol{\nabla} S_0|$, i.e.

$$\sqrt{2m(E - V(\boldsymbol{x}))}\,\frac{d\boldsymbol{x}}{ds} = \boldsymbol{\nabla} S_0, \qquad (s = \text{arc length}). \tag{8.51}$$

They thus satisfy (8.46). The vector $d\boldsymbol{x}/ds$ is the unit vector along the path.

Ray Optics

We write Fermat's principle (8.47) in the form

$$\delta \int_{(1)}^{(2)} ds\, k(\boldsymbol{x}) = 0, \quad k(\boldsymbol{x}) = \frac{\omega}{c} n(\boldsymbol{x}) \tag{8.52}$$

with k the wave number and ω the angular frequency of light. This corresponds to the principle of Euler-Maupertuis for the trajectories of energy E in the potential $V(\boldsymbol{x})$, provided that

$$2m(E - V(\boldsymbol{x})) = C^2 k^2(\boldsymbol{x}) \tag{8.53}$$

with C an arbitrary constant. From the discussion in mechanics, we can directly adopt the following result: Every solution $S_0(\boldsymbol{x})$ of the Hamilton-Jacobi equation (cf. (8.49))

$$(\boldsymbol{\nabla} S_0)^2 = C^2 k^2(\boldsymbol{x}) \tag{8.54}$$

describes a bundle of light rays; the orthogonal trajectories of the surfaces $S_0(\boldsymbol{x}) =$ const.. This bundle is determined by (vgl. (8.51))

$$Ck(\boldsymbol{x})\frac{d\boldsymbol{x}}{ds} = C\boldsymbol{k}(\boldsymbol{x}) = \boldsymbol{\nabla} S_0\,. \tag{8.55}$$

The choice $C = 1$ is convenient in terms of wave optics, as we will see shortly. In optics, $S_0(\boldsymbol{x})$ is then called the *eikonal* and (8.54) the *eikonal equation*.

Wave Optics → Ray Optics

In scalar wave optics, monochromatic light is described by a complex field $\psi(\boldsymbol{x}, t) = u(\boldsymbol{x})e^{-i\omega t}$, which satisfies the wave equation (with $C = 1$)

$$(C^2\Delta + k^2)u = 0 \tag{8.56}$$

We decompose

$$u(\boldsymbol{x}) = A(\boldsymbol{x})e^{i\tilde{S}_0(\boldsymbol{x})} \tag{8.57}$$

into the amplitude A and the phase $\tilde{S}_0 = S_0/C$. With

$$\begin{aligned}
\boldsymbol{\nabla}(Ae^{i\tilde{S}_0}) &= (\boldsymbol{\nabla} A + iA\boldsymbol{\nabla}\tilde{S}_0)e^{i\tilde{S}_0},\\
\Delta(Ae^{i\tilde{S}_0}) &= \operatorname{div}\boldsymbol{\nabla}(Ae^{i\tilde{S}_0}) = \Big(\Delta A + iA\Delta\tilde{S}_0 + 2i\boldsymbol{\nabla} A\cdot\boldsymbol{\nabla}\tilde{S}_0 - A(\boldsymbol{\nabla}\tilde{S}_0)^2\Big)e^{i\tilde{S}_0}
\end{aligned}$$

from (8.56) follow the two equations

$$C^2\Delta A - A(\boldsymbol{\nabla} S_0)^2 + Ak^2 = 0, \qquad A\Delta S_0 + 2\boldsymbol{\nabla} A\cdot\boldsymbol{\nabla} S_0 = 0. \tag{8.58}$$

The ray optics, in the form of the eikonal equation (8.54) with $C = 1$, is thus a good approximation in regions where

$$\left|\frac{\Delta A}{A}\right| \ll k^2. \tag{8.59}$$

Roughly speaking, this means that the eikonal approximation is good in regions where the amplitude $A(\boldsymbol{x})$ changes little over a wavelength $\approx k^{-1} = |\boldsymbol{\nabla} S_0|^{-1}$. The eikonal $S_0(\boldsymbol{x})$ is then the phase of the light wave, and the light rays are orthogonal trajectories of the surfaces of constant phase. The vector field $\boldsymbol{\nabla} S_0$ describes (cf. (8.55)) the local wave vector $\boldsymbol{k}(\boldsymbol{x}) = k(\boldsymbol{x})\,d\boldsymbol{x}/ds$.

Mechanics ← Wave Mechanics (Schrödinger)

Schrödinger's starting point was that the Hamilton-Jacobi equations (8.49) for $S_0(\boldsymbol{x})$ should be the ray-optical approximation of a (mechanical) wave equation of a scalar field $u(\boldsymbol{x})$. The comparison of (8.50) with (8.54) suggests that a particle with momentum $\boldsymbol{p}$ should be associated with a wave with the (de Broglie) wave vector ($C = \hbar$)

$$\boldsymbol{p} = \hbar \boldsymbol{k}, \tag{8.60}$$

The Hamilton-Jacobi equation (8.49)

$$(\boldsymbol{\nabla} S_0)^2 = 2m(E - V(\boldsymbol{x}))$$

is thus the ray-optical approximation to (8.56), i.e.

$$\hbar^2 \Delta u(\boldsymbol{x}) + 2m(E - V(\boldsymbol{x}))u(\boldsymbol{x}) = 0\,. \tag{8.61}$$

The equation (8.61) is the *time-independent Schrödinger equation*: It describes states u of energy E in wave mechanics.

Analogously, the time-dependent Hamilton-Jacobi equation for the system (8.48),

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\boldsymbol{\nabla} S)^2 + V(\boldsymbol{x}) = 0 \tag{8.62}$$

is the ray-optical approximation of the phase $\tilde{S} = S/\hbar$ of a complex wave $\psi(\boldsymbol{x}, t) = A(\boldsymbol{x}, t)e^{iS(\boldsymbol{x},t)/\hbar}$, which satisfies the *time-dependent Schrödinger equation*

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi \tag{8.63}$$

This is the *equation of motion of wave mechanics* for the system (8.48). It determines $\psi(\boldsymbol{x}, t)$ from the initial state $\psi(\boldsymbol{x}, 0)$. Equation (8.61) arises from (8.63) through the ansatz

$$\psi(\boldsymbol{x}, t) = u(\boldsymbol{x})e^{-i\omega t} \qquad \text{with } E = \hbar\omega. \tag{8.64}$$

For the phase $S/\hbar$, this corresponds exactly to the separation ansatz (8.41).

The statistical interpretation of the wave function $\psi(\boldsymbol{x}, t)$ comes from Born (1926): One normalizes $\psi(\boldsymbol{x}, t)$ (at a time t_0) so that

$$\int |\psi(\boldsymbol{x}, t_0)|^2 \, d^3x = 1.$$

Then

$$P(\Omega, t) = \int_\Omega |\psi(\boldsymbol{x}, t)|^2 \, d^3x$$

is the probability that the particle is located in the region $\Omega \subset \mathbb{R}^3$ at time t.

Chapter 9

Relativistic Mechanics

9.1 Einstein's Principle of Relativity

The Michelson-Morley experiment led Einstein (1905) to demand, in addition to the Galilean principle of relativity, which states that the physical laws are the same in all inertial systems, the additional *Einstein postulate*:

> *The speed of light has the same value* $c = 299\,792\,458\,m/s$ *in all inertial systems.*

This principle is the foundation of the *Special Theory of Relativity*. Inertial systems are defined such that free particles, as in classical mechanics, follow the law of inertia $\ddot{\boldsymbol{x}} = 0$. It is immediately clear that the invariance of physical laws can no longer be demanded under Galilean transformations, as these lead to the elementary velocity addition formulas. In Chapter 9.2, we will investigate which transformation group is compatible with the Einstein postulate. However, we will first learn a few simple consequences of the Einstein postulate.

Since we saw in Chapter 1.3 that assuming an absolute time leads us directly to the Galilean transformations as general transformations between inertial systems, we must abandon the concept of absolute time and demand that, in general, $t' \neq t$. The absolute value of the speed of light allows us to synchronize clocks 1 and 2 in two reference systems S and S' and thus to introduce at least partially a common time. The protocol for synchronizing two clocks in two different inertial systems S and S', which measure time through the same periodic process, is as follows: We imagine that in the reference system S, a light flash is emitted from clock 1 at time t_0. The beam is reflected by clock 2 (at rest in system S') and detected by clock 1 at time $t_0 + \Delta t$. Since the speed of light is absolute, we can identify the moment of arrival of the light beam at clock 2 with the time $t_0 + \Delta t/2$ in system S and thus synchronize the clocks. Furthermore, the universality of the speed of light allows us to relate length measurements to time measurements and to define the meter as the

1/299792458-th part of the distance that a light beam travels in one second. The problem of synchronizing the clocks is that the procedure is not transitive. If clocks 1 and 2 are synchronized and also 2 and 3 are synchronized, it does not follow that 1 and 3 are synchronized. This leads us to the concept of *relativity of simultaneity*.

Relativity of Simultaneity: From the absolute time in Galilean spacetime, it follows that two events that occur simultaneously in reference system S' also occur simultaneously in S. This fact is incompatible with the Einstein postulate of the universality of the speed of light. To see this, consider two reference systems that move relative to each other with speed $\boldsymbol{w} = (w, 0, 0)$, $w > 0$. We consider three clocks A, B, C on the 1-axis, which are at rest in S'. Clock B is located between clocks A and C, such that the distance from A to B is equal to the distance from B to C. An observer in system S' will note that a light flash emitted at time t'_{B} from B arrives simultaneously ($t'_{\mathrm{A}} = t'_{\mathrm{C}}$) at A and C.

From the perspective of an observer in the (moving) system S, the light flash is emitted at time t_{B} from B and spreads out with speed c in all directions. Since A moves towards the light flash and C moves away from it, the inequality $t_{\mathrm{A}} < t_{\mathrm{C}}$ holds from the perspective of S. Thus, clocks that are synchronized in system S' are not synchronized in system S, and therefore the concept of "simultaneity" depends on the reference system. The absence of an absolute concept of simultaneity is the cause of many of the initially paradoxical consequences of the principle of relativity.

9.2 Lorentz Transformation

In the last chapter, we saw that the Einstein postulate, supported by the Michelson-Morley experiment, forces us to rethink our notions of time and space. In particular, we must abandon the concept of absolute time. In this chapter, we want to develop the Einstein postulate into a mathematical theory that allows us to establish new laws of nature that are compatible with the Einstein postulate. To do this, we first need to consider what the new transformations between inertial systems are.

Similar to how the Galilean transformations leave Newton's equations invariant, the *Lorentz transformations* are the group of transformations that switch between systems that comply with the Einstein postulate. In this sense, the Lorentz transformations in the Special Theory of Relativity allow us to switch from one inertial system to any other inertial system. The task then remains to find a relativistic (i.e., Lorentz-invariant) formulation of mechanics, which we will work through in the next chapter.

9.2.1 Transformation Between Inertial Systems

At first glance, it seems that the Einstein postulate contradicts the law of inertia. This "feeling" is due to the fact that we are firmly rooted in our classical view of the world, and the Einstein postulate, as seen in the last chapter, comes with the loss of the absolute concept of simultaneity and thus leads to the introduction of a time per reference system. To obtain the group of Lorentz transformations, we consider two inertial systems S and S' that move relative to each other with speed $\boldsymbol{w}$.

We consider a light flash that is emitted at $(t_1, \boldsymbol{x}_1)$ and later arrives at $(t_2, \boldsymbol{x}_2)$. Due to the Einstein postulate, both

$$(\Delta \boldsymbol{x})^2 - c^2(\Delta t)^2 = 0 \qquad \text{and} \qquad (\Delta \boldsymbol{x}')^2 - c^2(\Delta t')^2 = 0 \tag{9.1}$$

hold, with $\Delta t = t_2 - t_1$ and $\Delta \boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1$. The motion of a free particle in an inertial system is given by straight lines. From the requirement that a Lorentz transformation should map inertial systems onto each other, it follows with the law of inertia $\ddot{\boldsymbol{x}} = \ddot{\boldsymbol{x}}' = 0$; thus, straight lines are mapped onto straight lines. Therefore, a Lorentz transformation is a linear mapping from $(\Delta t, \Delta \boldsymbol{x})$ to $(\Delta t', \Delta \boldsymbol{x}')$. Due to the linearity of the transformation, (9.1) is equivalent to[1]

$$(\Delta \boldsymbol{x}')^2 - c^2(\Delta t')^2 = \kappa(\boldsymbol{w})[(\Delta \boldsymbol{x})^2 - c^2(\Delta t)^2]. \tag{9.2}$$

Now consider another reference system S'', which moves with speed $-\boldsymbol{w}$ relative to S'. Then we obtain

$$(\Delta \boldsymbol{x}'')^2 - c^2(\Delta t'')^2 = \kappa(-\boldsymbol{w})\kappa(\boldsymbol{w})[(\Delta \boldsymbol{x})^2 - c^2(\Delta t)^2].$$

Since S'' is at rest relative to S, we have $\Delta \boldsymbol{x} = \Delta \boldsymbol{x}''$ and $\Delta t = \Delta t''$, from which it follows that $\kappa(-\boldsymbol{w})\kappa(\boldsymbol{w}) = 1$. Due to the isotropy of space, $\kappa(\boldsymbol{w})$ must also depend only on the magnitude w and not on the direction of the relative motion. From this, we can conclude that $\kappa(-\boldsymbol{w})\kappa(\boldsymbol{w}) = \kappa(w)^2 = 1$ and thus $\kappa(w) = 1$.[2]

The results can be best summarized by combining position and time into the *4-coordinates*

$$x \equiv (x^0, x^1, x^2, x^3) = (ct, \boldsymbol{x}) \tag{9.3}$$

of a particle. Thus, the Lorentz transformations are defined such that they leave the square of the distance

$$\Delta s^2 = \big(\Delta x', \Delta x'\big) = \big(\Delta x, \Delta x\big) = \eta_{\mu\nu}(\Delta x)^\mu(\Delta x)^\nu \tag{9.4}$$

invariant, with the Minkowski metric

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{9.5}$$

[1] A linear mapping maps a homogeneous polynomial of degree 2 to a homogeneous polynomial of degree 2. Since $(\Delta \boldsymbol{x})^2 - c^2(\Delta t)^2 = (\Delta \boldsymbol{x}')^2 - c^2(\Delta t')^2 = 0$, it follows from the homogeneity that (9.2) holds.

[2] The alternative solution $\kappa(w) = -1$ can be discarded since $\kappa = 1$ at $w = 0$.

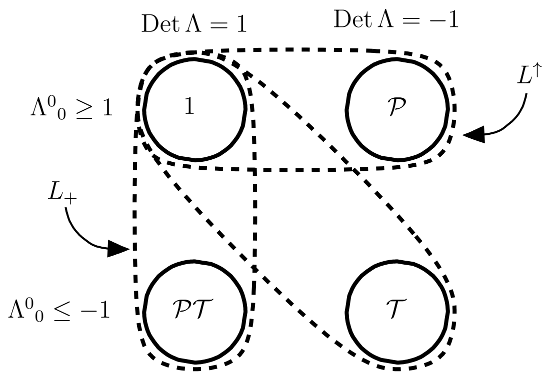


Figure 9.1: Structure of the Lorentz group: the Lorentz group L splits into four disjoint components characterized by $\mathrm{Det}\,\Lambda = \pm 1$ and $\mathrm{sgn}(\Lambda^0{}_0)$.

We use the *Einstein summation convention*: repeated indices are summed over $\{0, 1, 2, 3\}$.

A general affine transformation between the reference systems S and S' can be written as $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$ with $a \in \mathbb{R}^4$ and $\Lambda \in \mathrm{GL}(4, \mathbb{R})$. The condition (9.4) leads to the constraint

$$\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta (\Delta x)^\alpha (\Delta x)^\beta = \eta_{\mu\nu}(\Delta x')^\mu (\Delta x')^\nu = \eta_{\mu\nu}(\Delta x)^\mu (\Delta x)^\nu$$

which is equivalent to

$$\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \tag{9.6}$$

or in matrix notation $\Lambda^t \eta \Lambda = \eta$.[3] Transformations that satisfy the condition (9.6) are called elements of the *Lorentz group*, denoted by $L = \mathrm{O}(1, 3)$. The group of transformations that connect inertial systems with fixed scales is the group of inhomogeneous Lorentz transformations with $x' = \Lambda x + a$, also called the *Poincaré group*. Vectors v, which transform according to the homogeneous part of the Poincaré group, i.e., as $v' = \Lambda v$, are called *contravariant four-vectors*.

9.2.2 Lorentz Group

We consider an element $\Lambda \in L$. From the relation (9.6), a number of properties of Λ can be derived. The determinant calculation immediately yields $(\mathrm{Det}\,\Lambda)^2 = 1$. Thus, L splits into two components characterized by $\mathrm{Det}\,\Lambda = \pm 1$. Furthermore, for the (00)-components, we obtain the property $(\Lambda^0{}_0)^2 - \sum_{k=1}^3 (\Lambda^k{}_0)^2 = 1$, i.e., $(\Lambda^0{}_0)^2 \geq 1$. The Lorentz group thus has two disjoint components characterized by the sign of $\Lambda^0{}_0$.

[3] The inverse of Λ is obtained by multiplying from the left by η and from the right by Λ^{-1} as $\Lambda^{-1} = \eta \Lambda^t \eta$.

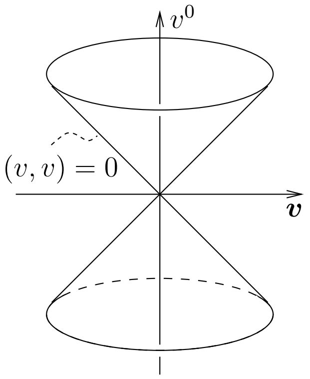


Figure 9.2: Vectors on the light cone with $(v, v) = 0$ are called null vectors. Outside the light cone are the spacelike vectors. The timelike vectors inside the cone split into two components depending on whether they point into the future (above) or the past (below).

That all four cases occur is shown by the reflections

$$1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{9.7}$$

$$\mathcal{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathcal{PT} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{9.8}$$

with $\mathcal{P}$ being the reflection at the coordinate origin (parity) and $\mathcal{T}$ being the time reversal. The reflections form a subgroup of L. Other subgroups include, for example,

$$\begin{aligned} L_+ &= \{\Lambda \in L \mid \operatorname{Det}\Lambda = 1\}, && \text{the proper Lorentz transformations,} \\ L^\uparrow &= \{\Lambda \in L \mid {\Lambda^0}_0 \geq 1\}, && \text{the orthochronous Lorentz transformations,} \\ L_+^\uparrow &= L_+ \cap L^\uparrow. \end{aligned} \tag{9.9}$$

A Lorentz transformation, by definition, maps the light cone with $\Delta s^2 = 0$ onto itself. Four-vectors on the light cone with $(v, v) = 0$ are also referred to as *null vectors*, see Fig. 9.2. We also define vectors that point outside the light cone as *spacelike* vectors v with $(v, v) < 0$ and those that point inside the light cone as *timelike* vectors with $(v, v) > 0$. Timelike vectors can additionally be subdivided into vectors that point into the timelike future ($v^0 > 0$) and those that point into the timelike past ($v^0 < 0$).

That $L^\uparrow$ is a group can be seen geometrically. A Lorentz transformation $\Lambda \in L$ maps the interior of the light cone onto itself. In doing so, the two sub-cones (future and past) $V^\pm = \{x \mid (x, x) > 0, \pm x^0 \geq 0\}$ either remain invariant or are swapped. The crucial factor is the sign $\operatorname{sgn}({\Lambda^0}_0)$, since $[\Lambda(1, \mathbf{0})]^0 = {\Lambda^0}_0$. Thus, $\operatorname{sgn}({\Lambda^0}_0)$

and naturally also $\mathrm{Det}\,\Lambda$ behave multiplicatively[4] under the group operation. The orthochronous transformations $L^{\uparrow}$ map the future onto the future and the past onto the past.

Due to the multiplicity of $\mathrm{sgn}(\Lambda^0{}_0)$ and $\mathrm{Det}\,\Lambda$, any $\Lambda \in L$ can be written as the product of an element of the proper orthochronous Lorentz group $L_+^{\uparrow}$ with a reflection $\{1, \mathcal{P}, \mathcal{T}, \mathcal{PT}\}$. Therefore, we will restrict ourselves in the following to the proper orthochronous subgroup.

Proper Orthohronous Lorentz Group The 4×4 matrix Λ has a total of 16 real entries. The equation (9.6) provides 10 independent equations.[5] Thus, elements $\Lambda \in L_+^{\uparrow}$ are determined by 6 real parameters.

Three of the parameters can be identified with the three-dimensional rotations. In fact, it holds that

$$\Lambda(R) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right) \tag{9.10}$$

immediately satisfies $\Lambda^t \eta \Lambda = \eta$ where $R \in \mathrm{SO}(3)$ with $R^t R = 1$. Thus, the rotations $\Lambda(R)$ form a subgroup of $L_+^{\uparrow}$.

Of greater interest are the remaining three-parameter transformations, which we expect to connect mutually moving reference systems. To this end, we investigate whether special solutions exist in block form

$$\Lambda = \left(\begin{array}{cc|cc} a & b & 0 & 0 \\ c & d & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \tag{9.11}$$

i.e., x^2, x^3 are not transformed. Substituting into (9.6) yields the conditions

$$a^2 - c^2 = 1, \qquad ab - cd = 0, \qquad b^2 - d^2 = -1.$$

Since we are only interested in solutions from $L_+^{\uparrow}$, we also require $a > 0$ and $ad - bc = 1$.

We can satisfy the first equation by introducing the parameter χ with $a = \cosh\chi$ and $c = -\sinh\chi$. From the second equation, we obtain $b = cd/a$. Substituting into $ad - bc = 1$ yields $1 = d/a$, i.e., $d = \cosh\chi$ and $b = -\sinh\chi$. Thus, we obtain the

[4] The latter follows from the general relation $\mathrm{Det}(AB) = (\mathrm{Det}\,A)(\mathrm{Det}\,B)$ for arbitrary square matrices A and B.

[5] Transposing (9.6) leads to the same system of equations. Therefore, only 10 of the 16 equations are independent.

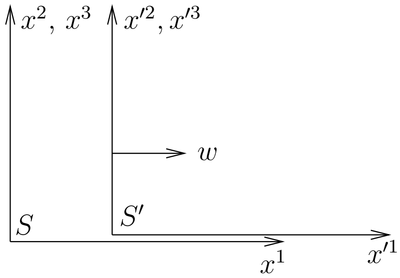


Figure 9.3: Relationship of the systems S and S' under a boost along the 1-axis.

special Lorentz transformations (boosts)

$$\Lambda(\chi) = \left(\begin{array}{cc|cc} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = \exp\left(\begin{array}{cc|c} 0 & -\chi & \\ -\chi & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right). \tag{9.12}$$

From the last identity, it immediately follows that the boosts form a subgroup with the multiplication law

$$\Lambda(\chi_1 + \chi_2) = \Lambda(\chi_1)\Lambda(\chi_2). \tag{9.13}$$

From this, it can be directly inferred that $\Lambda(\chi)^{-1} = \Lambda(-\chi)$.

Next, we want to explain the physical significance of $\Lambda(\chi)$. For this, we write the transformation law $x = \Lambda(\chi)^{-1}x'$ of a boost in components

$$\begin{aligned} ct &= (\cosh\chi)ct' + (\sinh\chi)x'^1, & x^2 &= x'^2, \\ x^1 &= (\sinh\chi)ct' + (\cosh\chi)x'^1, & x^3 &= x'^3. \end{aligned} \tag{9.14}$$

An object that is at rest in the S' system at position $\boldsymbol{x}' = \boldsymbol{x}'(t')$ will therefore move in system S with speed[6]

$$w = \frac{dx^1}{dt} = \frac{dx^1}{dt'}\frac{dt'}{dt} = \frac{dx^1}{dt'}\left(\frac{dt}{dt'}\right)^{-1} = c\tanh\chi \tag{9.15}$$

Thus, we obtain the interpretation that $\Lambda(\chi)$ transforms to a new inertial system that moves with speed w along the 1-axis relative to S, see Fig. 9.3. Simple hyperbolic relationships yield the expressions

$$\begin{aligned} \cosh\chi &= \frac{1}{\sqrt{1-\tanh\chi^2}} = \frac{1}{\sqrt{1-(w/c)^2}} \equiv \gamma, \\ \sinh\chi &= \tanh\chi\cosh\chi = \frac{w/c}{\sqrt{1-(w/c)^2}} = \gamma w/c. \end{aligned} \tag{9.16}$$

From (9.14), we thus obtain

$$\begin{aligned} t &= \gamma(t' + wx'^1/c^2), & x^2 &= x'^2, \\ x^1 &= \gamma(x'^1 + wt'), & x^3 &= x'^3. \end{aligned} \tag{9.17}$$

[6]In the literature, $\chi = \operatorname{artanh}(w/c)$ is referred to as rapidity.

For $w/c \to 0$, we obtain as limiting behavior the Galilean transformation

$$t = t', \quad x^1 = x'^1 + wt, \quad x^2 = x'^2, \quad x^3 = x'^3.$$

A transformation into a system S' that moves with a velocity $\boldsymbol{w}$ in any spatial direction relative to S can generally be obtained by a combination of a rotation (so that $\boldsymbol{w}'$ is along the 1-direction), a boost, and a reverse rotation. However, it is much simpler to think that one can split the vector $\boldsymbol{x}$ into a component $\boldsymbol{x}_\parallel$ along $\boldsymbol{w}$ and a component $\boldsymbol{x}_\perp$ orthogonal to it. Analogous to (9.17), one then obtains

$$t = \gamma(t' + (\boldsymbol{w} \cdot \boldsymbol{x}'_\parallel)/c^2), \qquad \boldsymbol{x}_\parallel = \gamma(\boldsymbol{x}'_\parallel + \boldsymbol{w}t'), \qquad \boldsymbol{x}_\perp = \boldsymbol{x}'_\perp. \tag{9.18}$$

Thus, a boost $x' = \Lambda(\boldsymbol{w})x$ generally depends on the three real parameters $\boldsymbol{w}$. The Lorentz transformations become singular for $w \to c$. The speed of light represents a maximum speed. No object (and no information) can move faster than light.

9.2.3 Addition of Velocities

It is immediately clear that the Galilean velocity addition $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{v}'$ leads to contradictions with the absoluteness of the speed of light. This demands that with $|\boldsymbol{v}| = c$, $|\boldsymbol{v}'| = c$ must also hold, regardless of $\boldsymbol{w}$. The relativistic formulas for velocity addition can be derived directly from the equations (9.18) for a general boost. We introduce two (axis-parallel) reference systems S and S' whose coordinates are linked by (9.18) (i.e., they move with $\boldsymbol{w}$ relative to each other). Now consider an object that moves along the path $\boldsymbol{x}'(t') = \boldsymbol{x}'_0 + \boldsymbol{v}'t'$ in S'. The same object will move in system S along the path $\boldsymbol{x}(t)$. In 4-vector notation, we have the coordinates $x(t) = (ct, \boldsymbol{x}(t))$ and $x'(t') = (ct', \boldsymbol{x}'(t'))$, which are linked by $x' = \Lambda(\boldsymbol{w})x$. Using the general formula (9.18) for a boost, we obtain

$$\frac{dt}{dt'} = \gamma(1 + (\boldsymbol{w} \cdot \boldsymbol{v}'_\parallel)/c^2) = \gamma(1 + (\boldsymbol{w} \cdot \boldsymbol{v}')/c^2)$$

and

$$\begin{aligned} \frac{d\boldsymbol{x}_\parallel}{dt'} &= \gamma(\boldsymbol{v}'_\parallel + \boldsymbol{w}), \\ \frac{d\boldsymbol{x}_\perp}{dt'} &= \boldsymbol{v}'_\perp \end{aligned}$$

The object thus moves in system S with the velocity (*velocity addition*)

$$\boldsymbol{v} = \frac{d(\boldsymbol{x}_\parallel + \boldsymbol{x}_\perp)}{dt} = \frac{\boldsymbol{w} + \boldsymbol{v}'_\parallel + \boldsymbol{v}'_\perp\sqrt{1 - (w/c)^2}}{1 + (\boldsymbol{w} \cdot \boldsymbol{v}')/c^2}. \tag{9.19}$$

The inverse relationship is obtained by replacing $\boldsymbol{w}$ with $-\boldsymbol{w}$. In the limit that $|\boldsymbol{w}|, |\boldsymbol{v}'| \ll c$, one obtains, as desired, the non-relativistic formula $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{v}'$.

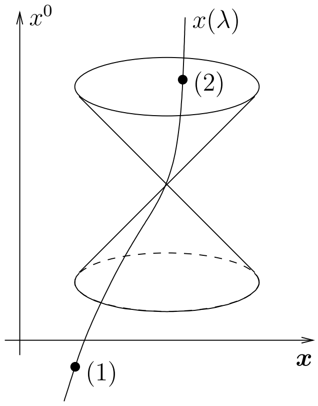


Figure 9.4: World line of a particle parametrized by λ. The motion occurs between the events $x^{(1)}$ and $x^{(2)}$.

9.3 Relativistic Mechanics

As we have seen, Newtonian mechanics is not compatible with the relativity postulate. In the special theory of relativity, we must therefore make the laws of mechanics *covariant* (i.e., invariant under Lorentz transformations). As a side condition, the old mechanics should be recovered in the limit $|\dot{\boldsymbol{x}}| \ll c$.

9.3.1 Four-Velocity

The motion of a particle is determined by the trajectory $\boldsymbol{x}(t)$ with velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{x}}(t)$. In relativistic notation, the trajectory becomes a world line

$$x^\mu(\lambda) = (ct(\lambda), \boldsymbol{x}(t(\lambda))) \tag{9.20}$$

with λ being an arbitrary parameter that parametrizes the world line, see Fig. 9.4. As explained in Chapter 9.2, transformations between different inertial systems are performed by a Poincaré transformation Λ with $x' = \Lambda x + a$.

One would now like to generalize the concept of velocity to a four-vector. To achieve this, $x(\lambda)$ must be parametrized by a Lorentz scalar τ, which in the limit $|\boldsymbol{v}| \ll c$ coincides with time. Due to the invariance of the distance (9.4), the proper time (starting from an arbitrary time $t^{(1)}$)

$$\tau(t^{(2)}) = \frac{1}{c}\int_{(1)}^{(2)} ds = \int_{(1)}^{(2)} \sqrt{\eta_{\mu\nu}\, dx^\mu\, dx^\nu}/c = \int_{t^{(1)}}^{t^{(2)}} dt\, \sqrt{1 - \boldsymbol{v}^2(t)/c^2} \tag{9.21}$$

between the events $x^{(1)}$ and $x^{(2)}$ is precisely a Lorentz scalar, i.e., τ is independent of the reference system. To make the parametrization in (9.20) reference system independent, it is therefore often sensible to set $\lambda = \tau$ by solving $\tau(t)$ from (9.21) for t. The derivative of the world line with respect to proper time yields the contravariant four-vector,

$$u(\tau) \equiv \frac{dx(\tau)}{d\tau} = \frac{dt}{d\tau}\frac{d(ct, \boldsymbol{x}(t))}{dt} = \gamma(t)(c, \boldsymbol{v}(t)) \tag{9.22}$$

with $\gamma(t) = (1 - \boldsymbol{v}^2/c^2)^{-1/2}$, which we refer to as *four-velocity.* For the square of the "length" of the four-vector, it holds that $(u, u) = \gamma^2(c^2 - \boldsymbol{v}^2) = c^2$. Thus, u is always normalized to c and only 3 parameters are independent.

9.3.2 Energy-Momentum Relation

With the four-velocity, we can immediately generalize the momentum $\boldsymbol{p}$ to a contravariant four-momentum

$$p \equiv (p^0, \boldsymbol{p}) = mu = \gamma m(c, \boldsymbol{v}) \tag{9.23}$$

In the non-relativistic limit $|\boldsymbol{v}| \ll c$, $\gamma \to 1$, the spatial components of p^μ coincide with the non-relativistic momentum. The spatial part thus remains conserved for a free particle in an inertial system. Since p^μ is a contravariant four-vector, the complete four-momentum p^μ remains conserved in every inertial system. Analogously, one can conclude that the total momentum $P^\mu = \sum_{j=1}^N (p_j)^\mu$ of N particles with four-momenta p_j is conserved in every inertial system.

The question now is, which conserved quantity corresponds to the time component p^0 in non-relativistic physics. Expanding p^0 in powers of $|\boldsymbol{v}|/c$ yields

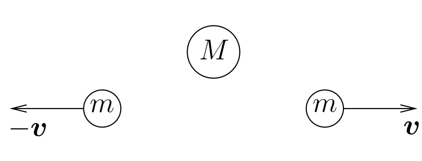
$$p^0 = \frac{mc}{\sqrt{1 - \boldsymbol{v}^2/c^2}} = mc + \frac{1}{2} m\boldsymbol{v}^2/c + \cdots \tag{9.24}$$

It is immediately clear that $p^0 c$ agrees with the kinetic energy T of Newtonian mechanics, up to the constant mc^2. Thus, the (relativistic) *kinetic energy* E is given by

$$E = p^0 c = E_0 + T \quad \text{with} \quad E_0 = mc^2. \tag{9.25}$$

The rest energy $E_0 = mc^2$ is precisely Einstein's famous formula. The relativistic energy contains, in contrast to non-relativistic physics, a rest energy contribution that depends only on the mass of the object. Unlike in non-relativistic physics, mass can be annihilated in relativity theory, and the corresponding energy can be converted into other forms of energy.

As an example, consider the (symmetric) decay of a particle into two parts. In the rest frame of the particle, one has initially the four-momentum $P = (Mc, \boldsymbol{0})$ with M being the total mass of the particle.



After the symmetric decay, the total momentum P is composed of the momentum of two particles with mass m, which move with speed $\pm\boldsymbol{v}$. The particles have the four-momenta $\gamma m(c, \pm\boldsymbol{v})$ with $\gamma = (1 - \boldsymbol{v}^2/c^2)^{-1/2}$, which add up to the total momentum $P = 2\gamma m(c, 0)$. From the conservation of four-momentum, we can now conclude that

$$2m = M\sqrt{1 - \boldsymbol{v}^2/c^2} < M, \tag{9.26}$$

i.e., the total mass is not conserved. The rest energy of the mass defect is given by

$$(M-2m)c^2 = 2mc^2(\gamma - 1) = 2\frac{1}{2}m\boldsymbol{v}^2 + \cdots \tag{9.27}$$

and thus for $|\boldsymbol{v}| \ll c$ equals the non-relativistic energy of the decay products. During the decay of the particle, rest energy is thus converted into kinetic energy of the decay products.

From the fact that $(u,u) = c^2$, one obtains the relativistic energy-momentum relation

$$(p,p) = \eta_{\mu\nu}\, p^\nu\, p^\mu = (E/c)^2 - \boldsymbol{p}^2 = m^2c^2, \tag{9.28}$$

which holds by construction in all inertial systems.

9.3.3 Equation of Motion

With this groundwork, it is now possible to formulate mechanics in a Lorentz-invariant manner. This is most easily done using the Lagrangian formalism. In this formalism, one assigns an action to each world line $x(\lambda)$

$$S[x(\lambda)] = \int_{(1)}^{(2)} d\lambda\, L(x, \partial_\lambda x, \lambda) \tag{9.29}$$

with L being the Lagrangian. The Hamiltonian principle demands that the path of a particle is characterized by an extremum of the action at fixed endpoints in $\mathbb{R}^4$. From the invariant condition $\delta S = 0$ on the path, it is directly evident that the equation of motion becomes covariant as long as the action S is a Lorentz scalar.[7]

Free Particle

In non-relativistic mechanics, a free particle is described by the Lagrangian $L_0 = \frac{1}{2}m\boldsymbol{v}^2$. For a relativistic generalization of the equations of motion of a free particle, we need an action that is a Lorentz scalar and reproduces the non-relativistic behavior in the limit $|\boldsymbol{v}| \ll c$. As we have seen in (9.21), proper time assigns a Lorentz scalar to a world line. A natural Ansatz is therefore $S_0 = -mc^2\tau$, where the prefactor E_0 is chosen such that S_0 has the unit of action. In fact, we obtain

$$S_0 = -mc^2\tau = -mc^2\int_{(1)}^{(2)} dt\, \sqrt{1-\boldsymbol{v}^2/c^2} = \int_{(1)}^{(2)} dt \left(-mc^2 + \frac{1}{2}m\boldsymbol{v}^2 + \cdots\right),$$

so that the relativistic Lagrangian

$$L_0 = -mc^2\sqrt{1-\boldsymbol{v}^2/c^2} \tag{9.30}$$

[7] In principle, the actions in the various inertial systems can also differ by a factor.

in the limit $|\boldsymbol{v}| \ll c$ agrees with the non-relativistic Lagrangian of a free particle, up to the constant mc^2. The relativistic equations of motion

$$\frac{d}{dt}(\gamma m\boldsymbol{v}) = \frac{d\boldsymbol{p}}{dt} = 0 \tag{9.31}$$

are obtained as Euler-Lagrange equations for L_0. However, in this form, it is not immediately evident that the equation (9.31) is covariant. To make the covariance explicit, we note that due to the energy-momentum relation (9.28) with (9.31), $\dot{p}^0$ is also determined. In fact, by differentiating (9.28) with respect to t, we obtain

$$c\frac{dp^0}{dt} = c\frac{\boldsymbol{p}}{p^0} \cdot \frac{d\boldsymbol{p}}{dt} = \boldsymbol{v} \cdot \frac{d\boldsymbol{p}}{dt} = 0, \tag{9.32}$$

so that we find the (manifestly) covariant equation $dp^\mu/d\tau = 0$.

Particle in a Potential

In Galilean physics, one often considers a particle in a scalar potential $V(\boldsymbol{x}, t)$. The corresponding problem is not really useful in relativistic mechanics. The problem is that starting from a scalar potential V, the potential in another reference system automatically becomes dependent on the velocity $\boldsymbol{v}$. Therefore, it is better to start directly from a vector potential that couples to the velocity. Another point is that there is no physical force that can be described by a scalar potential, as the gravitational force is not covariantly formulated and thus only the electromagnetic force fits into the relativistic concept. Electrodynamics comes directly with a scalar potential $\varphi(\boldsymbol{x}, t)$ and a vector potential $\boldsymbol{A}(\boldsymbol{x}, t)$. The action of the electromagnetic fields on a particle with charge e is described in classical mechanics by the Lagrangian, see (4.39),

$$L = L_0 + \frac{e}{c}\boldsymbol{v} \cdot \boldsymbol{A} - e\varphi = L_0 - \frac{e}{c}\eta_{\mu\nu}\, u^\mu A^\nu \sqrt{1 - \boldsymbol{v}^2/c^2}\,. \tag{9.33}$$

with $A = (\varphi, \boldsymbol{A})$. If one now replaces the free Lagrangian $L_0 = m\boldsymbol{v}^2/2$ with the relativistic generalization (9.30), one sees that the action

$$S = \int_{(1)}^{(2)} dt\, L = -\int_{(1)}^{(2)} d\tau \left(mc^2 + \frac{e}{c}\,(u, A)\right) \tag{9.34}$$

is already covariant if A^μ transforms like a four-vector.[8]

The equations of motion of a charged particle in the electromagnetic field are the Euler-Lagrange equations

$$\frac{d}{dt}\frac{m\boldsymbol{v}}{\sqrt{1 - \boldsymbol{v}^2/c^2}} = e\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right)\,, \tag{9.35}$$

[8]This fact will be proven in electrodynamics.

for (9.34), since in the non-relativistic calculation from Chapter 4.4 one merely has to replace

$$\frac{\partial}{\partial \boldsymbol{v}} \frac{m\boldsymbol{v}^2}{2} = m\boldsymbol{v} \quad \mapsto \quad -mc^2 \frac{\partial}{\partial \boldsymbol{v}} \sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}} = \frac{m\boldsymbol{v}}{\sqrt{1 - \boldsymbol{v}^2/c^2}} \tag{9.36}$$

i.e., $m\boldsymbol{v} \mapsto \boldsymbol{p}$. From (9.35), dp^0/dt is also determined. The differentiation of (9.28) with respect to t yields

$$c\frac{dp^0}{dt} = c\frac{\boldsymbol{p}}{p^0} \cdot \frac{d\boldsymbol{p}}{dt} = \boldsymbol{v} \cdot \frac{d\boldsymbol{p}}{dt} = e\boldsymbol{v} \cdot \boldsymbol{E}\,. \tag{9.37}$$

This is the energy equation, since the right side represents the power of the Lorentz force and the left side represents the change in kinetic energy.

Notation

$\boldsymbol{x}_i$	Cartesian coordinates of the i-th particle in an inertial system
$\boldsymbol{y}_i$	body-fixed coordinates of the i-th particle
$\dot{\square}$	time derivative
q^α	(generalized) position coordinates
p_α	(generalized) momenta
x_i	coordinates in phase space
$\{\cdot,\cdot\}$	Poisson bracket
$\boldsymbol{X}$	center of mass
$\boldsymbol{P}$	total momentum
$\boldsymbol{F}$	force
$\boldsymbol{L}$	angular momentum with magnitude $l = \lvert\boldsymbol{L}\rvert$
$\boldsymbol{M}$	torque
$\boldsymbol{\omega}$	angular velocity in body-fixed coordinates
Θ	moment of inertia
$\boldsymbol{S}$	angular momentum in body-fixed coordinates
d^3y	infinitesimal volume element
E	(total) energy
T	kinetic energy
V	potential, potential energy
$L(q,\dot{q},t)$	Lagrangian function
$H(q,p,t)$	Hamiltonian function
S	action
$S_0 = S + Et$	reduced action