Path integrals for quantum optics

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Please report mistakes and typos to F. Hassler.

Contents

Chapter 1

Path integrals in quantum mechanics

1.1 Derivation of the path integral

In the following, we recapitulate the known results from the introductory lecture on quantum mechanics.

Problems in quantum mechanics are described by the Schrödinger equation

$$i\hbar\partial_t\psi(x,t) = \hat{H}\psi(x,t)\,. \tag{1.1}$$

We restrict ourselves to problems in 1D with $x \in \mathbb{R}$ for simplicity. The generalization of the results to higher dimensional problems is rather straightforward. We formally introduce to position basis $|x\rangle$ with the normalization

$$\langle x|x'\rangle = \delta(x-x'), \tag{1.2}$$

where δ is the Dirac delta function. With this, we write $\psi(x,t) = \langle x|\psi(t)\rangle$, where $|\psi(t)\rangle$ denotes the (abstract) wavefunction in Dirac notation. Using Eq. (1.2), we find

$$\begin{aligned}\psi(x,t) = \langle x|\psi(t)\rangle &= \int dx'\,\delta(x-x')\langle x'|\psi(t)\rangle = \int dx'\,\langle x|x'\rangle\langle x'|\psi(t)\rangle \\ &= \langle x|\left(\int dx'|x'\rangle\langle x'|\right)|\psi(t)\rangle \end{aligned} \tag{1.3}$$

From the above equation, we get

$$\int dx\,|x\rangle\langle x| = 1, \tag{1.4}$$

which describes the completeness relation of the position basis. Note that 1 is the identity operator.

We can also define a momentum basis $|p\rangle$ via the inner product

$$\langle x|p\rangle = \overline{\langle p|x\rangle} = e^{ipx/\hbar} \tag{1.5}$$

with the position basis. Using Eq. (1.4), we can express the momentum basis in terms of the position basis

$$|p\rangle = \int dx\, |x\rangle\langle x|p\rangle = \int dx\, e^{ipx/\hbar}|x\rangle \tag{1.6}$$

The momentum basis fulfills the orthogonality relation

$$\begin{aligned}\langle p'|p\rangle = \langle p'|\left(\int dx\, |x\rangle\langle x|\right)|p\rangle = \int dx\, \langle p'|x\rangle\langle x|p\rangle = \int dx\, e^{-ip'x/\hbar}e^{ipx/\hbar} \\ = 2\pi\hbar\delta(p-p').\end{aligned} \tag{1.7}$$

Moreover, the momentum basis is also complete due to the fact that

$$\int dp\, \langle x'|p\rangle\langle p|x\rangle = \int dp\, e^{ip(x-x')/\hbar} = 2\pi\hbar\delta(x-x') \tag{1.8}$$

As the states $|x\rangle, |x'\rangle$ are arbitrary, the completeness relation

$$\int \frac{dp}{2\pi\hbar}|p\rangle\langle p| = 1 \tag{1.9}$$

follows.

The states $|x\rangle$ and $|p\rangle$ are eigenstates of the position and momentum operators $\hat{x}$ and $\hat{p}$ with

$$\hat{x}|x\rangle = x|x\rangle \quad \text{and} \quad \hat{p}|p\rangle = p|p\rangle. \tag{1.10}$$

The operators have the spectral decomposition

$$\hat{x} = \int dx\, x|x\rangle\langle x|, \qquad \hat{p} = \int \frac{dp}{2\pi\hbar}p|p\rangle\langle p| \tag{1.11}$$

which is equivalent to the defining relations (1.10).

We can use the completeness of the position basis (1.4) to write the momentum operator in the position basis,

$$\begin{aligned}\hat{p} &= \int dx \int dx' \int \frac{dp}{2\pi\hbar}p|x\rangle\langle x|p\rangle\langle p|x'\rangle\langle x'| = \int dx \int dx' \left(\int \frac{dp}{2\pi\hbar}pe^{ip(x-x')/\hbar}\right)|x\rangle\langle x'| \\ &= \int dx \int dx' i\hbar\partial'_x\delta(x-x')|x\rangle\langle x'| = \int dx|x\rangle(-i\hbar\partial_x)\langle x|\,.\end{aligned} \tag{1.12}$$

Importantly, the operators $\hat{x}$ and $\hat{p}$ do not commute but fulfill the canonical commutation relation

$$[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar\,. \tag{1.13}$$

This fact follows from Eqs. (1.10) and (1.12) can be seen as follows

$$[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = \int dx|x\rangle\Big[x(-i\hbar\partial_x) - (-i\hbar\partial_x)x\Big]\langle x| = i\hbar\int dx\,|x\rangle\langle x| = i\hbar\,1\,.$$

The dynamics is generated by the Hamiltonian $\hat{H}$ which generically can be written as a function of position and momentum operators $\hat{H} = \hat{H}(t) \equiv \hat{H}(\hat{x},\hat{p};t)$. As the Schrödinger equation is a linear equation, it is useful to define the fundamental solution $\hat{U}(t,t_0)$ which is given as the solution of the initial value problems (t_0 is arbitrary)

$$i\hbar\partial_t\hat{U}(t,t_0) = \hat{H}(t)\hat{U}(t,t_0), \qquad \hat{U}(t_0,t_0) = 1\,. \tag{1.14}$$

With this, the solution of the Schrödinger equation $i\hbar\partial_t|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$ with the initial condition $|\psi(t_0)\rangle = |\psi_0\rangle$ is given by

$$|\psi(t)\rangle = \hat{U}(t,t_0)|\psi_0\rangle\,; \tag{1.15}$$

which can be checked by simply plugging into the Schrödinger equation. Importantly, we have $\hat{U}(t,t_0) = \hat{U}(t,t')\hat{U}(t',t_0)$ for an arbitrary $t' \in [t_0,t]$ which simplify is the mathematical statement that the time-evolution from t_0 to t is given by the time-evolution from t_0 to t' followed by the one from t' to t. Note the ordering of the operators $\hat{U}$ though!

The fundamental solution can be formally written as

$$\begin{aligned}
\hat{U}(t,t_0) &= \mathcal{T}\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t')\,dt'\right)\\
&= \mathcal{T}\Bigg[1 + \left(-\frac{i}{\hbar}\right)\int_{t_0}^{t}dt_1\,\hat{H}(t_1) + \frac{1}{2}\left(-\frac{i}{\hbar}\right)^2\int_{t_0}^{t}dt_1\,\hat{H}(t_1)\int_{t_0}^{t}dt_2\,\hat{H}(t_2)\\
&\qquad + \frac{1}{6}\left(-\frac{i}{\hbar}\right)^2\int_{t_0}^{t}dt_1\,\hat{H}(t_1)\int_{t_0}^{t}dt_2\,\hat{H}(t_2)\int_{t_0}^{t}dt_3\,\hat{H}(t_3) + \cdots\Bigg]\\
&= 1 - \frac{i}{\hbar}\int_{t_0}^{t}dt_1\,\hat{H}(t_1) + \left(-\frac{i}{\hbar}\right)^2\int_{t_0}^{t}dt_1\,\hat{H}(t_1)\int_{t_0}^{t_1}dt_2\,\hat{H}(t_2)\\
&\qquad + \left(-\frac{i}{\hbar}\right)^3\int_{t_0}^{t}dt_1\,\hat{H}(t_1)\int_{t_0}^{t_1}dt_2\,\hat{H}(t_2)\int_{t_0}^{t_2}dt_3\,\hat{H}(t_3) + \cdots\,,
\end{aligned} \tag{1.16}$$

where $\mathcal{T}$ denotes the time-ordering operator which is defined as

$$\mathcal{T}\left(\hat{O}(t_1)\hat{O}(t_2)\right) = \hat{O}(t_1)\hat{O}(t_2)\theta(t_1-t_2) + \hat{O}(t_2)\hat{O}(t_1)\theta(t_2-t_1) \tag{1.17}$$

where $\theta(\cdot)$ denotes Heaviside's step function. Note that $U(t_0,t_0) = \mathcal{T}\exp(0) = 1$ and[1]

$$-i\hbar\partial_t\hat{U}(t,t_0) = \mathcal{T}\hat{H}(t)\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t')\,dt'\right) = \hat{H}(t)\hat{U}(t,t_0) \tag{1.18}$$

[1] Note that the time-ordering is crucial to move $\hat{H}(t)$ in front of all the other operators in $\exp(\cdot)$.

as required.

In the position basis, the fundamental solution is given by the function

$$F(x_b, x_a; t_b, t_a) = \langle x_b | \hat{U}(t_b, t_a) | x_a \rangle \tag{1.19}$$

which is the central object evaluated in the path-integral description, see below. In terms of F, the solution of the Schrödinger equation (1.1) is given as

$$\begin{aligned}\psi(x,t) = \langle x|\psi(t)\rangle = \langle x|\hat{U}(t,t_0)|\psi_0\rangle &= \int dx' \, \langle x|\hat{U}(t,t_0)|x'\rangle\langle x'|\psi_0\rangle \\ &= \int dx' \, F(x,x';t,t_0)\psi_0(x')\,. \end{aligned}\tag{1.20}$$

In the following, we will need the notion of *normal-ordering*. We define the normal-ordered Hamiltonian as the function

$$H(p,x;t) = \frac{\langle p|\hat{H}(\hat{x},\hat{p};t)|x\rangle}{\langle p|x\rangle} \tag{1.21}$$

If in the Hamilton operator $\hat{H}$ all the momentum are to the left of the position operators, we simply have $H(p,x;t) = \hat{H}(p,x;t)$. In general, the two expressions for H and $\hat{H}$ are different though. For example, let us consider the two Hamiltonians

$$\hat{H}_1(\hat{p},\hat{x};t) = \hat{p}\hat{x} \quad \text{and} \quad \hat{H}_2(\hat{p},\hat{x};t) = \hat{x}\hat{p}$$

We see that $\hat{H}_1$ is in the normal-ordered form while $\hat{H}_2$ is not. Using Eq. (1.13), we can write $\hat{H}_2 = \hat{H}_1 + i\hbar$. So we find the normal-ordered expressions

$$H_1(p,x;t) = xp \quad \text{and} \quad H_2(p,x;t) = xp + i\hbar\,. \tag{1.22}$$

1.2 Path integral

Now we will introduce the concept of path integral.

Usually, in quantum mechanics, we are interested in finding the probability amplitude that a particle will go from point x_a at time t_a to point x_b at time x_b. This is expressed by the probability amplitude $F(x_b, x_a; t_b, t_a)$. In general, it is very difficult to obtain the full evolution $\hat{U}(t_b, t_a)$ from t_a to t_b. The problem can be simplified by splitting the evolution up into small time-steps of size ϵ with the time-steps given by $t_j = t_a + \epsilon j$. In particular, we are interested in small ϵ and let $\epsilon \to 0$ in the end. This is akin to introducing the Riemann integral as a limit of sums.

For small ϵ, the Hamiltonian $\hat{H}(t)$ is essentially constant in the interval $[t_j, t_{j+1}]$ such that

$$\begin{aligned}\hat{U}_j = \hat{U}(t_{j+1}, t_j) = \mathcal{T}\exp\left(-\frac{i}{\hbar}\int_{t_j}^{t_{j+1}} dt\, \hat{H}(t)\right) &\approx \mathcal{T}\exp\left(-\frac{i}{\hbar}\int_{t_j}^{t_{j+1}} dt\;\; \hat{H}(t_j)\right) \\ = e^{-i\epsilon\hat{H}(t_j)/\hbar} \end{aligned}\tag{1.23}$$

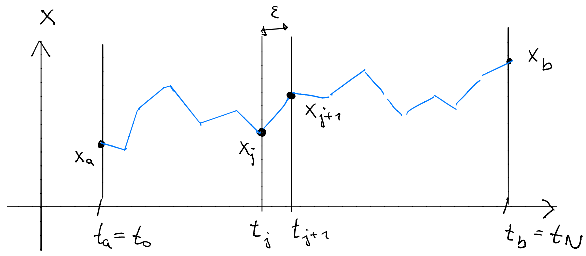

Figure 1.1: The 'path integral' is obtained by slicing the time-evolution at times $t_j = t_a + \epsilon j$ separated by ϵ. At each time step, the particle has the position $x_j = x(t_j)$. At intermediate times with $j \neq 0, N$ any position is allowed which leads to the integral over all possible path with the endpoints $x_0 = x_a$, $x_N = x_b$ fixed.

up to terms of order ϵ^2. In a similar spirit, the complete time evolution, after subdivision (note the ordering of the operators $\hat{U}$ from earlier to later times)

$$\hat{U}(t_b, t_a) = \prod_{j=0}^{N-1} \hat{U}_j = \hat{U}(t_N, t_{N-1}) \cdots \hat{U}(t_1, t_0) \tag{1.24}$$

can be simplified to

$$\hat{U}(t_b, t_a) \approx \prod_{j=0}^{N-1} e^{-i\hat{H}(t_j)\epsilon/\hbar} = e^{-i\hat{H}(t_{N-1})\epsilon/\hbar} \cdots e^{-i\hat{H}(t_0)\epsilon/\hbar} \tag{1.25}$$

As a next step, we would like to introduce a complete basis after each time-step. In particular, we need the matrix element

$$\langle p_j|\hat{U}_j|x_j\rangle \equiv \langle p_j|e^{-i\hat{H}(t_j)\epsilon/\hbar}|x_j\rangle \approx \langle p_j|1 - i\hat{H}(t_j)\epsilon/\hbar|x_j\rangle = \langle p_j|x_j\rangle e^{-iH(p_j,x_j;t_j)\epsilon/\hbar} \tag{1.26}$$

which is valid to fist order in ϵ and relates the propagator $\hat{U}_j$ to the normal-ordered Hamiltonian $H(p_j, x_j; t_j)$ of Eq. (1.21).

With these preliminary results, it is possible to calculate the propagator $F(x_b, x_a; t_b, t_a)$ by introducing a completeness in momentum before and a completeness of the position basis after each factor in Eq. (1.25). We obtain (with $x_0 = x_a, x_N = x_b$)

$$F(x_b, x_a; t_b, t_a) = \int \left(\frac{dp_0}{2\pi\hbar}\right) \prod_{j=1}^{N-1} \iint \left(\frac{dx_j\, dp_j}{2\pi\hbar}\right) \langle x_b|p_{N-1}\rangle\langle p_{N-1}|\hat{U}_{N-1}|x_{N-1}\rangle$$
$$\cdots \langle p_1|\hat{U}_1|x_1\rangle\langle x_1|p_0\rangle\langle p_0|\hat{U}_0|x_a\rangle \tag{1.27}$$

$$\approx \int \left(\frac{dp_0}{2\pi\hbar}\right)\left(\prod_{j=1}^{N-1}\frac{dx_j\,dp_j}{2\pi\hbar}\right)\prod_{j=0}^{N-1}\langle x_{j+1}|p_j\rangle\langle p_j|x_j\rangle e^{-iH(p_j,x_j;t_j)\epsilon/\hbar}$$
$$= \int \left(\frac{dp_0}{2\pi\hbar}\right)\left(\prod_{j=1}^{N-1}\frac{dx_j\,dp_j}{2\pi\hbar}\right) e^{\frac{i}{\hbar}\left[\sum_{j=0}^{N-1}(x_{j+1}-x_j)p_j-H(p_j,x_j;t_j)\epsilon\right]}\,.$$

This expression becomes exact in the limit $\epsilon \to 0$ which corresponds to $N \to \infty$. In this limit, we have infinitely many time-slices. The position of the particle is then described by a function $x(t)$ which interpolates between the values $x(t_j) = x_j$ and similarly for $p(t)$. The integrals in (1.27) can then be interpreted as a path-integral, *i.e.*, an integral over all functions $x(t), p(t)$. To this end, we introduce the notation (interpreted under the limit $\epsilon \to 0$)

$$\mathcal{D}[x(t)]\mathcal{D}[p(t)] = \left(\frac{dp_0}{2\pi\hbar}\right)\left(\prod_{j=1}^{N-1}\frac{dx_j\,dp_j}{2\pi\hbar}\right) \tag{1.28}$$

for the measure and

$$S[p(t),x(t)] = \sum_{j=0}^{N-1}(x_{j+1}-x_j)p_j - H(p_j,x_j,t_j)\epsilon \approx \int_{t_a}^{t_b} dt\ [\dot{x}p - H(p,x;t)]\ . \tag{1.29}$$

for the exponent. With this, the propagator of the quantum particle can be written as the path-integral

$$F(x_b,x_a;t_b,t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}[x(t)]\mathcal{D}[p(t)]e^{iS[p(t),x(t)]/\hbar} \tag{1.30}$$

with $S = \int dt\mathcal{L}$ the classical action that corresponds to each path $x(t), p(t)$ over the time $[t_a, t_b]$. Note that $\mathcal{L} = \dot{x}p - H$ is the Lagrangian of the problem.

1.3 Intermezzo: Gaussian integral

In the following, it is very important to be able to perform Gaussian integrals. In general, given a symmetric matrix $A \in \mathbb{R}^{n\times n}$, $A = A^T$, we can find an orthogonal transformation O, $OO^T = I_n$, with $|\operatorname{Det} O| = 1$, such that $A = O^T\Lambda O$, with the diagonal matrix of eigenvalues $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$.

In the case that A is positive definite (all $\lambda_j > 0$), we define the Gaussian integral over $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as

$$\int dx_1 \cdots x_n\, e^{-\boldsymbol{x}^T A\boldsymbol{x}/2} = \int d^n x\, e^{-\boldsymbol{x}^T A\boldsymbol{x}/2} = \frac{(2\pi)^{n/2}}{\sqrt{\operatorname{Det} A}} \tag{1.31}$$

This relation can be shown by a change of variables from $\boldsymbol{x}$ to $\boldsymbol{y} = O\boldsymbol{x}$ which yields

$$\int d^n x\, e^{-\boldsymbol{x}^T A \boldsymbol{x}/2} = \frac{1}{|\operatorname{Det} O|} \int d^n y\, e^{-\boldsymbol{y}^T \Lambda \boldsymbol{y}/2} = \prod_{j=1}^{n} \int dy_j e^{-\lambda_j y_j^2/2}$$
$$= \prod_{j=1}^{n} \left(\frac{2\pi}{\lambda_j}\right)^{1/2} = \frac{(2\pi)^{n/2}}{\sqrt{\operatorname{Det} A}} \tag{1.32}$$

where we have used that $\operatorname{Det} A = \lambda_1 \cdots \lambda_n$.

In the exercises, we will show Wick's theorem which is based on the important concept of the generating function

$$Z(\boldsymbol{\lambda}) = \int d^n x\, e^{-\boldsymbol{x}^T A \boldsymbol{x}/2 + \boldsymbol{\lambda}\cdot\boldsymbol{x}}, \qquad \lambda \in \mathbb{R}^n \tag{1.33}$$

The generating function can be evaluated by completing the square

$$\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x} - \boldsymbol{\lambda}\cdot\boldsymbol{x} = \frac{1}{2}(\boldsymbol{x} - A^{-1}\boldsymbol{\lambda})^T A (\boldsymbol{x} - A^{-1}\boldsymbol{\lambda}) + \frac{1}{2}\boldsymbol{\lambda}^T A^{-1}\boldsymbol{\lambda}\,. \tag{1.34}$$

With this and changing the variable to $\boldsymbol{y} = \boldsymbol{x} - A^{-1}\boldsymbol{\lambda}$, we can evaluate the generating function as

$$Z(\boldsymbol{\lambda}) = e^{\boldsymbol{\lambda}^T A^{-1} \boldsymbol{\lambda}/2} \int d^n y\, e^{-\boldsymbol{y} A \boldsymbol{y}/2} = \frac{(2\pi)^{n/2}}{\sqrt{\operatorname{Det} A}} e^{\boldsymbol{\lambda}^T A^{-1} \boldsymbol{\lambda}/2}\,. \tag{1.35}$$

1.4 Harmonic oscillator

In this section, we discuss the 'simple' problem of a harmonic oscillator with frequency ω_0 more explicitly. The (normal-ordered) Hamiltonian is given by

$$H(p, x) = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2\,. \tag{1.36}$$

In the following, we introduce the dimensionless variables $q = \sqrt{m\omega_0/\hbar}\, x$ and $\tilde{p} = p/\sqrt{\hbar m \omega_0}$ in terms of which the Hamiltonian assumes the form

$$H = \frac{\hbar\omega_0}{2}(\tilde{p}^2 + q^2)\,. \tag{1.37}$$

In the following, we will set $\hbar = 1$ and write p instead of $\tilde{p}$ for simplicity.

As an example of the path integral formalism, we would like to evaluate $F(q_b, q_a; \tau = t_b - t_a)$. Note that because the system is autonomous (the Hamiltonian does not depend on time) the propagator only depends on the time difference τ. The Hamiltonian action corresponding to the path $q(t), p(t)$ is given by

$$S[p, q] = \int_0^\tau \left[p\dot{q} - \frac{\omega_0}{2}(p^2 + q^2)\right] dt\,. \tag{1.38}$$

The classical path $q_{\rm cl}, p_{\rm cl}$ corresponds to the saddle point $\delta S[p_{\rm cl}, q_{\rm cl}] = 0$. In the exercises, we will show that they correspond to solutions of Hamilton's equations

$$\begin{aligned} \dot{q}_{\rm cl} &= \omega_0 p_{\rm cl} && \left(= \frac{\partial H}{\partial p}\right), \\ \dot{p}_{\rm cl} &= -\omega_0 q_{\rm cl} && \left(= -\frac{\partial H}{\partial q}\right). \end{aligned} \tag{1.39}$$

The solution corresponding to the boundary conditions $q_{\rm cl}(0) = q_a, q_{\rm cl}(\tau) = q_b$ is given by

$$q_{\rm cl} = q_a \cos(\omega_0 t) + \frac{q_b - q_a \cos(\omega_0 \tau)}{\sin(\omega_0 \tau)} \sin(\omega_0 t), \qquad p_{\rm cl} = \frac{\dot{q}_{\rm cl}}{\omega_0}. \tag{1.40}$$

This solution corresponds to the energy

$$E = H = \frac{\omega_0 \left[q_a^2 + q_b^2 - 2 q_a q_b \cos(\omega_0 \tau)\right]}{2 \sin^2(\omega_0 \tau)}, \tag{1.41}$$

which is a conserved quantity.

The action along the classical path is defined as

$$\begin{aligned} S_{\rm cl} = S[p_{\rm cl}, q_{\rm cl}] &= \int_0^\tau (p_{\rm cl}\dot{q}_{\rm cl} - H)\, dt = \frac{\omega_0}{2} \int_0^\tau p_{\rm cl}^2\, dt - E\tau \\ &= \frac{1}{2\sin(\omega_0\tau)} \left[(q_a^2 + q_b^2)\cos(\omega_0\tau) - 2q_a q_b\right]. \end{aligned} \tag{1.42}$$

Because $\delta S_{\rm cl} = 0$, we find writing $q = q_{\rm cl} + s$, $p = p_{\rm cl} + r$

$$S[p, q] = S_{\rm cl} + S[s, r]. \tag{1.43}$$

The propagator is thus given by the path integral expression

$$F(q_b, q_a; \tau) = e^{iS_{\rm cl}} \int_{s(0)=0}^{s(\tau)=0} \mathcal{D}[r]\mathcal{D}[s] \exp\left[i \int_0^\tau dt \left(r\dot{s} - \frac{1}{2}(r^2 + s^2)\right)\right]. \tag{1.44}$$

Note in particular that the second factor is independent of q_a, q_b. It can be evaluated using the time-sliced expression from which we obtain $(\omega_0/2\pi i \sin\omega_0\tau)^{1/2}$, see exercises. In summary, the propagator for the harmonic oscillator in the position basis is given by

$$F(q_b, q_a; \tau) = e^{iS_{\rm cl}} \sqrt{\frac{\omega_0}{2\pi i \sin(\omega_0\tau)}}. \tag{1.45}$$

1.5 Expectation values

In this section we can answer the question, how can we evaluate quantity like

$$\langle \hat{A}(t)\hat{B}(t')\rangle$$

that is the expectation value of two operators acting at different times. Here, we consider the operators in the Heisenberg picture with

$$\hat{A}(t) = \hat{U}(t_0,t)\hat{A}\hat{U}(t,t_0) = \hat{U}(t,t_0)^\dagger \hat{A}\hat{U}(t,t_0) \tag{1.46}$$

with t_0 some (arbitrary) reference time; note that $\hat{U}(t_0,t_1)^\dagger = \hat{U}(t_1,t_0)$ for t_0, t_1 arbitrary. In particular, we are interested in matrix elements

$$\begin{aligned}\langle x_b, t_b|\hat{A}(t)\hat{B}(t')|x_a,t_a\rangle &= \langle x_b|\hat{U}(t_0,t_b)^\dagger \hat{A}(t)\hat{B}(t')\hat{U}(t_0,t_a)|x_a\rangle \\ &= \langle x_b|\hat{U}(t_b,t)\hat{A}\hat{U}(t,t')\hat{B}\hat{U}(t',t_a)|x_a\rangle\,,\end{aligned} \tag{1.47}$$

where $|x_a,t_a\rangle = U(t_0,t_a)|x_a\rangle$ corresponds to the particle at position x_a at time t_a.

The path integral approach, via the time-slices, corresponds very naturally to time-ordered correlations. An observable at time t_j is most naturally included in the overlaps $\langle x_j|p_{j-1}\rangle$ of Eq. (1.27). Because of this, and observable $\hat{A}$ is replaced by the anti-normal-ordered expression

$$A(x,p) = \frac{\langle x|\hat{A}|p\rangle}{\langle x|p\rangle}.$$

Splitting the evolution into small time-slices, similar to the last section, we can straightforwardly show that

$$\langle x_b,t_b|\mathcal{T}\hat{A}(t)\hat{B}(t')|x_a,t_a\rangle = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}[x]\mathcal{D}[p]\, A(x(t),p(t))B(x(t'),p(t'))\, e^{iS/\hbar}$$

Example

As an example, we will calculate the expectation value of q^2 in the case of the harmonic oscillator which starts and ends at $q = 0$. Assuming that $0 < t_1 < \tau$, we arrive at (using $F(q, 0; t)$ from (1.45))

$$\begin{aligned}G(t_1;\tau) &= \langle q=0,\tau|q^2(t_1)|q=0,t=0\rangle \\ &= \int_{q(0)=0}^{q(\tau)=0} \mathcal{D}[q]\,\mathcal{D}[p]\,q(t_1)^2 e^{iS[p,q]} = \\ &= \int dq_1\, q_1^2 \left(\int_{q(0)=0}^{q(t_1)=q_1} \mathcal{D}[q]\mathcal{D}[p] e^{i\int_0^{t_1}(p\dot{q}-H(p,q))dt}\right) \\ &\qquad \times \left(\int_{q(t_1)=q_1}^{q(t)=0} \mathcal{D}[q]\mathcal{D}[p] e^{i\int_{t_1}^{\tau}(p\dot{q}-H(p,q))dt}\right)\end{aligned}$$

$$
\begin{aligned}
&= \omega_0 \int dq_1\, q_1^2 \, \frac{\exp\left(\frac{i}{2}\cot(\omega_0 t_1) q_1^2\right)}{\sqrt{2\pi i \sin(\omega_0 t_1)}} \frac{\exp\left(\frac{i}{2}\cot[\omega_0(\tau - t_1)] q_1^2\right)}{\sqrt{2\pi i \sin[\omega_0(\tau - t_1)]}} \\
&= -\frac{\omega_0}{2\pi \sin(\omega_0 t_1)\sin[\omega_0(\tau - t_1)]} \int dq_1\, q_1^2 \exp\left[\frac{i}{2}\Big(\cot(\omega_0 t_1) + \cot[\omega_0(\tau - t_1)]\Big) q_1^2\right] = \\
&= -\frac{\omega_0}{2\pi \sin(\omega_0 t_1)\sin[\omega_0(\tau - t_1)]} \frac{\sqrt{2\pi}}{\Big(i\cot(\omega_0 t_1) + i\cot[\omega_0(\tau - t_1)]\Big)^{3/2}} ;
\end{aligned}
$$

here, we have introduced $q_1 \equiv q(t_1)$ over which we have integrated last. The final result of our calculation is

$$
G(t_1, \tau) = \omega_0 \left(\frac{\sin(\omega_0 t)\sin[\omega_0(\tau - t_1)]}{2\pi i \sin^3(\omega_0 \tau)} \right)^{1/2} ; \tag{1.48}
$$

We can check the results in the limits $t_1 \to 0$ and $t_1 \to \tau$. In both cases, we obtain $G = 0$ which makes sense as $\hat{q}|q = 0\rangle = 0$.

1.6 Canonical commutation relations

We know that in quantum mechanics, the position and momentum operators satisfy the canonical commutation relation

$$
[\hat{x}, \hat{p}] = i \,.
$$

In the path integral description, the commutator seems absent. In fact, all the observables are replaced by the anti-normal-ordered expressions which are simply functions. Moreover, the ordering of the observables is given by the time-ordering.

In this section, we discuss where the commutation relations such as the canonical commutation relation between position and momentum is 'hidden' in the path-integral description. To this end, we introduce the time-ordered correlator

$$
\hat{C}(t, t') \equiv \mathcal{T}\hat{x}(t)\hat{p}(t') = \hat{x}(t)\hat{p}(t')\theta(t - t') + \hat{p}(t')\hat{x}(t)\theta(t' - t) \tag{1.49}
$$

where, for the second equation we used Eq. (1.17).

If we derive the equation of motion which $C(t, t')$ fulfills, we find

$$
\begin{aligned}
\frac{\partial \hat{C}(t, t')}{\partial t} &= \dot{\hat{x}}(t)\hat{p}(t')\theta(t - t') + \hat{p}(t')\dot{\hat{x}}(t)\theta(t' - t) \\
&\quad + \hat{x}(t)\hat{p}(t')\delta(t - t') - \hat{p}(t')\hat{x}(t)\delta(t' - t) \,.
\end{aligned} \tag{1.50}
$$

The terms in the last line are called *contact-terms* as they are local in time and arise due to the time-ordering procedure. Using Heisenberg's equation $\dot{\hat{x}}(t) = -i[\hat{x}, \hat{H}](t)$

and the canonical-commutation relation, we obtain the *Schwinger-Dyson equation*

$$\begin{aligned}\partial_t \mathcal{T}\hat{x}(t)\hat{p}(t') = \frac{\partial \hat{C}(t,t')}{\partial t} &= \mathcal{T}\dot{\hat{x}}(t)\hat{p}(t') + \delta(t-t')\left(\hat{x}(t)\hat{p}(t) - \hat{p}(t)\hat{x}(t)\right) \\ &= -i\mathcal{T}[\hat{x},\hat{H}](t)\hat{p}(t') + i\delta(t-t')\,; \end{aligned} \tag{1.51}$$

note that the contact-term encodes the canonical commutation relation. Note that the Schwinger-Dyson equation states that the time-ordered correlator of $\hat{x}$ with the conjugate variable $\hat{p}$ does not simply fulfill the 'classical' equation of motion $\dot{\hat{x}} = -i[\hat{H},\hat{x}]$ as additional contact-terms encoding the commutation relation appear. These contact terms are the difference between the classical and the quantum-field theory.

It is thus crucial to understand at which point the commutation relation is encoded in the path integral description. This is rather subtle but important in order to produce reliable results. To this end, we derive the Schwinger-Dyson equation from the path-integral description. It arises from the statement, that the path-integral is unchanged under the substitution $p(t) \mapsto p(t) + \lambda(t)$. In particular, the expression

$$\int \mathcal{D}[x]\mathcal{D}[p]\,(p(t') + \lambda(t'))e^{iS[p+\lambda,x]} = \int \mathcal{D}[x]\mathcal{D}[p]\,(p(t') + \lambda(t'))e^{i\int dt[(p+\lambda)\dot{x} - H(p+\lambda,x)]} \tag{1.52}$$

does not depend on $\lambda(t)$; here, we have used that $\mathcal{D}[p+\lambda] = \mathcal{D}[p]$. This implies that the change of the expression to first order in $\lambda(t)$, *i.e.*, the variation,

$$0 = \int \mathcal{D}[x]\mathcal{D}[p]\,\int dt\left[\delta(t-t') + ip(t')\dot{x}(t) - ip(t')\frac{\partial H}{\partial p}(t)\right]\lambda(t)e^{iS[p,q]} + \mathcal{O}(\lambda^2) \tag{1.53}$$

has to vanish.

As $\lambda(t)$ is arbitrary, the expression in the rectangular bracket is zero. This yields

$$\partial_t x(t)p(t') = \frac{\partial H}{\partial p}(t)p(t') + i\delta(t-t')\,. \tag{1.54}$$

In order to compare this result to Eq. (1.51), we have to remind ourselves that the time-ordering is implicit in the path integral description and thus the left-hand side is the same. For the right hand side, we write $\hat{x} \equiv i\partial_p$ in the momentum representation. Thus, $-i[\hat{x},\hat{H}] = \partial_p \hat{H}$ and Eq. (1.54) reproduces (1.51).[2]

We give a second derivation how the path-integral description encodes the nontrivial commutator using the discrete time-sliced description. This derivation might be initially easier to understand.

[2]More accurately, we should calculate the normal-ordered representation of $[\hat{x},\hat{H}]$. We have $H(p,x) = \langle p|\hat{H}|x\rangle/\langle p|x\rangle$ by definition. With $\hat{x}|x\rangle = x|x\rangle$, $\hat{x}|p\rangle = i\partial_p|p\rangle$ and $\langle p|x\rangle = e^{-ipx}$, we find $\langle p|x\rangle i\partial_p H(p,x) = i\partial_p\langle p|\hat{H}|x\rangle - \langle p|\hat{H}|x\rangle(i\partial_p\langle p|x\rangle/\langle p|x\rangle) = \langle p|\hat{x}\hat{H}|x\rangle - \langle p|\hat{H}|x\rangle x = \langle p|[\hat{x},\hat{H}]|x\rangle$; thus, $i\partial_p H(p,x)$ is the normal-ordered representation of $[\hat{x},\hat{H}]$.

In the discrete version (1.27), we calculate the expectation value of p_n, where n is an arbitrary time-slice; p_n corresponds to $p(t')$ in the proof above. We have

$$\langle p_n \rangle = \int \left(\frac{dp_0}{2\pi\hbar} \right) \left(\prod_{j=1}^{N-1} \frac{dx_j\, dp_j}{2\pi\hbar} \right) p_n \, e^{i \sum_{j=0}^{N-1} [(x_{j+1}-x_j)p_j - H(p_j,x_j;t_j)\epsilon]} \tag{1.55}$$

This expression remains invariant under the shift $p_n \mapsto p_n + \lambda$ which simply corresponds to a variable substitution of the integral dp_n. Writing down only the contribution that depends on λ, this implies

$$0 = \frac{\partial}{\partial \lambda} \int dp_n \, (p_n + \lambda) e^{i[(x_{n+1}-x_n)(p_n+\lambda) - H(p_n+\lambda, x_n; t_n)\epsilon]} \times \begin{pmatrix} \text{stuff} \\ \text{which is} \\ \text{indep. of} \\ \lambda \end{pmatrix} \tag{1.56}$$

Taking the derivative, we have

$$0 = \int dp_n \left[1 + i(x_{n+1} - x_n)p_n - i\epsilon p_n \frac{\partial H(p_n + \lambda, x_n; t_n)}{\partial p_n} \right] \times \begin{pmatrix} \text{non-} \\ \text{vanishing} \\ \text{stuff} \end{pmatrix} \tag{1.57}$$

Note that the term in the rectangular bracket has to vanish as before as λ is arbitrary. As $\epsilon \to 0$, the first two terms remain which implies

$$x_{n+1}p_n - p_n x_n = i \, . \tag{1.58}$$

This corresponds to the commutator $\hat{x}\hat{p} - \hat{p}\hat{x} = [\hat{x}, \hat{p}] = i$ due to the fact that the time-ordering is implicit and for equal times the expressions are normal-ordered and thus $\hat{p}$ is to the left of $\hat{x}$.

1.7 Coherent states

The problem of the harmonic oscillator is essential in the field of quantum optics. This is because, as we have learned in optics and electromagnetic courses, light is an electromagnetic wave that oscillates. When light passes through the material, the electric field affects the electrons by a force $e\mathcal{E}$ where $\mathcal{E}$ is the electric field. The oscillation of each mode of the electromagnetic field corresponds then to a harmonic oscillator.

In the previous section, we used rescaled the variables of the harmonic oscillator such that the Hamiltonian could be written in the form

$$\hat{H} = \frac{\omega_0}{2}(\hat{p}^2 + \hat{q}^2);$$

where the operators $\hat{q}$, $\hat{p}$ fulfilled the canonical commutation relation $[\hat{q}, \hat{p}] = i$. The path-integral was then derived using the position and momentum basis together with a normal-ordering procedure that placed the momentum left of the position operators.

For quantum optics applications, the position and momentum correspond to different quadratures of the light-field. In this case, it is often most convenient to treat $\hat{q}$, and $\hat{p}$ on equal footings. To this end, we introduce the creation operator

$$\hat{a}^\dagger = \frac{\hat{q} - i\hat{p}}{\sqrt{2}} \tag{1.59}$$

and the associated annihilation operator

$$\hat{a} = \frac{\hat{q} + i\hat{p}}{\sqrt{2}} \,. \tag{1.60}$$

It can be checked that they fulfill the algebra $[\hat{a}, \hat{a}^\dagger] = 1$ of ladder operators. Moreover, the Hamiltonian assumes the form

$$H = \omega_0(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}) = \omega_0(\hat{n} + \tfrac{1}{2}) \tag{1.61}$$

where we have introduced the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$. The number operator is Hermitian and has the eigenvalues $0, 1, 2, \ldots$ which we will call the number of photons in the oscillator. The corresponding eigenvectors $|n\rangle$ are called the Fock basis. From the quantum mechanics course, we know that

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad \hat{n}|n\rangle = n|n\rangle. \tag{1.62}$$

In particular, $|0\rangle$ is the vacuum state with $\hat{a}|0\rangle = 0$. From Eq. (1.60) we see that the annihilation operator $\hat{a}$ combines both the position $\hat{q}$ as well as the momentum $\hat{p}$ in a 'complex' quantity; in a sense, we have '$q = \sqrt{2}\,\mathrm{Re}\, a$' and '$p = \sqrt{2}\,\mathrm{Im}\, a$'.

It is thus useful to find eigenstates of the operator $\hat{a}$ which will serve as simultaneous eigenstates of $\hat{q}$ and $\hat{p}$, of course without violating Heisenberg's uncertainty principle, treating them on equal footing. We call the eigenstates $|\alpha\rangle$ of the annihilation operator $\hat{a}$ with

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \qquad \alpha \in \mathbb{C} \tag{1.63}$$

a *coherent state*. For concreteness, we define

$$|\alpha\rangle = e^{\alpha a^\dagger}|0\rangle = \sum_{n=0}^{\infty} \frac{(\alpha\hat{a}^\dagger)^n}{n!}|0\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \,. \tag{1.64}$$

This implies that $\langle n|\alpha = \alpha^n/\sqrt{n!}$. We can directly show that $|\alpha\rangle$ is an eigenstate of $\hat{a}$. In particular, we have (with $m = n-1$)

$$\hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\hat{a}|n\rangle = \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{(n-1)!}}|n-1\rangle = \alpha\sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}}|m\rangle = \alpha|\alpha\rangle \,. \tag{1.65}$$

Note that the coherent states that we have defined are not normalized. This is different from the definition of coherent states in other sources. In particular, we find the overlap

$$\langle\alpha_1|\alpha_2\rangle = \langle 0|e^{\bar{\alpha}_1 a}e^{\alpha_2 a^\dagger}|0\rangle = e^{\bar{\alpha}_1\alpha_2[\hat{a},\hat{a}^\dagger]}\langle 0|e^{\alpha_2\hat{a}^\dagger}e^{\alpha_1\hat{a}}|0\rangle = e^{\bar{\alpha}_2\alpha_1} \,, \tag{1.66}$$

where we have used the Baker–Campbell–Hausdorff relation. Because of this, the normalized coherent states are given by

$$|\alpha) = e^{-|\alpha|^2/2}|\alpha\rangle\,. \tag{1.67}$$

However, in the following, it will be more convenient to work with the unnormalized states $|\alpha\rangle$.

The coherent states are an (over-)complete basis set. The completeness relation is given by

$$\int \frac{d^2\alpha}{\pi}\,|\alpha)(\alpha| = \int \frac{d^2\alpha}{\pi}\,e^{-|\alpha|^2}|\alpha\rangle\langle\alpha| = 1\,; \tag{1.68}$$

here and in the following, we use the notation $d^2\alpha = d\,\mathrm{Re}\,\alpha\,d\,\mathrm{Im}\,\alpha$ for the measure on the complex plane of coherent states. To show that (1.68) is indeed true, we can evaluate the matrix elements in the (complete) Fock basis. We find (with $\alpha = |\alpha|e^{i\theta}$)

$$\begin{aligned}\int\frac{d^2\alpha}{\pi}e^{-|\alpha|^2}\langle n|\alpha\rangle\langle\alpha|m\rangle &= \int\frac{d^2\alpha}{\pi}e^{-|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}\frac{\bar{\alpha}^m}{\sqrt{m!}} = \int d|\alpha|^2\frac{|\alpha|^{n+m}e^{-|\alpha|^2}}{\sqrt{n!\,m!}}\int_0^{2\pi}\frac{d\theta}{2\pi}\,e^{i\theta(n-m)}\\ &= \frac{\delta_{n,m}}{n!}\int d|\alpha|^2 e^{-|\alpha|^2}|\alpha|^{2n} = \delta_{n,m} = \langle n|1|m\rangle\,.\end{aligned} \tag{1.69}$$

From the completeness relation (1.68), we can also find a convenient expression to calculate the trace of an operator $\hat{A}$. It is given by

$$\begin{aligned}\mathrm{Tr}(\hat{A}) &= \sum_n\langle n|\hat{A}|n\rangle = \int\frac{d^2\alpha}{\pi}\sum_n\langle n|\hat{A}|\alpha)(\alpha|n\rangle = \int\frac{d^2\alpha}{\pi}\sum_n(\alpha|n\rangle\langle n|\hat{A}|\alpha)\\ &= \int\frac{d^2\alpha}{\pi}(\alpha|\hat{A}|\alpha) = \int\frac{d^2\alpha}{\pi}e^{-|\alpha|^2}\langle\alpha|\hat{A}|\alpha\rangle\,.\end{aligned} \tag{1.70}$$

As we treat $\hat{q}$ and $\hat{p}$ on equal footing, we cannot use the former definition of normal-ordering. For coherent states, we thus introduce a different definition of normal-ordering. In particular, the normal-ordered Hamiltonian, from now on, is given by

$$H(\bar{\alpha}_1,\alpha_2;t) = \frac{\langle\alpha_1|\hat{H}(t)|\alpha_2\rangle}{\langle\alpha_1|\alpha_2\rangle} = \langle\alpha_1|\hat{H}(t)|\alpha_2\rangle e^{-\bar{\alpha}_1\alpha_2} \tag{1.71}$$

This way of normal-ordering corresponds to moving all creation operators $\hat{a}^\dagger$ to the left of the annihilation operators $\hat{a}$ and then substituting $\hat{a}\mapsto\alpha_2$ and $\hat{a}^\dagger\mapsto\bar{\alpha}_1$. Note that $H(\bar{\alpha}_1,\alpha_2)$ is only a function of $\bar{\alpha}_1$ and α_2 and not of $\alpha_1,\bar{\alpha}_2$. Because of this, we can get the normal-ordered Hamiltonian from the 'diagonal' element $H(\alpha;t) = H(\bar{\alpha},\alpha;t) = \langle\alpha|\hat{H}(t)|\alpha\rangle e^{-|\alpha|^2}$ by simply replacing $\bar{\alpha}\mapsto\bar{\alpha}_1$ and $\alpha\mapsto\alpha_2$. The normal-ordering with respect to $\hat{a},\hat{a}^\dagger$ is (slightly) different from the normal-ordering in $\hat{q},\hat{p}$, with the differences being of the order of $\hbar$ due to the commutator.

The harmonic oscillator has the normal-ordered Hamiltonian

$$H(\alpha;t) = \omega_0(|\alpha|^2+\tfrac{1}{2})\,. \tag{1.72}$$

Often, we forget about the zero-point motion and simply write $H = \omega_0|\alpha|^2$.

1.8 Coherent state path integral

In this section, we follow the same steps as in chapter 1. As previously, we will split the time into small time-slices. The difference is that now instead of position and momentum, we put the coherent states we studied in this chapter. Because of that the element that we should study now is

$$\langle \alpha_{j+1}|\hat{U}(t_{j+1},t_j)|\alpha_j\rangle \approx e^{-iH(\bar{\alpha}_{j+1},\alpha_j;t_j)\epsilon+\bar{\alpha}_{j+1}\alpha_j}\,. \tag{1.73}$$

The total time-evolution is then given by

$$\begin{aligned} F(\bar{\alpha}_b,\alpha_a;t_b,t_a) &= \langle \alpha_b|U(t_b,t_a)|\alpha_a\rangle \\ &= \int \prod_{j=1}^{N-1} \frac{d^2\alpha_j}{\pi} e^{\bar{\alpha}_b\alpha_N + i\epsilon\sum_{j=0}^{N-1}\left[i\bar{\alpha}_{j+1}\left(\frac{\alpha_{j+1}-\alpha_j}{\epsilon}\right)-H(\bar{\alpha}_{j+1},\alpha_j;t_j)\right]} \\ &= \int_{\alpha(t_a)=\alpha_a}^{\bar{\alpha}(t_b)=\bar{\alpha}_b} \mathcal{D}[\bar{\alpha}(t)]\mathcal{D}[\alpha(t)]e^{iS+\bar{\alpha}_b\alpha(t_b)}, \end{aligned} \tag{1.74}$$

where we defined

$$\mathcal{D}[\bar{\alpha}]\mathcal{D}[\alpha] \equiv \prod_{j=1}^{N-1}\frac{d^2\alpha_j}{\pi} \quad \text{and} \quad S = \int_{t_a}^{t_b} dt\,[i\bar{\alpha}(t)\dot{\alpha}(t) - H(\bar{\alpha}(t),\alpha(t);t)]\,. \tag{1.75}$$

Note that in all these expressions mapping from the operator language to the path integral normal- and time-ordering is implied.

Examples

As a first example, we treat our favorite system, the harmonic oscillator. We want to evaluate the propagator $F(\bar{\alpha}_b,\alpha_a;\tau) = \langle \alpha_b|\hat{U}(\tau)|\alpha_a\rangle$ for a harmonic oscillator with Hamiltonian $H = \omega_0|\alpha|^2$. Note that in the Hamiltonian we have neglected the zero point energy. We need to evaluate

$$F(\bar{\alpha}_b,\alpha_a;\tau) = \int_{\alpha(0)=\alpha_a}^{\bar{\alpha}(\tau)=\bar{\alpha}_b} \mathcal{D}[\bar{\alpha}]\mathcal{D}[\alpha]e^{iS+\bar{\alpha}_b\alpha(\tau)}, \text{ with } S = \int_0^\tau dt\,\left[i\bar{\alpha}(t)\dot{\alpha}(t) - \omega_0|\alpha(t)|^2\right]\,. \tag{1.76}$$

As before, we calculate the classical solution (the saddle-point) and split an arbitrary path as the sum over the classical path α_{cl} and a deviation δ. The classical equation of motion is given by $\delta S/\delta\bar{\alpha} = 0$ (and its complex conjugate) which leads to

$$i\dot{\alpha}_{\text{cl}} = \omega_0\alpha_{\text{cl}} \quad \text{and} \quad -i\dot{\bar{\alpha}}_{\text{cl}} = \omega_0\bar{\alpha}_{\text{cl}}\,.$$

The equations for α_{cl} and $\bar{\alpha}_{\text{cl}}$ are decoupled. The solution with the boundary conditions $\alpha_{\text{cl}}(0) = \alpha_a$ and $\bar{\alpha}_{\text{cl}}(\tau) = \bar{\alpha}_b$ is given by

$$\alpha_{\text{cl}}(t) = e^{-i\omega_0 t}\alpha_a, \qquad \bar{\alpha}_{\text{cl}}(t) = e^{i\omega_0(t-\tau)}\bar{\alpha}_b\,. \tag{1.77}$$

Note that the initial condition is propagated forward in time by the solution $\alpha_{\rm cl}(t)$ while the final condition is propagated backward in time by the solution $\bar\alpha_{\rm cl}(t)$. In general, $\bar\alpha_{\rm cl}$ is *not* the complex conjugate of $\alpha_{\rm cl}$. This is due to the fact that by setting $\alpha_a = (q_a + ip_a)/\sqrt{2}$ and $\bar\alpha_b = (q_b - ip_b)/\sqrt{2}$, we have overspecified the problem as we have specified position and momentum both at the initial and the final time. However, for consistent intial and final conditions (fulfilling $\alpha_{\rm cl}(\tau) = \alpha_b$), we do have that $\bar\alpha_{\rm cl}(t)$ is the complex conjugate of $\alpha_{\rm cl}(t)$. In terms of the operator description, the fact that $\alpha(t_b)$ is not the complex conjugate of $\bar\alpha_b$, corresponds to the final overlap $\langle\alpha_b|\alpha(t_b)\rangle = e^{\bar\alpha_b\alpha(t_b)}$ which for coherent state is not a delta distribution. In general, for the coherent state path integral, it is most useful to treat $\bar\alpha$ and α as two independent variables corresponding to the bra and the ket of an arbitrary matrix element. In the case that they are complex conjugates of each other, this then corresponds to the diagonal element, *i.e.*, the expectation value in the coherent state $|\alpha\rangle$.

For the harmonic oscillator, we calculate the path integral by parametrizing a general path as

$$\alpha(t) = \alpha_{\rm cl}(t) + \delta(t) \quad \text{and} \quad \bar\alpha(t) = \bar\alpha_{\rm cl}(t) + \bar\delta(t)\,. \tag{1.78}$$

The action is described by

$$S = S_{\rm cl} + \int_0^\tau dt\,\left(i\bar\delta\dot\delta - \omega_0\bar\delta\delta\right) \tag{1.79}$$

with the classical action

$$S_{\rm cl} = \int_0^\tau dt\,\left(\omega_0\bar\alpha_b\alpha_a e^{-i\omega_0\tau} - \omega_0\bar\alpha_b\alpha_a e^{-i\omega_0\tau}\right) = 0\,. \tag{1.80}$$

The propagator is then given by

$$F(\bar\alpha_b, \alpha_a; \tau) = e^{\bar\alpha_b\alpha_q e^{-i\omega_0\tau}} \times \int_{\delta(0)=0}^{\bar\delta(0)=0} \mathcal{D}[\bar\delta]\mathcal{D}[\delta] e^{i\int_0^\tau\left(i\bar\delta\dot\delta - \omega_0|\delta|^2\right)dt} \tag{1.81}$$

Note that the remaining path integral, the second factor, is independent of $\bar\alpha_b$ and α_a. Performing the path-integral, *e.g.* via the time-sliced description, it can be shown that

$$\int_{\delta(0)=0}^{\bar\delta(\tau)=0} \mathcal{D}[\bar\delta]\mathcal{D}[\delta] e^{i\int_0^\tau\left(i\bar\delta\dot\delta - \omega_0|\delta|^2\right)dt} = 1. \tag{1.82}$$

Thus, the propagator of the harmonic oscillator is given by

$$F(\bar\alpha_b, \alpha_a; \tau) = e^{\bar\alpha_b\alpha_a e^{-i\omega_0\tau}} \tag{1.83}$$

As a second example, we will calculate the expectation value

$$\mathcal{G}(t_1;\tau) = \langle 0, \tau|\hat q(t_1)^2|0, t=0\rangle, \quad \text{where } \tau > t_1 > 0\,. \tag{1.84}$$

Note that in this case $\alpha_a = \bar{\alpha}_b = 0$ denotes to the vacuum state $|0\rangle$. In this sense, $\mathcal{G}$ is different from $G(t_1;\tau)$ evaluated in chapter 1 where we evaluated $q(t_1)^2$ in the position eigenstate $|q=0\rangle$. For the path integral formulation, we need to anti-normal-order the observable

$$\hat{q}^2 = \tfrac{1}{2}(\hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{a}a^\dagger - 1)\,, \tag{1.85}$$

where we have used the definition $\hat{q} = (\hat{a}+\hat{a}^\dagger)/\sqrt{2}$. Thus, the anti-normal-ordered represenation of $\hat{q}^2$ is given by

$$q^2(\alpha,\bar{\alpha}) = \tfrac{1}{2}(\alpha^2 + \bar{\alpha}^2 + 2\alpha\bar{\alpha} - 1) = \tfrac{1}{2}[(\alpha+\bar{\alpha})^2 - 1]\,. \tag{1.86}$$

The path integral description of (1.84) is given by

$$\mathcal{G}(t_1;\tau) = \int_{\alpha(0)=0}^{\bar{\alpha}(\tau)=0} \mathcal{D}[\bar{\alpha}]\mathcal{D}[\alpha] e^{iS[p,q]}\,\frac{1}{2}\Big[(\alpha(t_1)+\bar{\alpha}(t_1))^2 - 1\Big]\,. \tag{1.87}$$

As before, we can restrict the paths by introducing

$$\alpha_1 = \alpha(t_1), \quad \bar{\alpha}_1 = \bar{\alpha}(t_1). \tag{1.88}$$

over which we integrate at the very end. With this, the path integral takes the form

$$\begin{aligned}\mathcal{G}(t_1,\tau) = \int \frac{d^2\alpha_1}{2\pi}[(\alpha_1+\bar{\alpha}_1)^2-1]e^{-\bar{\alpha}_1\alpha_1}&\left[\int_{\alpha(0)=0}^{\bar{\alpha}(t_1)=\bar{\alpha}_1}\mathcal{D}[\bar{\alpha}]\mathcal{D}[\alpha]e^{i\int_0^{t_1}dt(i\bar{\alpha}\dot{\alpha}-H)+\bar{\alpha}_1\alpha(t_1)}\right]\\ &\times\left[\int_{\alpha(t_1)=\alpha_1}^{\bar{\alpha}(\tau)=0}\mathcal{D}[\bar{\alpha}]\mathcal{D}[\alpha]e^{i\int_{t_1}^{\tau}dt(i\bar{\alpha}\dot{\alpha}-H)}\right];\end{aligned} \tag{1.89}$$

the factor $\exp(-\bar{\alpha}_1\alpha_1)$ is introduced such that the factor cancel the factor $\exp[\bar{\alpha}_1\alpha(t_1)]$ that is part of the definition of $F(\bar{\alpha}_1, 0; t_1)$. So, in the end, we need to calculate

$$\mathcal{G}(t_1,\tau) = \int \frac{d^2\alpha_1}{2\pi}[(\alpha_1+\bar{\alpha}_1)^2 - 1]e^{-|\alpha_1|^2} = \frac{1}{2}\,. \tag{1.90}$$

Comparing $\mathcal{G}$ to G from Eq. (1.48), we see that $\mathcal{G}$ does not depend on t_1 (nor τ). This is due to the fact that the vacuum state is a stationary state that does not evolve in time. Our result thus simply corresponds to

$$\langle 0|\hat{q}^2|0\rangle = \frac{1}{2}\,, \tag{1.91}$$

which is the zero-point motion of the vacuum state.

Chapter 2

Path integral for open systems

2.1 Density matrix and environment

For many quantum mechanical problems, it is not sufficient to consider an isolated system. Instead, we have to examine the dynamics of a system coupled to an environment. The environment generally consists of many degrees of freedom, resulting in a complicated, multi-dimensional problem. However, in most cases the only information of interest is the systems dynamics under the influence of the environment described by a reduced density matrix.

In this section, we consider an arbitrary one-dimensional system with Hamiltonian $\hat{H}_S(\hat{p}, \hat{x})$ coupled to an environment modeled by N harmonic oscillators with the creation and annihilation operators $\hat{a}_j^\dagger$ and $\hat{a}_j$. The dynamics of the total system can be described by the Hamiltonian ($\hbar = 1$ from now on)

$$\hat{H} = \hat{H}_S(\hat{p}, \hat{x}) + \sum_{j=1}^{N} \omega_j \hat{a}_j^\dagger \hat{a}_j, \tag{2.1}$$

$$\text{with the constraint} \quad \hat{x} = \sum_{j=1}^{N} \left(c_j \hat{a}_j + \overline{c_j} \hat{a}_j^\dagger \right). \tag{2.2}$$

Below, we use a Lagrange multiplier $\lambda(t)$ to encode the constraint, coupling the dynamics of the one-dimensional system with the environmental degrees of freedom. Here, the complex constants c_j correspond to the coupling strength of each oscillator.

The time-evolution of the density matrix of the total system is given by

$$\hat{\rho}(t_b) = e^{-i\hat{H}(t_b - t_a)} \hat{\rho}(t_a) e^{i\hat{H}(t_b - t_a)}, \tag{2.3}$$

with the conventional time-evolution operator $\hat{U} = e^{-i\hat{H}(t_b - t_a)}$. To obtain the reduced density matrix, we have to trace out the degrees of freedom of the harmonic oscillators. Using the over-complete coherent state basis of the N oscillators $|\boldsymbol{\alpha}\rangle = |\alpha_1, \alpha_2, \cdots, \alpha_N\rangle$, we obtain $\hat{\rho}_S(t_b) = \int \frac{d^{2N}\alpha}{\pi^N} \langle \boldsymbol{\alpha}_b | \hat{\rho}(t_b) | \boldsymbol{\alpha}_b \rangle e^{-|\boldsymbol{\alpha}_b|^2}$

For the following calculation, we assume that the global system at the initial time t_a is in a product state. Furthermore, we assume that the environment and the system are decoupled and that the environment is in thermal equilibrium. We write

$$\hat{\rho}(t_a) = \hat{\rho}_S(t_a) \otimes \hat{\rho}_E \tag{2.4}$$

with $\hat{\rho}_E = \prod_j \hat{\rho}_j$ and $\hat{\rho}_j = (1-e^{-\beta\omega_j})e^{-\beta\omega_j \hat{a}_j^\dagger \hat{a}_j}$, where we introduced the abbreviation $\beta = 1/k_B T$. In the coherent state basis, we can write

$$\hat{\rho}_j = \int \frac{d^2\alpha_j}{\pi} \rho_j(\alpha_j, \overline{\alpha_j}) |\alpha_j\rangle\langle\alpha_j| \quad \text{with} \quad \rho_j(\alpha_j, \overline{\alpha_j}) = (e^{\beta\omega_j} - 1)e^{-e^{\beta\omega_j}|\alpha_j|^2}. \tag{2.5}$$

To obtain the matrix elements $\rho_S(x_b^+, x_b^-, t_b) = \langle x_b^+|\hat{\rho}_S(t_b)|x_b^-\rangle$, we insert identities to the left and right of the initial density matrix in Eq. (2.3) in coordinate representation. Under the assumption of a product state at initial time t_a, we can write

$$\rho_S(x_b^+, x_b^-, t_b) = \int dx_a^+ \, dx_a^- \, J_S(x_b^+, x_b^-, t_b; x_a^+, x_a^-, t_a) \rho_S(x_a^+, x_a^-, t_a). \tag{2.6}$$

The propagating function $J_S(x_b^+, x_b^-, t_b; x_a^+, x_a^-, t_a)$ can be expressed as a path integral

$$\begin{aligned} J_S(x_b^+, x_b^-, t_b; x_a^+, x_a^-, t_a) = &\int_{x^+(t_a)=x_a^+}^{x^+(t_b)=x_b^+} \mathcal{D}[x^+]\mathcal{D}[p^+] \int_{x^-(t_a)=x_a^-}^{x^-(t_b)=x_b^-} \mathcal{D}[x^-]\mathcal{D}[p^-] \\ &\times e^{i(S_S[p^+,x^+]-S_S[p^-,x^-])} G[x^+, x^-], \end{aligned} \tag{2.7}$$

where $S_S[p, x]$ corresponds to the arbitrary Hamilton action of the system. Since we are evaluating the time-evolution of a density matrix $\hat{\rho}_S(t_a) = \int dx_a^+ dx_a^- \rho_S(x_a^+, x_a^-, t_a)$ $|x_a^+\rangle\langle x_a^-|$ and not of a single quantum state, we have to perform the time evolution twice: once forward for the 'ket' (x^+) and once backward for the 'bra' (x^-), corresponding to the time evolution operators $\hat{U}$ and $\hat{U}^\dagger$. This doubles the number of path integrals necessary to calculate the total time evolution. In the absence of an environment ($G = 1$), the path integrals for both time evolutions separate and the propagating function turns into the product of propagators $J_S(x_b^+, x_b^-, t_b; x_a^+, x_a^-, t_a) = F_S(x_b^+, t_b; x_a^+, t_a)\overline{F_S(x_b^-, t_b; x_a^-, t_a)}$, which correspond to the independent propagation of the 'ket' and 'bra'.

The influence functional $G[x^+, x^-]$ contains the information on the influence of the environment on the system and couples the forward and backward time evolution. As a result, the propagating function does not separate anymore and a density matrix is necessary to describe the time evolution of the reduced system. The influence functional only depends on the environment not on the dynamics of the system embedded in the environment. The goal of the remainder of this section is the evaluation of the influence functional of a general environment—modeled by an arbitrary number of harmonic oscillators with individual coupling strength c_j—that is of general interest for a variety of quantum mechanical problems.

To couple the environment and the system, we introduce the Lagrange multiplier $\lambda(t)$ which is multiplied to the constraint function $\hat{x} - \sum_{j=1}^{N}(c_j \hat{a}_j + \overline{c_j}\hat{a}_j^\dagger)$. By performing the path integral over this auxiliary, dynamic-less function for the forward and backward path integral, we ensure that the intended constraint function is zero at all times. We obtain

$$\begin{aligned} G[x^+, x^-] = &\int \frac{d^{2N}\alpha_b}{\pi^N} e^{-|\boldsymbol{\alpha}_b|^2} \int \frac{d^{2N}\alpha_a}{\pi^N} \int \mathcal{D}[\lambda^+]\mathcal{D}[\lambda^-] e^{-i\int_{t_a}^{t_b} dt(\lambda^+ x^+ - \lambda^- x^-)} \\ &\times J_E([\lambda^+],[\lambda^-]; \boldsymbol{\alpha}_b, \overline{\boldsymbol{\alpha}_b}, t_b; \boldsymbol{\alpha}_a, \overline{\boldsymbol{\alpha}_a}, t_a)\rho_E(\boldsymbol{\alpha}_a, \overline{\boldsymbol{\alpha}_a}), \end{aligned} \tag{2.8}$$

with $\rho_E(\boldsymbol{\alpha}_a, \overline{\boldsymbol{\alpha}_a}) = \prod_j \rho_j(\alpha_{j,a}, \overline{\alpha_{j,a}})$ as defined in Eq. (2.5). The propagating function J_E of the environment under the influence of the external force $\lambda(t)$ has a product form $J_E = \prod_j J_j$. Since the individual oscillators are themselves not connected to an additional environment, their propagating function turns into the product of two propagators $J_j([\lambda^+],[\lambda^-]; \alpha_b, \overline{\alpha_b}, t_b; \alpha_a, \overline{\alpha_a}, t_a) = F_j([\lambda^+]; \overline{\alpha_b}, t_b; \alpha_a, t_a)\overline{F_j([\lambda^-]; \overline{\alpha_b}, t_b; \alpha_a, t_a)}$. The propagators correspond to the matrix elements $\langle\alpha_b| e^{-i\hat{H}_j(t_b - t_a)}|\alpha_a\rangle$ with $\hat{H}_j = \omega_j \hat{a}_j^\dagger \hat{a}_j - \lambda^\pm(t)(c_j\hat{a}_j + \overline{c_j}\hat{a}_j^\dagger)$, where the Lagrange multiplier is equivalent to an external force. The propagator can be calculated either directly in the operator formalism or via the path integral

$$F_j([\lambda]; \overline{\alpha_b}, t_b; \alpha_a, t_a) = \int_{\alpha(t_a)=\alpha_a}^{\overline{\alpha(t_b)}=\overline{\alpha_b}} \mathcal{D}[\alpha]\mathcal{D}[\overline{\alpha}] e^{\overline{\alpha_b}\alpha(t_b) + iS_j[\alpha, \overline{\alpha}; \lambda]}, \tag{2.9}$$

with the Hamilton action $S_j[\alpha, \overline{\alpha}; \lambda] = \int_{t_a}^{t_b} dt\,(i\overline{\alpha}\dot{\alpha} - H_j)$. The path integral can be calculated by solving the corresponding classical Hamilton's equations of motion. We obtain

$$F_j([\lambda]; \overline{\alpha_b}, t_b; \alpha_a, t_a) = \exp\left[\overline{\alpha_b}\alpha(t_b) + ic_j\int_{t_a}^{t_b} dt\,\lambda(t)\alpha(t)\right], \tag{2.10}$$

$$\text{with} \quad \alpha(t) = i\overline{c_j}\int_{t_a}^{t} dt'\, e^{-i\omega_j(t-t')}\lambda(t') + \alpha_a e^{-i\omega_j(t-t_a)}. \tag{2.11}$$

After performing the Gaussian integrals over the initial and final coherent state, we obtain the influence functional

$$G[x^+, x^-] = \int \mathcal{D}[\lambda^+]\mathcal{D}[\lambda^-] e^{iS_E[\lambda^c, \lambda^q] - i\int_{t_a}^{t_b} dt\,(\lambda^c x^q + \lambda^q x^c)}, \tag{2.12}$$

$$\begin{aligned} \text{with} \quad S_E[\lambda^c, \lambda^q] = &\frac{i}{2}\sum_{j=1}^{N}|c_j|^2\left(2n_{\omega_j}+1\right)\int_{t_a}^{t_b} dt\int_{t_a}^{t_b} dt'\,\cos[\omega_j(t-t')]\lambda^q(t)\lambda^q(t') \\ &+ 2\sum_{j=1}^{N}|c_j|^2\int_{t_a}^{t_b} dt\int_{t_a}^{t} dt'\,\sin[\omega_j(t-t')]\lambda^q(t)\lambda^c(t'), \end{aligned} \tag{2.13}$$

where we introduced the center of mass variables $x^c = \frac{1}{2}(x^+ + x^-)$ and $x^q = x^+ - x^-$ and the corresponding Lagrange multipliers. In the context of path integration this is known as the Keldysh rotation with the classical variable x^c and the quantum variable x^q. The rationale for this notation will become clear shortly. The function $n_{\omega_j} = (e^{\beta\omega_j} - 1)^{-1}$ is the Bose-Einstein occupation of the j^{th} oscillator. The final step that remains is the calculation of the path integrals over the Lagrange multipliers. However, for the current form of the action in Eq. (2.13) this requires a complicated inversion of integrals. To simplify the calculation, we want to neglect the switching on and off of the coupling for a moment. To this end, we take the limit $t_a \to -\infty$ and $t_b \to \infty$ and multiply with a small convergence factor $e^{-0^+|t|}$ when necessary. In frequency space, with $x_\omega = \int dt\, x(t) e^{i\omega t}$, we obtain

$$S_E[\lambda^c, \lambda^q] = \int \frac{d\omega}{2\pi} \left[\frac{iZ_\omega}{\omega} \lambda^c_\omega \lambda^q_{-\omega} + \frac{i \operatorname{Re} Z_\omega}{2\omega} (2n_\omega + 1) \lambda^q_\omega \lambda^q_{-\omega} \right]. \tag{2.14}$$

Here, we defined the impedance $Z_\omega = i \sum_{j=1}^N 2\omega_j |c_j|^2 \omega / [(\omega + i0^+)^2 - \omega_j^2]$. For this action, calculating the path integral is straight forward and we obtain $G[x^c, x^q] = e^{iS_G[x^c, x^q]}$ where

$$S_G[x^c, x^q] = \int \frac{d\omega}{2\pi} \left[i\omega Y_\omega x^c_\omega x^q_{-\omega} + \frac{i}{2} \omega \operatorname{Re} Y_\omega (2n_\omega + 1) x^q_\omega x^q_{-\omega} \right] \tag{2.15}$$

with the admittance $Y_\omega = Z_\omega^{-1}$. To learn more about the causality structure of a density matrix path integral, it is instructive to rewrite this action in matrix form with $\boldsymbol{x} = (x^c, x^q)^T$. We obtain

$$S_G[x^c, x^q] = \frac{1}{2} \int \frac{d\omega}{2\pi} \boldsymbol{x}^T_{-\omega} A_\omega \boldsymbol{x}_\omega, \quad \text{with} \quad A_\omega = i\omega \begin{bmatrix} 0 & -Y_{-\omega} \\ Y_\omega & \operatorname{Re} Y_\omega (2n_\omega + 1) \end{bmatrix}. \tag{2.16}$$

The structure of A_ω is generic: In particular, the *c*-*c* component is always zero, independent of the system connected to the environment. It reflects the fact that for a purely classical dynamics ($x^q = 0$) the action is zero. Indeed, in the classical case, we have $x^+ = x^-$ and the action on the $+$ and $-$ part of the evolution cancel each other. In order to understand the structure of the *c*-*q* and *q*-*c* components, it is important to note that Y_ω is causal (retarded) and only has poles on the lower half of the complex plain, while $Y_{-\omega}$ is anti-causal (advanced) and only has poles on the upper half of the complex plain. In general, the *c*-*q* and *q*-*c* components are mutually Hermitian conjugated (advanced and retarded) matrices. This property is responsible for the causality of the response functions. The *q*-*q* component is an anti-Hermitian matrix with a positive-definite imaginary spectrum. It is responsible for the convergence of the path integral. It retains the information about the thermal distribution of the bath and is otherwise completely determined by the dissipative dynamics (given by $\operatorname{Re} Y_\omega$) of the *c*-*q* and *q*-*c* components. This correspondence ensures that the fluctuation-dissipation theorem is fulfilled at all times.

Note that the time-dependent correlations of the environmental degrees of freedom $\langle x^k(\tau) x^l(0) \rangle = i \int (d\omega/2\pi) e^{-i\omega\tau} (A_\omega^{-1})_{kl}$ follow directly from the matrix representation in Eq. (2.16), where $k,\, l = (c,\, q)$. The inverse of the matrix is given by

$$A_\omega^{-1} = -\frac{i}{\omega}\begin{bmatrix}\mathrm{Re}\, Z_\omega(2n_\omega+1) & Z_\omega \\ -Z_{-\omega} & 0\end{bmatrix} \equiv \begin{bmatrix} G_\omega^K & G_\omega^R \\ G_\omega^A & 0\end{bmatrix}. \tag{2.17}$$

In accordance with the general properties above, the entry G_ω^R is causal with $G^R(\tau) = -i\langle x^c(\tau)x^q(0)\rangle \propto \Theta(\tau)$. It is commonly referred to as the retarded Green's function. Analogously, the advanced Green's function $G^A(\tau) \propto \Theta(-\tau)$ is anti-causal while the Keldysh Green's function $G_\omega^K = \frac{1}{2}(G_\omega^R - G_\omega^A)(2n_\omega + 1)$ determines the fluctuation strength of the environment.

Furthermore, we can transform the action back to the time frame where we obtain

$$S_G[x^c, x^q] = \int dt \int dt' \,\left[-x^q(t)Y(t-t')\dot{x}^c(t') + \tfrac{i}{2}x^q(t)K(t-t')x^q(t')\right]. \tag{2.18}$$

with the time-dependent admittance $Y(t) = \int(d\omega/2\pi)Y_\omega\, e^{-i\omega t} \propto \Theta(t)$. Additionally, we defined the correlator $K(t) = \int(d\omega/2\pi)\,\omega\,\mathrm{Re}\, Y_\omega\,(2n_\omega+1)\, e^{-i\omega t} = K(-t)$. Note that while we have neglected the switching on and off of the coupling between the environment and the system in Eq. (2.13), this does not prevent us from including the switching on and off of the system variable in our final action by holding $x^\pm(t)$ at a fixed value until an initial time t_a. In particular, the choice $x^\pm(t) = 0$ ensures that both systems remain decoupled in the time interval $(-\infty, t_a)$.

2.2 Langevin equation

In this section, we want to discuss the (quasi-)classical equations of motion that follow from the interaction of a system with its environment. This resulting equation is often useful as it allows to efficiently simulate dissipative quantum mechanical systems with a classical stochastic equation. For concreteness, we investigate a 'particle' of mass m in a general potential described by the Hamiltonian $\hat{H}_S[\hat{p}, \hat{x}] = \frac{1}{2m}\hat{p}^2 + V(\hat{x})$. We corresponding action S is given by the sum of two terms $S = S_K + S_G$ with

$$\begin{aligned} S_K[p^c, p^q, x^c, x^q] &= S_S[p^c + \tfrac{1}{2}p^q, x^c + \tfrac{1}{2}x^q] - S_S[p^c - \tfrac{1}{2}p^q, x^c - \tfrac{1}{2}x^q] \\ &= \int dt\, \left[p^c\dot{x}^q + p^q\dot{x}^c - \tfrac{1}{m}p^c p^q - V(x^c + \tfrac{1}{2}x^q) + V(x^c - \tfrac{1}{2}x^q)\right]. \end{aligned} \tag{2.19}$$

and S_G from Eq. (2.18).

As discussed in the previous section, the quantum variables x^q and p^q encode the quantum character of the dynamics with $x^q = p^q = 0$ on a purely classical path. It is therefore useful to expand the total action S in powers of the quantum variables. To linear order, we obtain

$$S = \int dt\, \left\{p^q\left[\dot{x}^c - \tfrac{1}{m}p^c\right] - x^q\left[\dot{p}^c + V'(x^c) + \int dt'\, Y(t-t')\dot{x}^c(t')\right]\right\}. \tag{2.20}$$

If we perform the path integrals over the quantum variables, we obtain Dirac delta functionals of both square brackets in Eq. (2.20). They ensure that the classical equations of motion [we drop the superscript c for the classical variables in the following] $\dot{x} - p/m = 0$ and $\dot{p} + V'(x) + \int dt'\, Y(t-t')\dot{x}(t') = 0$ are fulfilled at each discrete time step. These two equation can be combined into the single equation $m\ddot{x} + V'(x) + \int dt'\, Y(t-t')\dot{x}(t') = 0$. The influence of the environment is modeled by the admittance which generally includes a dissipative term. In the absence of an additional driving force, any initial condition will therefore eventually decay to a minimum of the potential $V(x)$. This dynamic is independent of the temperature of the environment and thus violates the fluctuation-dissipation theorem.

The fluctuations are encoded in the q-q component of the action in Eq. (2.18). As this term is quadratic in x^q, it has been neglected in the derivation of the equations of motion above. However, as we will see in the following, this term is essential to obtain a correct description of the system. The flucutations can be included by employing a Hubbard-Stratonovich transformation and interpreting $K(t-t')$ as the correlation function $\langle \xi(t)\xi(t')\rangle_\xi$ of a classical random force $\xi(t)$ with vanishing average $\langle \xi(t)\rangle_\xi = 0$. In particular, note that the quadratic term of the action in Eq. (2.18) can be rewritten as

$$\exp\left[-\frac{1}{2}\int dt \int dt' x^q(t) K(t-t') x^q(t')\right]$$
$$= \int \mathcal{D}[\xi]\, \exp\left[i\int dt\, \xi(t)x^q(t) - \frac{1}{2}\int dt \int dt'\, \xi(t)K^{-1}(t-t')\xi(t')\right] \tag{2.21}$$
$$\equiv \left\langle \exp\left[i\int dt\, \xi(t)x^q(t)\right]\right\rangle_\xi ; \tag{2.22}$$

here and below, $\langle \cdot \rangle_\xi$ denote the (classical) ensemble average over paths $\xi(t)$ with a Gaussian probability distribution with $\langle \xi(t)\rangle_\xi = 0$ and $\langle \xi(t)\xi(t')\rangle_\xi = K(t-t')$. The correlator $K(t)$ is in general a complicated function of time (not necessarily positive). In the following, it is useful to introduce two characteristic scales: the noise intensity

$$D = \int_0^\infty dt\, |K(t)| \tag{2.23}$$

and the correlation time

$$\tau_c = \frac{\int_0^\infty dt\, t|K(t)|}{D} . \tag{2.24}$$

As a rough approximation, we can thus assume

$$|K(t)| \approx \frac{D}{\tau_c} e^{-|t|/\tau_c} \approx \frac{D}{\tau_c}\, \Theta(|t| < \tau_c) ; \tag{2.25}$$

such that $K(0) \simeq D/\tau_c$ and, similarly, we find $K^{-1}(0) \simeq 1/D\tau_c$.

Using the extended description with $\xi(t)$, the terms in S_G of Eq. (2.18) are linear in x^q and assume the form

$$S_G[x^c, x^q, \xi] = -\int dt \int dt' x^q(t) Y(t-t') \dot{x}^c(t') + \int dt\, \xi(t) x^q(t). \qquad (2.26)$$

The remaining path integral over x^q can be performed and leads to the (quasi-classical) Langevin equation [we denote $x(t) = x^c(t)$ for simplicity]

$$m\ddot{x}(t) + V'(x(t)) + \int_{-\infty}^{t} dt'\, Y(t-t')\dot{x}(t') = \xi(t); \qquad (2.27)$$

note that in order to obtain physical observables, we have to take the ensemble average $\langle \cdot \rangle_\xi$ after solving for $x(t)$.

In the classical description of (2.27), we are still able to calculate the advanced and retarded Green's functions introduced in Eq. (2.17) even though we can no longer access the quantum variable $x^q(t)$. We can, however, express the quantum variable as a function of the newly introduced random force $\xi(t)$ with the saddle-point solution of Eq. (2.21). We obtain $x^q(t) = -i \int ds\, K^{-1}(t-s)\xi(s)$. Thus, we can obtain the advanced and retarded Green's functions from the correlation between the random force ξ and the classical observable x. In particular, we find

$$G^R(\tau = t_2 - t_1) = -i\langle x^c(t_2) x^q(t_1)\rangle \equiv -\int ds\, K^{-1}(t_1 - s)\langle x(t_2)\xi(s)\rangle_\xi\,. \qquad (2.28)$$

The causality is established after the average over the noise and thus only holds in a statistical sense.

The Langevin equation is valid as long as the linear approximation of the potential in Eq. (2.20) is a good approximation for the dynamics of the system. In particular, this equation is exact for the case of a quantum harmonic oscillator. For a nonlinear potential, the higher order contributions can be neglected if $\hbar^2|V'''(x)|\langle|x^q|^2\rangle_\xi \simeq \hbar^2|V'''(x)|K^{-1}(0)$ is much smaller than either $|V'(x)|$ or the typical scale of the random force $\langle|\xi|\rangle \simeq K(0)^{1/2}$; both terms which we have retained in the quasi-classical Langevin equation.[1] As a result, the Langevin equation describes the quantum properties of the dynamics well for systems with $\hbar^2|V'''| \ll |V'|D\tau_c$ (small nonlinearities) or for $\hbar^2|V'''| \ll D^{3/2}\tau_c^{1/2}$ (strong fluctuations due to dissipation).

In the classical limit, where the relevant frequencies of the dynamics are much smaller than the temperature energy scale $\omega \ll k_B T$, we can approximately set $\mathrm{Re}\, Y_\omega = m\gamma$ and $\omega(2n_\omega + 1) = 2k_B T$ for the relevant frequencies. As a result, we obtain a classical white noise source with a correlation function $\langle \xi(t)\xi(t')\rangle = 2m\gamma k_B T\delta(t-t')$ yielding $D = m\gamma k_B T$. In this limit, the Langevin equation describes the classical dynamics of the system valid for times greater than $\tau_c \simeq 1/k_B T$. It is valid either for a system with small nonlinearities $\hbar^2|V'''| \ll |V'|m\gamma$ or at large temperatures with $(m\gamma)^{3/2}k_B T \gg \hbar^2|V'''|$.

[1]The estimate $\langle|x^q|^2\rangle_\xi \simeq K(0)^{-1}$ can be obtained using the correspondence between x^q and ξ mentioned above. Also, we have reinstated the dependence on $\hbar$ in order to stress that the approximation is valid in the quasi-classical limit of $\hbar$ small.

Example: In the following, we want to consider the example of a harmonic oscillator with $V'(x) = m\omega_0^2 x$ coupled to a purely dissipative environment with admittance $Y(t) = \gamma\delta(t)$ both in the quantum limit ($k_B T \ll \omega_0$) and the classical limit ($k_B T \gg \omega_0$). The corresponding Langevin equation is of the form

$$m(\ddot{x} + \gamma\dot{x} + \omega_0^2 x) = \xi(t),$$

with the solution $m x_\omega = \xi_\omega/(-\omega^2 - i\gamma\omega + \omega_0^2)$ which corresponds to

$$x(t) = \frac{1}{m\sqrt{\omega_0^2 - (\frac{\gamma}{2})^2}} \int_{-\infty}^{t} dt'\, e^{-\frac{\gamma}{2}(t-t')} \sin\Big[\sqrt{\omega_0^2 - (\tfrac{\gamma}{2})^2}\,(t-t')\Big]\xi(t'). \tag{2.29}$$

in the time domain. For completeness, we also give the expression for the velocity

$$\dot{x}(t) = \frac{1}{m}\int_{-\infty}^{t} dt'\, e^{-\frac{\gamma}{2}(t-t')} \cos\Big[\sqrt{\omega_0^2 - (\tfrac{\gamma}{2})^2}\,(t-t')\Big]\xi(t') - \tfrac{\gamma}{2}x(t). \tag{2.30}$$

The equal-time correlator is given by

$$\begin{aligned}\langle x(t)^2\rangle_\xi &= \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} e^{-i(\omega_2+\omega_1)t}\langle x_{\omega_1} x_{\omega_2}\rangle_\xi \\ &= \frac{1}{m^2}\int \frac{d\omega_1 d\omega_2}{(2\pi)^2} \frac{e^{-i(\omega_2+\omega_1)t}\langle \xi_{\omega_1}\xi_{\omega_2}\rangle_\xi}{(\omega_0^2 - \omega_1^2 - i\gamma\omega_1)(\omega_0^2 - \omega_2^2 - i\gamma\omega_2)}\,.\end{aligned} \tag{2.31}$$

In the quantum limit $k_B T \ll \omega_0$, the noise correlator is of the form $\langle \xi_{\omega_1}\xi_{\omega_2}\rangle_\xi = 2\pi K_{\omega_1}\delta(\omega_1 + \omega_2)$ with $K_\omega = m\gamma|\omega|$. This immediately yields the average fluctuations

$$\langle x(t)^2\rangle_\xi = \frac{1}{m}\int \frac{d\omega}{2\pi}\frac{\gamma|\omega|}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} = \frac{1}{2m\omega_0}\left(1 - \frac{\gamma}{\pi\omega_0}\right) \tag{2.32}$$

to leading order in γ.

The zeroth order term corresponds to the quantum fluctuations in the ground state of the harmonic oscillator with a potential energy $\frac{1}{2}m\omega_0^2\langle x(t)^2\rangle_\xi = \frac{1}{4}\omega_0$ which is half of the ground state energy (with the other half in the kinetic energy). The dissipation can be interpreted as a constant 'measurement' of the system as energy is constantly leaking out of system allowing a small insight into its current state. Therefore, dissipation reduces the fluctuation strength of $x(t)$ and $\dot{x}(t)$.

We can use the classical limit $k_B T \gg \omega_0$ to check the validity of the fluctuation-dissipation theorem that we briefly mentioned in the previous section. In this limit the noise correlator is given by $K_\omega = 2m\gamma k_B T$. The average kinetic and potential energy in the system is given by $\frac{1}{2}m\omega_0^2\langle x(t)^2\rangle_\xi = \frac{1}{2}, m\langle \dot{x}(t)^2\rangle_\xi = \frac{1}{2}k_B T$, which is exactly what we would predict using the equipartition theorem.

2.3 Lindblad master equation

The difficulty in the general expression of the path-integral of an open system is not so much the doubling of the degrees of freedom, which corresponds to going from $\Psi(x;t)$ to $\rho(x,x';t)$, but rather the fact that the action S_G is in general not local in time. In particular, this is due to the fact that K_ω is frequency dependent which translates to a nonlocal noise source. However, in many physical-situations one is only interested to the evolution of the system with frequencies close to a bare frequency ω_0. In this case, it is possible to approximate K_ω by the constant K_{ω_0}.

In particular, we investigate the periodically driven harmonic oscillator with the Hamiltonian

$$\hat{H}_S[\hat{p},\hat{x}] = \frac{\hat{p}^2}{2m} + \frac{m\omega_0^2}{2}\hat{x}^2 + \sqrt{2m\omega_0}\, f \sin[(\omega_0 - \Delta)t] \tag{2.33}$$

coupled to a bath with admittance $Y(t)$. Here, we introduced the driving strength f and the detuning Δ between the resonance frequency ω_0 and the driving frequency. We are interested in the limit of small dissipation $\gamma \ll \omega_0$ where the quality factor of the resonator is high such that the relevant frequencies of the system dynamics are centered very closely around the resonance frequency. For small detuning $|\Delta| \ll \omega_0$, we can therefore perform a *rotating-wave approximation* by introducing the coherent state basis in the rotating frame via, compare to equations following (1.36),

$$x(t) = \sqrt{\frac{2}{m\omega_0}}\,\mathrm{Re}\left[\alpha(t)e^{-i(\omega_0-\Delta)t}\right], \qquad p(t) = \sqrt{2m\omega_0}\,\mathrm{Im}\left[\alpha(t)e^{-i(\omega_0-\Delta)t}\right]. \tag{2.34}$$

and assuming $\alpha(t) \in \mathbb{C}$ to be a slow variable with frequency components α_ω which are only nonzero for $|\omega| \ll \omega_0$. This corresponds to neglecting all fast-oscillating terms in time.

We first perform the rotating-approximation on the system action yielding

$$\begin{aligned} S_S = \int dt\,[p\dot{x} - H_S(p,x)] &\approx \int dt\left[\tfrac{i}{2}(\overline{\alpha}\dot{\alpha} - \alpha\dot{\overline{\alpha}}) - \Delta|\alpha|^2 + \tfrac{i}{2}f(\alpha - \overline{\alpha})\right] \\ &= \int dt\left[i\overline{\alpha}\dot{\alpha} - \Delta|\alpha|^2 + \tfrac{i}{2}f(\alpha - \overline{\alpha})\right]. \end{aligned} \tag{2.35}$$

In this expression, the first term indicate that $\alpha \equiv \hat{a}$ and $\overline{\alpha} \equiv \hat{a}^\dagger$ are canonically conjugated variables albeit with a commutator $[\hat{a},\hat{a}^\dagger] = 1$ (which differs by the factor i when compared to $[\hat{x},\hat{p}] = i$). The second term is the Hamiltonian in the rotating frame which corresponds to

$$\hat{H}_{\mathrm{rw}} = \Delta\hat{a}^\dagger\hat{a} - \tfrac{i}{2}f(\hat{a} - \hat{a}^\dagger) \tag{2.36}$$

in the operator language. For the evolution of the density matrix, we equivalently obtain $S_K = S_S[p^+,x^+] - S_S[p^-,x^-]$.

Most importantly, in this approximation the action S_G that describes the interaction of the system becomes local in time in this approximation. In particular, we may set $Y_\omega \approx Y_{\omega_0}$. We split $Y_{\omega_0} = m(\gamma + i\Delta_Y)$ in its real and imaginary part. It can be shown that the imaginary-part can be incorporated in an (unimportant) shift $\Delta \mapsto \Delta + \Delta_Y$ of the detuning; this is also called the Lamb shift in the literature. Thus, we will only consider $Y_{\omega_0} = m\gamma$ in the following. We obtain $Y(t) = m\gamma\delta(t-0^+)$ and $K(t) = \frac{1}{2}m\gamma(2n_{\omega_0}+1)[\delta(t-0^+)+\delta(t+0^+)]$ which we can insert into Eq. (2.18) and obtain the time-local action

$$\begin{aligned} S_G &\approx i\gamma \int dt \left[\tfrac{1}{2}(\overline{\alpha^q}\alpha^c - \overline{\alpha^c}\alpha^q) + (n_0 + \tfrac{1}{2})|\alpha^q|^2\right] \\ &= i\gamma \int dt \left[-n_0\overline{\alpha^+}\alpha^- - (n_0+1)\overline{\alpha^-}\alpha^+ + (n_0+\tfrac{1}{2})(|\alpha^+|^2 + |\alpha^-|^2)\right]; \end{aligned} \tag{2.37}$$

here, we introduced the (short) notation $n_0 = n_{\omega_0}$.

As a result, the propagator assumes the form

$$\begin{aligned} J_S(\overline{\alpha_b}, \alpha_b, t_b; \overline{\alpha_a}, \alpha_a, t_a) = &\int_{\alpha^+(t_a)=\alpha_a}^{\overline{\alpha^+(t_b)}=\overline{\alpha_b}} \mathcal{D}[\alpha^+]\mathcal{D}[\overline{\alpha^+}] \int_{\overline{\alpha^-(t_a)}=\overline{\alpha_a}}^{\alpha^-(t_b)=\alpha_b} \mathcal{D}[\alpha^-]\mathcal{D}[\overline{\alpha^-}] \\ &\times e^{\overline{\alpha_b}\alpha^+(t_b) + \overline{\alpha_a}\alpha^-(t_a) + iS[\alpha^+,\overline{\alpha^+},\alpha^-,\overline{\alpha^-}]}, \end{aligned} \tag{2.38}$$

with the total action $S = S_K + S_G$ given by

$$\begin{aligned} S = \int_{t_a}^{t_b} dt \Big[&i(\overline{\alpha^+}\dot{\alpha}^+ - \overline{\alpha^-}\dot{\alpha}^-) - \Delta(|\alpha^+|^2 - |\alpha^-|^2) + \tfrac{i}{2}f(\alpha^+ - \overline{\alpha^+} - \alpha^- + \overline{\alpha^-}) \\ &- i\gamma(n_0+1)\overline{\alpha^-}\alpha^+ - i\gamma n_0\overline{\alpha^+}\alpha^- + i\gamma(n_0+\tfrac{1}{2})(|\alpha^+|^2 + |\alpha^-|^2)\Big]. \end{aligned} \tag{2.39}$$

It is evident that the dissipation is responsible for coupling the forward and backward branch of the path integral as terms proportional to γ depend both on the backward and forward value of the integration variable. The time-evolution thus involves the density matrix $\hat{\rho}$ and cannot be understood on the level of the wave-function Ψ.

As the action S is local in time, it corresponds to a simple differential equation in the operator formulation; this equation goes under the name *Lindblad master equation*. We can derive this equation by investigating how $\hat{\rho}$ evolves in time. However, here we shortcut this calculation by recalling the correspondence of the Schrodinger equation and its path integral description. In the equation

$$i\partial_t \Psi = \hat{H}(\hat{p}, \hat{x})\Psi \tag{2.40}$$

the Hamiltonian $\hat{H}$ plays the role of the generator of time-translation with the evolution given by $\mathcal{T}\exp(-i\int^t dt'\hat{H})$. The corresponding path integral assumes the form

$$\Psi = \int \mathcal{D}[p]\mathcal{D}[x] \exp\Big[i \underbrace{\int^t (p\dot{x} - H)}_{=S}\Big] \tag{2.41}$$

the first term in the exponent encodes the fact that $[\hat{x}, \hat{p}] = i$. The second term is the generator of time-translations as indicated above.

Given this insight, we extract the time-evolution of the density operator $\hat{\rho}$ from the time-local action S of (2.39). For this, we have to recall the crucial fact that $\pm$ correspond to applying the operators to the left/right of the density matrix; we introduce the notation $\hat{a}_\pm$ for these (super-)operators in the operator description. The first two terms, just encode the fact that $[\hat{a}, \hat{a}^\dagger] = 1$ both for the operators to the left and to the right of $\hat{\rho}$. Thereby, the relative minus sign is simply due to the fact that applying the operators to the right of $\hat{\rho}$ effectively inverts their order. In particular,

$$[\hat{a}_+, \hat{a}_+^\dagger](\hat{\rho}) = \hat{a}_+\hat{a}_+^\dagger(\hat{\rho}) - \hat{a}_+^\dagger\hat{a}_+(\hat{\rho}) = \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{a}^\dagger\hat{a}\hat{\rho} = [\hat{a}, \hat{a}^\dagger]\rho = \hat{\rho} \tag{2.42}$$

while

$$[\hat{a}_-, \hat{a}_-^\dagger](\hat{\rho}) = \hat{a}_-\hat{a}_-^\dagger(\hat{\rho}) - \hat{a}_-^\dagger\hat{a}_-(\hat{\rho}) = \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger = \rho[\hat{a}^\dagger, \hat{a}] = -\hat{\rho}\,. \tag{2.43}$$

The remaining terms in (2.39) then correspond to the generator $\mathcal{L}$ (Lindbladian superoperator). In the density matrix setting it is customary to denote the time evolution as $\mathcal{T}\exp(\int^t dt'\mathcal{L})$ which differs by a factor $-i$ from the Hamiltonian evolution. In particular, we have [2]

$$\begin{aligned}\mathcal{L} = &-i(\hat{H}_{\mathrm{rw}})_+ + i(\hat{H}_{\mathrm{rw}})_- \\ &+ \gamma(n_0+1)\hat{a}_-^\dagger\hat{a}_+ + \gamma n_0\hat{a}_+^\dagger\hat{a}_- - \tfrac{1}{2}\gamma(n_0+\tfrac{1}{2})(\{\hat{a}_+^\dagger, \hat{a}_+\} + \{\hat{a}_-^\dagger, \hat{a}_-\})\,.\end{aligned} \tag{2.44}$$

The time-evolution of the density matrix is then given by the Lindblad equation

$$\dot{\hat{\rho}} = \mathcal{L}(\hat{\rho}) = -i[\hat{H}_{\mathrm{rw}}, \hat{\rho}] + \gamma(n_0+1)\left(\hat{a}\hat{\rho}\hat{a}^\dagger - \tfrac{1}{2}\{\hat{a}^\dagger\hat{a}, \hat{\rho}\}\right) + \gamma n_0\left(\hat{a}^\dagger\hat{\rho}\hat{a} - \tfrac{1}{2}\{\hat{a}\hat{a}^\dagger, \hat{\rho}\}\right), \tag{2.45}$$

where $n_0 = (e^{\omega_0/k_BT} - 1)^{-1}$ corresponds to the Bose-Einstein occupation of the harmonic oscillator. The first term of the Lindblad master equation describes the reversible part of the dynamics. The remaining terms describe the irreversible dynamics due to the coupling to the environment. By introducing the jump operator $\mathcal{J}[\hat{O}]$ with $\mathcal{J}[\hat{O}]\hat{\rho} = \hat{O}\hat{\rho}\hat{O}^\dagger - \frac{1}{2}\{\hat{O}^\dagger\hat{O}, \hat{\rho}\}$, we can rewrite the master equation in the form

$$\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] + \gamma(n_0+1)\mathcal{J}[\hat{a}]\hat{\rho} + \gamma n_0\mathcal{J}[\hat{a}^\dagger]\hat{\rho} \tag{2.46}$$

that is most conventional in the quantum optics literature. For a general, normal-ordered rotating-wave Hamiltonian $\hat{H}(\hat{a}^\dagger, \hat{a})$ the equivalent rotating-wave action is

[2] Note that the anti-commutator in the last term arises due to the fact that $K(t) \propto \delta(t - 0^+) + \delta(t + 0^+)$. The expression (2.44) is missing the term $+\gamma/2$. That this term has to be present can be seen by checking that the trace has to be preserved under $\mathcal{L}$. Alternatively, one can directly go from (2.37) to the operator formalism by setting $\alpha^q \mapsto \hat{a}_+ - \hat{a}_-$, $\alpha^c \mapsto \frac{1}{2}(\hat{a}_+ + \hat{a}_-)$ and remembering that the quantum variables are to the left of the classical ones.

given by

$$S = \int_{t_a}^{t_b} dt \Big[i(\overline{\alpha^+}\dot{\alpha}^+ - \overline{\alpha^-}\dot{\alpha}^-) - H(\overline{\alpha^+}, \alpha^+) + H(\overline{\alpha^-}, \alpha^-) - i\gamma(n_0+1)\overline{\alpha^-}\alpha^+ \\ - i\gamma n_0 \overline{\alpha^+}\alpha^- + i\gamma(n_0 + \tfrac{1}{2})(|\alpha^+|^2 + |\alpha^-|^2)\Big]. \quad (2.47)$$

2.3.1 Correlators

As the action S in Eq. (2.39) is quadratic in the fields $\overline{\alpha^\pm}, \alpha^\pm$, we can explicitly evaluate all correlators of interest. To this end, we rewrite the action as

$$S = \int dt \begin{bmatrix} \overline{\alpha^+(t)} & \overline{\alpha^-(t)} \end{bmatrix} M(t) \begin{bmatrix} \alpha^+(t) \\ \alpha^-(t) \end{bmatrix} - f \int dt \, \mathrm{Re}[\alpha^+(t) - \alpha^-(t)] \quad (2.48)$$

with the local kernel

$$M(t) = \begin{bmatrix} i\partial_t - \Delta + i\gamma(n_0 + \frac{1}{2}) & -i\gamma n_0 \\ -i\gamma(n_0+1) & -i\partial_t + \Delta + i\gamma(n_0 + \frac{1}{2}) \end{bmatrix}. \quad (2.49)$$

In order to obtain correlation function, we have to invert $M(t)$ which can be easily achieve in frequency space with $\partial_t \mapsto -i\omega$. In the following, we investigate for simplicity the situation without driving with $f = \Delta = 0$; the results can be straightforwardly extended to the general case.

In particular, we obtain

$$\langle \hat{a}^\dagger \hat{a} \rangle = \mathrm{Tr}(\hat{a}\hat{\rho}\hat{a}^\dagger) \equiv \langle a_+(0)\overline{a_-(0)} \rangle = i(M^{-1}(t=0))_{12} = \int \frac{d\omega}{2\pi} \frac{n_0 \gamma}{\omega^2 + (\frac{\gamma}{2})^2} = n_0 \quad (2.50)$$

for the average number of photons. The result given by the Bose-Einstein distribution is to be expected as we are investigating a harmonic oscillator in thermal equilibrium.

In a next step, we evaluate the first-order correlation function

$$g^{(1)}(t) = \frac{\langle \hat{a}^\dagger(t)\hat{a}(0) \rangle}{n_0} \equiv \frac{\langle a_+(0)\overline{a_-(t)} \rangle}{n_0} = \frac{i(M^{-1}(-t))_{12}}{n_0} = \int \frac{d\omega}{2\pi} \frac{e^{i\omega t}\gamma}{\omega^2 + (\frac{\gamma}{2})^2} = e^{-\gamma|t|/2} \quad (2.51)$$

which decays with the rate $\frac{1}{2}\gamma$. Note that $g^{(1)}(0) = 1$ irrespective of temperature (i.e., the thermal occupation n_0).

Additionally, we can evaluate the second-order coherence of $\hat{n}(t) = \hat{a}^\dagger(t)\hat{a}(t)$

$$g^{(2)}(t) = \frac{\langle : \hat{n}(t)\hat{n}(0) : \rangle}{n_0^2} \equiv \frac{\langle \overline{a_-(t)} a_+(t) \overline{a_-(0)} a_+(0) \rangle}{n_0^2} \quad (2.52)$$

where the normal ordering $: \cdot :$ is traded for the Keldysh ordering (creation operators get a $-$ and annihilation operators a $+$ label). Wick's theorem yields two terms.

Pairing the terms at the same time argument yields the occupation n_0^2 which cancels with the denominator. The other pairing gives a product of first order coherences with the final result

$$g^{(2)}(t) = 1 + |g^{(1)}(t)|^2 = 1 + e^{-\gamma|t|}\,. \tag{2.53}$$

This is always greater than 1 which is a result of the bosonic statistics and yields the Hanbury and Brown Twiss effect. In fact, the second-order coherence starts at $g^{(2)}(0) = 2$ for vanishing time delay and then decays to 1 (for $t \to \infty$).

2.4 Martin-Siggia-Rose action

In a previous section, we investigated the classical limit of the Langevin equation. Here, we want to discuss how we can take the classical limit of a quantum action directly. Analogously to Sec. 2, we investigate a 'particle' in a general potential described by the Hamiltonian $\hat{H}_S[\hat{p},\hat{x}] = \frac{1}{2m}\hat{p}^2 + V(\hat{x})$ coupled to an environment described by the admittance $Y(t)$. The corresponding total action S is given by the sum of two terms $S = S_K + S_G$ with S_K given by Eq. (2.19) and S_G given by Eq. (2.18). For the classical limit, it is necessary to keep track of Planck's constant $\hbar$. We have the total action

$$\begin{aligned} S = \frac{1}{\hbar}\int dt\, &\left[p^c\dot{x}^q + p^q\dot{x}^c - \tfrac{1}{m}p^cp^q - V(x^c + \tfrac{1}{2}x^q) + V(x^c - \tfrac{1}{2}x^q)\right] \\ + \frac{1}{\hbar}\int dt \int dt'\, &\left[-x^q(t)Y(t-t')\dot{x}^c(t') + \tfrac{i}{2}x^q(t)K(t-t')x^q(t')\right], \end{aligned} \tag{2.54}$$

where in addition to the prefactor of the action, Planck's constant also appears in the Bose-Einstein occupation of the correlator $K(t)$. The quantum variables x^q and p^q are a measure of the quantum fluctuations. We have previously discussed the classical limit of the Langevin equation, where we performed the path integral over the quantum variables and showed that the resulting equation of motion is exact in the high-temperature limit of strong fluctuations. Here, we want to take the classical limit of the action directly by rescaling the quantum variables such that they become a measure of the thermal fluctuations in the classical limit. We introduce the rescaled quantum variables $\tilde{x} = ip^q/\hbar$ and $\tilde{p} = -ix^q/\hbar$, such that we obtain conjugate variables with $[\tilde{x}, x] = [\tilde{p}, p] = 1$ which implies $\tilde{x} \equiv \partial_x$ and $\tilde{p} \equiv \partial_p$.

Analogously to the derivation of the Langevin equation, any non-linear terms in $\tilde{x}$ and $\tilde{p}$ vanish in the classical limit $\hbar \to 0$. The only exception is the fluctuation term where, in the classical limit, we find $\hbar\omega(2n_\omega + 1) \to 2k_BT$. In the end, we obtain

$$iS = \int dt\, \left\{\tilde{x}\,[\dot{x} - p/m] + \tilde{p}\left[\dot{p} + V'(x) + \int dt'\, \left[Y(t-t')\dot{x}(t') + \tfrac{1}{2}K(t-t')\tilde{p}(t')\right]\right]\right\}, \tag{2.55}$$

with the classical correlator $K(t) = 2k_BT\int(d\omega/2\pi)\,\mathrm{Re}\,Y_\omega e^{-i\omega t}$ and the shorthand $x^c = x$ and $p^c = p$. In literature, this classical action is known as a Martin-Siggia-Rose action and the rescaled quantum variables $\tilde{x}, \tilde{p}$ — which are now a measure of the thermal fluctuation strength — are referred to as response fields. Following the same steps as in the derivation of the Langevin equation, we find that the action corresponds to the coupled stochastic equations

$$m\dot{x}(t) = p(t) \tag{2.56}$$

$$\dot{p}(t) + V'(x(t)) + \int_{-\infty}^{t} dt'\, Y(t-t')\dot{x}(t') = \xi(t), \tag{2.57}$$

with the colored noise source $\xi(t)$ and the correlator $\langle \xi(t)\xi(t') \rangle_\xi = K(t-t')$.

In order to obtain an action that is local in time, we consider a constant, purely-dissipative admittance $Y(t) = m\gamma\delta(t)$. In this case, the action in Eq. (2.55) is given by

$$iS = \int_{t_a}^{t_b} dt\, \left\{ \tilde{x}\left[\dot{x} - p/m\right] + \tilde{p}\left[\dot{p} + V'(x) + \gamma p + mk_B T\gamma\tilde{p}\right] \right\}, \tag{2.58}$$

where we have used the equation of motion $p = m\dot{x}$ in the dissipative term for convenience.

2.5 Fokker-Planck equation

In the classical limit, the density matrix becomes diagonal and corresponds to a probability distribution $P(x, p, t)$ of finding the 'particle' at position x and momentum $p = m\dot{x}$ at a certain time t. Note that we can specify both coordinates in phase space simultaneously as we consider the high temperature limit where quantum mechanics is irrelevant (which can be seen by the fact that $\hbar$ does not appear in the action any more).

Analogous to the Lindblad master equation, the local action can be interpreted as a differential equation for $P(x, p, t)$. As before, the terms $\tilde{x}\dot{x}$ and $\tilde{p}\dot{p}$ encode the commutator relation such that the response fields correspond to differential operators with $\tilde{x} \equiv \partial_x$ and $\tilde{p} \equiv \partial_q$. Due to the causality of the admittance, the differential operators always stand to the left of the classical variables x and p. The remaining terms lead to the Fokker-Planck equation

$$\dot{P}(x, p, t) + \nabla \cdot \boldsymbol{J} = 0, \quad \text{with} \quad \nabla = \begin{bmatrix} \partial_x \\ \partial_p \end{bmatrix}$$
$$\text{and} \quad \boldsymbol{J} = \begin{bmatrix} p/m \\ -V'(x) - \gamma p - k_B T m\gamma\partial_p \end{bmatrix} P(x, p, t), \tag{2.59}$$

which is sometimes referred to as the Kramers' equation.

Another common Fokker-Planck equation is the Smoluchowski equation which is valid in the overdamped regime, where the relaxation time $1/\gamma$ is much smaller than any other timescale of the system. In this limit, we can neglect the interial term $\ddot{x} = \dot{p}$ in the local action of Eq. (2.58). As a result, the dynamic of the variable p becomes trivial and we can integrate over $\tilde{x}$ to obtain the overdamped action

$$iS = \int_{t_a}^{t_b} dt\, \tilde{p} \left[m\gamma\dot{x} + V'(x) + k_B T m\gamma\tilde{p} \right]. \tag{2.60}$$

As a result of the overdamped approximation, the remaining variables x ad $\tilde{p}$ are now conjugate with $[\tilde{p}, x] = 1/m\gamma$ and the response field corresponds to the differential operator $\tilde{p} = (1/m\gamma)\partial_x$. We obtain the corresponding Fokker-Planck-Smoluchowski equation

$$\dot{P}(x,t) + \frac{1}{m\gamma}\partial_x \Big[-V'(x) - k_B T\partial_x \Big] P(x,t) = 0. \tag{2.61}$$

While it is possible to use second order perturbation theory to perform the same overdamped approximation directly from the Kramers' equation in Eq. (2.59), the derivation is quite involved. Here, we therefore limit ourselves to demonstrating how both equations lead to the same equilibrium distribution.

In equilibrium with $\dot{P} = 0$, Eq. (2.61) is a second order differential equation for $P(x)$. For a confining potential with $V(x \to \pm\infty) = \infty$, we can solve the differential equation to obtain the normalizable solution $P(x) = c\exp[-V(x)/k_B T]$ with a normalization constant c such that $\int dx\, P(x) = 1$. The probability distribution is commonly known as the Boltzmann distribution.

In order to derive the equilibrium solution to Kramers' equation in Eq. (2.59), we employ the detailed balance conditions. They state that under time reversal T, the (stationary) probability current and the probability distribution have to fulfill the conditions $\boldsymbol{J}(Tx, Tp) = T\boldsymbol{J}$ and $P(Tx, Tp) = P(x, p)$. Here, the time-reversal operator T maps $x \mapsto x$, $p \mapsto -p$, $J_x \mapsto -J_x$, and $J_p \mapsto J_p$. Most of them are trivially fulfilled. However, the relation $J_p(x, -p) = J_p(x, p)$ leads to the condition $(\gamma p + k_B T m\gamma\partial_p)P(x, p) = 0$; this equation has the solution $P(x, p) = f(x)\exp(-p^2/2mk_B T)$ with arbitrary $f(x)$.

Inserting the solution fulfilling detailed balance into Eq. (2.59) leads to a first order differential equation for $f(x)$ that can be solved to obtain

$$P(x,p) = c\,\tilde{c}\, e^{-V(x)/k_B T - p^2/2mk_B T} \tag{2.62}$$

with an additional normalization constant $\tilde{c}$ such that $\int dp\, \tilde{c}\exp(-p^2/2k_B T) = 1$. This solution corresponds to the Maxwell-Boltzmann distribution in phase-space. Note that the equilibrium distribution of the Smoluchowski equation is obtained when integrating over p with $P(x) = \int dp P(x, p)$.

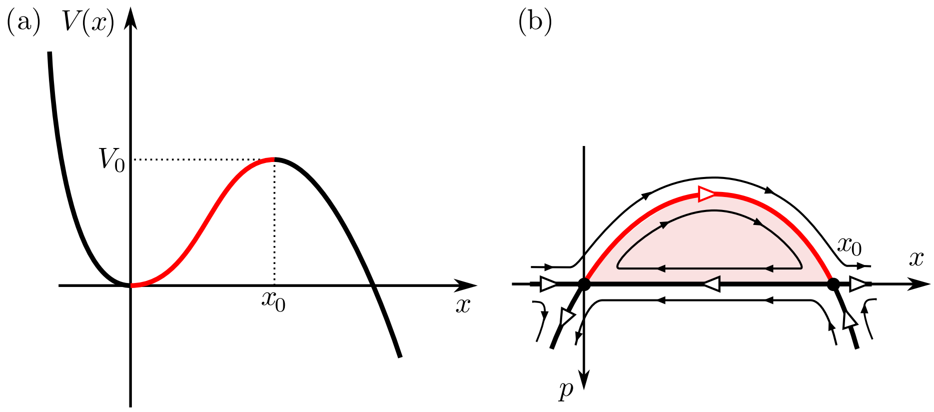

Figure 2.1: (a) Potential with a meta-stable minimum at $x = 0$. The height of the potential barrier is given by $V(x_0) = V_0$. (b) Phase portrait of the Hamilton function $H(p, x)$. The thick lines correspond to the solutions at $H = 0$. An escape from the meta-stable minimum is only possible via the thermally activated solution at $p \neq 0$ (highlighted in red).

2.6 Thermal activation over a potential barrier

One instructive application of the Martin-Siggia-Rose formulation is the thermal activation over a potential barrier. We consider a 'particle' in a meta-stable minimum of a potential at $x = 0$ as sketched in Fig. 2.1(a). Additionally, we assume that the particle's motion is over-damped with $\gamma \gg \sqrt{V''(0)}$. Then, the system can be described by the action in Eq. (2.60). By substitution $p = m\gamma\tilde{p}$, we obtain the action

$$iS = \int_{t_a}^{t_b} dt \left[p\dot{x} + p\frac{V'(x)}{m\gamma} + \frac{k_B T}{m\gamma}p^2 \right] = \int_{t_a}^{t_b} dt \, [p\dot{x} - H(p, x)] \,, \tag{2.63}$$

such that the saddle-point equations corresponds to Hamilton's equations of motion $\dot{x} = \partial_p H$ and $\dot{p} = -\partial_x H$ with the 'Hamilton function' $m\gamma H(p, x) = -pV'(x) - k_B T p^2$. Note that that H is not an energy and p does not correspond to the physical moment. It is rather an auxiliary variable that encodes the noise. Nevertheless, the saddle-point solutions conserve the value of the Hamilton function analogous to the energy conservation in the conventional Hamilton formalism.

For the activation over the barrier, a special role is played by the trajectories for $H = 0$. As indicated in the phase portrait in Fig. 2.1(b), the stationary solutions $V'(x) = 0$ without noise ($p = 0$) lie on this value of H. These solutions are connected by first the line $p = 0$, which corresponds to the relaxational dynamics $m\gamma\dot{x} = -V'(x)$, that predicts the viscous relaxation towards the meta-stable minimum at $x = 0$. As such, this solution does not offer the possibility to escape from the meta-stable state over the barrier at x_0. The second solution of $H = 0$ is $p = -V'(x)/k_B T$, $m\gamma\dot{x} = V'(x)$ that allows for an escape from the meta-stable minimum. In order to obtain the relative weight of finding the particle at a position $x_* \gg x_0$ relative to the metastable position $x = 0$, we need to find the optimal trajectory that brings the system to the position x_*. If no time restrictions are imposed, the optimal path is given by the zero energy solution, which, in fact, requires an infinite amount of time to depart from the minimum at $x = 0$. It is obvious from the phase portrait in Fig. 2.1(b) that in order to escape the meta-stable state the system must follow the fluctuation curve $p = -V'(x)/k_B T$ to the maximum at $x = x_0$ and then follow the noiseless solution $p = 0$ to $x \to \infty$. The action is only accumulated along the fluctuating part of the trajectory. To exponential accuracy, the escape rate is thus given by e^{iS_0}, where S_0 is the action of the optimal trajectory from $x = 0$ to $x = x_0$ given by the shaded area in Fig. 2.1(b). We obtain

$$iS_0 = \int dt\, p\dot{x} = \int_0^{x_0} dx\, p(x) = -\frac{V_0}{k_B T}. \tag{2.64}$$

To exponential accuracy, we thus obtain the classical Boltzmann distribution which is independent of the individual form of the potential.

Chapter 3

Photon counting statistics

3.1 Binomial process

We consider the following model for photon emission. At each discrete time step δ, the system can emit a single photon with probability K (which is independent on previous steps). In turn, the probability to emit no photon at each discrete time step δ is given by $1-K$. The process is repeated N times such that the total measurement time is given by $\tau = N\delta$. The probability to measure n photons during the total measurement time τ is binomially distributed with $P(n) = \binom{N}{n} K^n (1-K)^{N-n}$.

As a different approach to the problem, we can introduce a counting factor $e^{i\chi}$. The idea is that in each step we have the two outcomes, 0-photons with probability $p_0 = 1-K$ and 1-photon emitted with probability $p_1 = K$. In each step, we decorate the photon emission with the label $e^{i\chi}$ which yields the factor $p_0+e^{i\chi}p_1 = 1-K+Ke^{i\chi}$. The different steps are independent. For N steps we obtain the characteristic function

$$Z(\chi) = (Ke^{i\chi} + 1 - K)^N. \tag{3.1}$$

In this expression $e^{i\chi}$ counts the number of emitted photons. The terms corresponding to no photon emission in any of the steps, do not depend on χ. The terms that correspond to a single photon emission in any of the steps, are multiplied with the factor $e^{i\chi}$. Two photons, obtain a factor $e^{i\chi}e^{i\chi} = e^{2i\chi}$. Overall, the characteristic function is given by $Z(\chi) = \sum_{n=0}^{N} P(n)e^{in\chi} = \langle e^{in\chi} \rangle$ with $P(n)$ the probability to emit n-photons in N-steps during the measurement time τ in total. This relation can be inverted with the result

$$P(n) = \int_0^{2\pi} \frac{d\chi}{2\pi} Z(\chi) e^{-in\chi}. \tag{3.2}$$

As another useful property, characteristic functions provide us with the ability to calculate the moment $\langle n^j \rangle$ of the photon counting statistics by taking the j^{th} derivative of the characteristic functions with regards to $i\chi$ and setting $\chi = 0$ in the end of the calculation. We obtain

$$\langle n^j \rangle = \sum_{n=0}^{N} n^j P(n) = \left[\frac{d^j}{d(i\chi)^j} \sum_{n=0}^{N} P(n) e^{in\chi} \right]_{\chi=0} = \left. \frac{d^j Z(\chi)}{d(i\chi)^j} \right|_{\chi=0} . \tag{3.3}$$

In different words, the moment $\langle n^j \rangle$ corresponds to the j^{th} coefficient in a Taylor expansion of the characteristic function $Z(\chi)$ around $\chi = 0$ with

$$Z(\chi) = \sum_{j=0}^{\infty} \langle n^j \rangle \frac{(i\chi)^j}{j!} . \tag{3.4}$$

For the binomial process of photon emission described above, we obtain the average number of photons $\langle n \rangle = KN$. Analogously, we obtain $\langle n^2 \rangle = KN[1 + K(N-1)]$ and $\langle n^3 \rangle = KN[1 + 3K(N-1) + K^2(N-1)(N-2)]$. In general, we can immediately obtain from Eq. (3.1) that the j^{th} moment $\langle n^j \rangle$ is proportional to N^j in the limit of large N. Due to this, we have $\langle n^j \rangle = \langle n \rangle^j (1 + \mathcal{O}(N^{-1}))$ such that all the moments are dominated by $\langle n \rangle$ and thus all give the same information about the distribution. On the other hand, it is well-known that the variance

$$\mathrm{Var}(n) = \langle n^2 \rangle - \langle n \rangle^2 \tag{3.5}$$

gives us additional information about the width of the distribution. This is due to the fact, that the dominating term $\propto N^2$ is canceled between the two terms on the right hand side such that only the subleading term $\propto N$ remains. The question is to generalize this procedure and to define 'better' higher order moments that give us new information about the distribution and are not simply swamped by the average (and the variance) for $N \to \infty$.

This tasked is solved by the cumulants $\langle\!\langle n^j \rangle\!\rangle$ which are defined as the Taylor expansion coefficients of the *logarithm* of the characteristic function — also known as the cumulant generating function $\lambda(\chi) = \ln Z(\chi)$. We define the cumulants $\langle\!\langle n^j \rangle\!\rangle = d^j \lambda(\chi)/d(i\chi)^j|_{\chi=0}$. The first four cumulants are given by

$$\langle\!\langle n \rangle\!\rangle = \langle n \rangle = \bar{n}, \tag{3.6}$$

$$\langle\!\langle n^2 \rangle\!\rangle = \mathrm{Var}(n) = \langle (n - \bar{n})^2 \rangle \tag{3.7}$$

$$\langle\!\langle n^3 \rangle\!\rangle = \langle n^3 \rangle - 3\langle n^2 \rangle \langle n \rangle + 2\langle n \rangle^3 = \langle (n - \bar{n})^3 \rangle, \tag{3.8}$$

$$\langle\!\langle n^4 \rangle\!\rangle = \langle n^4 \rangle - 4\langle n^3 \rangle \langle n \rangle - 3\langle n^2 \rangle^2 + 12\langle n^2 \rangle \langle n \rangle^2 - 6\langle n \rangle^4 \neq \langle (n - \bar{n})^4 \rangle. \tag{3.9}$$

The first cumulant corresponds to the mean, the second cumulant is the variance, and the third cumulant is identical to the third central moment. However, the higher cumulants are neither moments nor central moments, but rather more complicated polynomial functions of the moments. In general, the moments and cumulants are connected via the recursion relation

$$\langle\!\langle n^j \rangle\!\rangle = \langle n^j \rangle - \sum_{k=1}^{j-1} \binom{j-1}{k} \langle\!\langle n^k \rangle\!\rangle \langle n^{j-k} \rangle . \tag{3.10}$$

For the binomial process of photon emission, we obtain

$$\lambda(\chi) = N \ln(Ke^{i\chi} + 1 - K) \tag{3.11}$$

and the first three cumulants are given by $\langle n \rangle = NK$, $\langle\langle n^2 \rangle\rangle = NK(1-K)$, and $\langle\langle n^3 \rangle\rangle = NK(1-3K+2K^2)$. In contrast to the moments, each cumulant is proportional to N independent of its order.

In general, cumulants have several advantages over moments. For example, only a few cumulants are necessary to sufficiently describe most probability distributions. In particular, the well-known Gaussian distribution can be entirely described by the first two cumulants. Furthermore, the cumulants of the sum of two statistically independent variables separate with $\lambda_{n+m}(\chi) = \ln\langle e^{i(n+m)\chi}\rangle = \ln\langle e^{in\chi}\rangle + \ln\langle e^{im\chi}\rangle = \lambda_n(\chi) + \lambda_m(\chi)$.

Another commonly used property used to characterized photon statistics is the Fano factor $F = \langle\langle n^2 \rangle\rangle / \langle n \rangle$. It is a measure for the average number of correlated photons. In particular, a Fano factor $F > 1$ describes bunched photons while a Fano factor $F < 1$ corresponds to an anti-bunched counting statistics. A Fano factor of 1 describes a Poissonian statistics where each photon is emitted independent of previous emissions. For the binomial process, we obtain an anti-bunched statistics with $F = 1 - K$.

For completeness, we also mention the factorial moments and cumulants. The factorial moment generating function corresponds to $F(s) = \langle s^n \rangle$ with the factorial moments given by $\langle n^j \rangle_f = d^j F(s)/ds^j|_{s=1}$. The factorial moments are connected to the regular moments via $\langle n^j \rangle_f = \langle n(n-1)\ldots(n-j+1)\rangle$. The corresponding factorial cumulant generating function is given by $\mu(s) = \ln F(s)$ with $\langle\langle n^j \rangle\rangle_f = d^j \mu(s)/ds^j|_{s=1}$. The factorial moment generating function can be obtained from the regular moment generating function via the substitution $e^{i\chi} \mapsto s$. The Poissonian distribution can be described entirely by the first factorial cumulant. As such, factorial cumulants are particularly useful to describe probability distributions that closely resemble a Poissonian statistics. For the binomial distribution, the second and third factorial cumulants are given by $\langle\langle n^2 \rangle\rangle_f = -NK^2$ and $\langle\langle n^3 \rangle\rangle_f = 2NK^3$.

3.2 Continuum limit and rate equations

In this section, we want to consider the continuum limit of the binomial photon-emission process. It corresponds to taking the two limits $\delta \to 0$ and $K \to 0$ while leaving the emission rate $K/\delta = \Gamma$ constant. In this limit, we can expand the cumulant-generating function of the binomial process of Eq. (3.11) in K to obtain

$$\lambda(\chi) = \tau\Gamma(e^{i\chi} - 1). \tag{3.12}$$

This cumulant-generating function of photon emission during a measurement time τ describes a Poissonian process where the cumulants are given by $\langle\langle n^j \rangle\rangle = \tau\Gamma$. In

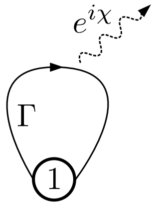

Figure 3.1: Diagram of a Poissonian process that emits photons at a rate Γ. The counting term $e^{i\chi}$ provides access to the counting statistics of emitted photons. The probability P_1 to remain in the single available state remains one throughout the time evolution.

terms of the factorial cumulants, this means that only the first factorial cumulant is nonzero. We can write down the rate equation

$$\dot{P}_1(t) = e^{i\chi(t)}\Gamma P_1(t) - \Gamma P_1(t) \tag{3.13}$$

involving only a single state that corresponds to this process. An equivalent diagrammatic description can be found in Fig. 3.1. As the system consists of a single state, the probability P_1 to be in this state remains unchanged throughout the time evolution. This can be obtained directly by setting the counting field $\chi(t) = 0$ in the differential equation above such that we find $\dot{P}_1 = 0$. However, the system is still able to emit photons at a rate Γ without changing its overall state as is characteristic for a Poissonian process. In the rate eqaution, the first term corresponds to the emission of the photon and has thus been decorated with the counting factor $e^{i\chi(t)}$. A common choice is a piecewise-constant counting field, with $\chi(t) = \chi$ during the measurement time τ and $\chi(t) = 0$ at all other times. Then, the characteristic function can be obtained by solving the rate equation in Eq. (3.13) for finite χ over the measurement time τ. The characteristic function for counting in the interval τ is then given by $Z(\chi) = P_1(\tau)$. The result coincides with the previously obtained characterisitc function in Eq. (3.12).

More generally, the rate equation can be straightforwardly generalized to multi-state dynamics with P_n the probability to be in state N. The evolution is then given by a general Markov master equation $\dot{\boldsymbol{P}} = M\boldsymbol{P}$ with M the transition matrix. It is also possible to have processes that involve more than a single photon. In this case, each term in M corresponding to a process where k photons can be detected is multiplied by the counting factor $e^{ik\chi(t)}$ analogous to Eq. (3.13). In this way, the transition matrix M depends on χ. It is even possible to set $k < 0$ which corresponds to 'negative' counting, i.e., removing photons from the detector. In the general case, the characteristic function is given by the sum over all probabilities $Z(\chi) = \sum_n P_n(\tau) = \sum_n [e^{M(\chi)\tau}\boldsymbol{P}(0)]_n$.

3.3 Lindblad master equation

Analogous to counting fields in rate equations, we can also introduce counting fields to the Lindblad master equation. As an important example, we consider a system with a characteristic frequency ω_0 and the corresponding rotating-wave Hamiltonian $\hat{H}_{\mathrm{rw}}$ that is coupled to a bath at temperature T by the dissipation rate γ. We couple this system to a detector with a coupling rate γ_d. A diagram of the model is displayed in Fig. 3.2. Since we want to model an ideal detector that only absorbs photons, we set the temperature of the detector to zero. The system is described by the Lindblad master equation

$$\dot{\hat{\rho}} = -i[\hat{H}_{\mathrm{rw}}, \hat{\rho}] + \gamma(n_0+1)\mathcal{J}[\hat{a}]\hat{\rho} + \gamma n_0 \mathcal{J}[\hat{a}^\dagger]\hat{\rho} + \gamma_d \left(e^{i\chi(t)}\hat{a}\hat{\rho}\hat{a}^\dagger - \tfrac{1}{2}\{\hat{a}^\dagger\hat{a}, \hat{\rho}\}\right), \quad (3.14)$$

with n_0 the Bose-Einstein occupation at frequency ω_0 and temperature T. The first three terms are equivalent to the evolution of the system without a detector, while the the last term encodes the coupling of the system to the detector. The counting factor $e^{i\chi(t)}$ is coupled to the term $\gamma_d \hat{a}\hat{\rho}\hat{a}^\dagger$ which corresponds to the emission of a single photon into the detector. For a piecewise-constant counting field, the characteristic function can consequently be obtained by solving the master equation for finite χ over the time τ and taking the trace of the final density matrix such that $Z(\chi) = \mathrm{Tr}\,\hat{\rho}(\tau)$. In the long time limit, the leading contribution to the characteristic function is independent of the initial condition $\hat{\rho}(0)$. We can rewrite Eq. (3.14) in the form

$$\dot{\hat{\rho}} = -i[\hat{H}_{\mathrm{rw}}, \hat{\rho}] + \Gamma(\bar{n}+1)\mathcal{J}[\hat{a}]\hat{\rho} + \Gamma\bar{n}\mathcal{J}[\hat{a}^\dagger]\hat{\rho} + \eta\Gamma(e^{i\chi(t)}-1)\hat{a}\hat{\rho}\hat{a}^\dagger, \quad (3.15)$$

with the total dissipation rate $\Gamma = \gamma + \gamma_d$, the detection efficiency $\eta = \gamma_d/\Gamma$, and the effective Bose-Einstein occupation $\bar{n} = \gamma n_0/\Gamma = (1-\eta)n_0$. The effective Bose-Einstein occupation is reduced due to the additional coupling of the system to the detector at $T = 0$, as photons emitted to the detector reduce the effective temperature of the system. Dependent on the coupling rates, the effective Bose-Einstein occupation of the system takes a value between 0 and n_0. For $\gamma = 0$, photons can only leave the system via the detector and we obtain a detection efficiency $\eta = 1$. Furthermore, since the system is only connected to the detector at $T = 0$, the effective Bose-Einstein occupation is given by $\bar{n} = 0$. For $\gamma > 0$, photons can also leave the system undetected via the thermal bath. This leads to a finite detection efficiency $\eta < 1$.

As previously discussed, the Lindblad master equation can be straightforwardly mapped to the path integral formalism with a local-in-time action. Using the path integral formalism, we obtain the characteristic function

$$Z(\chi) = \int \mathcal{D}[\alpha^+]\mathcal{D}[\overline{\alpha^+}]\mathcal{D}[\alpha^-]\mathcal{D}[\overline{\alpha^-}]e^{iS[\overline{\alpha^+},\overline{\alpha^-},\alpha^+,\alpha^-,\chi]}, \quad (3.16)$$

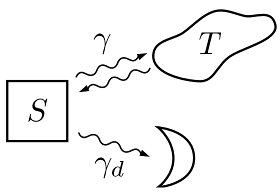

Figure 3.2: Diagram of an open-system detector model. The system is coupled to a bath at temperature T by the coupling rate γ. Additionally, the system is coupled to an ideal detector at $T = 0$ with the dissipation rate γ_d.

with a local-in-time action S that depends on the counting field $\chi(t)$ via the relation

$$\begin{aligned} S = \int dt \Big[& i(\overline{\alpha^+}\dot{\alpha}^+ - \overline{\alpha^-}\dot{\alpha}^-) - H_{\mathrm{rw}}(\overline{\alpha^+}, \alpha^+) + H_{\mathrm{rw}}(\overline{\alpha^-}, \alpha^-) \\ & - i\Gamma(\bar{n}+1)\overline{\alpha^-}\alpha^+ - i\Gamma\bar{n}\overline{\alpha^+}\alpha^- + i\Gamma(\bar{n}+\tfrac{1}{2})(|\alpha^+|^2 + |\alpha^-|^2) \\ & - i\eta\Gamma(e^{i\chi(t)} -)\overline{\alpha^-}\alpha^+ \Big]. \end{aligned} \tag{3.17}$$

3.4 Harmonic oscillator in thermal equilibrium

In this section, we want to calculate the counting statistics of a harmonic oscillator with resonance frequency ω_0 in thermal equilibrium. In experiments, a harmonic oscillator corresponds to a cavity in which the photons are trapped. Such cavities make sure that only a single mode of the light-field at frequency ω_0 is relevant to describe the system. Such cavities can be made experimentally with quality factors of the order of 10^{10}. This means that each photon bounces 10^{10} times between the mirrors before it is essentially lost. Due to the very weak coupling to the environment, a rotating-wave approximation is very accurate. For a harmonic oscillator is not connected to any external drive, we obtain the rotating-wave Hamiltonian $\hat{H}_{\mathrm{rw}} = 0$.

We count the photons emitted at the detector in the measurement interval $(0, \tau)$. Analogously to the previous section, we set $\chi(t) = \chi$ during the measurement interval and $\chi(t) = 0$ at all other times. We are interested in the limit of long measurement times $\tau \gg 1/\Gamma$, such that the measurement time is much longer than the typical waiting time of the photon emission. In this limit, we can disregard the contribution associated with the boundary conditions at the ends of the interval. In other words, we are only interested in the contribution to the cumulants that increases linearly with the measurement time. So instead of evaluating the situation counting field χ is finite in the interval $[0, \tau]$ and zero outside, we consider the system to be periodic in time (with period τ). Because of this, we remove boundary effect due to the turning-on and -off of the counting field.

Due to this, we introduce discrete frequencies spaced equally apart with $2\pi/\tau$ such that $\alpha(t) = \sum_{k=-\infty}^{\infty} \alpha_{\omega_k} e^{-i\omega_k t}$ with $\omega_k = 2\pi k/\tau$. In frequency space, we obtain the quadratic action

$$S(\chi) = i\tau \sum_{k=-\infty}^{\infty} \overline{\boldsymbol{\alpha}_{\omega_k}}^T A_{\omega_k}(\chi) \boldsymbol{\alpha}_{\omega_k}, \text{ with} \tag{3.18}$$

$$A_{\omega_k}(\chi) = \begin{bmatrix} -i\omega_k + \Gamma(\bar{n}+\frac{1}{2}) & -\Gamma\bar{n} \\ -\Gamma(\bar{n}+1) - \eta\Gamma(e^{i\chi}-1) & i\omega_k + \Gamma(\bar{n}+\frac{1}{2}) \end{bmatrix}, \tag{3.19}$$

where we introduced $\boldsymbol{\alpha}_{\omega_k} = (\alpha^+_{\omega_k}, \alpha^-_{\omega_k})^T$. Since the action separates for each discrete frequency, evaluating the path integrals amounts to calculating the determinant of the matrix A_{ω_k} at each frequency. Instead of keeping track of the normalization constants, we simply demand that a multiplicative constant is fixed by the relation $Z(0) = 1$. This is equivalent to dividing the result for finite χ by the result for $\chi = 0$ (which would be 1 if all the constants would be taken into account). We obtain the characteristic function $Z(\chi) = \prod_{k=-\infty}^{\infty} \mathrm{Det}\, A_{\omega_k}(\chi = 0)/\, \mathrm{Det}\, A_{\omega_k}(\chi)$.

In the limit of long measurement times $\tau \gg 1/\Gamma$, we turn the product of determinants into the exponent of a sum and transform the sum to a frequency integral such that

$$Z(\chi) = e^{\lambda(\chi)}, \quad \text{with} \quad \lambda(\chi) = -\tau \int \frac{d\omega}{2\pi} \ln\left[1 - \frac{4\eta\bar{n}(e^{i\chi} - 1)\Gamma^2}{\Gamma^2 + 4\omega^2}\right]. \tag{3.20}$$

with the cumulant-generating function $\lambda(\chi)$. Using partial integration, we can perform the integral to obtain

$$\lambda(\chi) = \frac{\tau\Gamma}{2}\left[1 - \sqrt{1 - 4\eta\bar{n}(e^{i\chi} - 1)}\right]. \tag{3.21}$$

For the average number of photons, we obtain $\langle n\rangle = \eta\bar{n}\Gamma\tau$ with the variance given by $\langle\!\langle n^2\rangle\!\rangle = \eta\bar{n}(1+2\eta\bar{n})\Gamma\tau$. For the Fano factor of the radiation, we find $F = \langle\!\langle n^2\rangle\!\rangle/\langle n\rangle = 1 + 2\eta\bar{n}$. Therefore, the emitted photons are bunched. The Fano factor is also related to the second-order coherence $g^{(2)}(t)$ of the radiation by the integral

$$F = 1 + \frac{\langle n\rangle}{\tau}\int dt\left[g^{(2)}(t) - 1\right]. \tag{3.22}$$

As we derived in Eq. (2.53), the second-order coherence of the thermal radiation is given by $g^{(2)}(t) = 1 + e^{-\Gamma|t|}$ which leads to the Fano factor calculated above.

Furthermore, the j^{th} factorial cumulant is given by

$$\langle\!\langle n^j\rangle\!\rangle_f = \frac{(2j-2)!}{(j-1)!}\eta^j\bar{n}^j\Gamma\tau. \tag{3.23}$$

It is evident that for $\eta\bar{n} \ll 1$ higher order factorial cumulants become very small. In this limit, we can expand the cumulant-generating function in Eq. (3.21) to obtain a Poissonian distribution with $\lambda(\chi) = \tau\Gamma\eta\bar{n}(e^{i\chi} - 1)$. In the limit of small detection efficiency or small temperatures, the thermal radiation from a single oscillatory mode thus exhibits a Poissonian statistics.

Comment on the quantization of photons:

In the high temperature limit with $\bar{n} \gg 1$, the average number of photons emitted during the time interval $(0, \tau)$ is large. In this limit, we can neglect the discreetness of the photon flow and replace $e^{i\chi} - 1 \mapsto i\chi$, valid for small χ, in the cumulant-generating function such that $\lambda(\chi) \approx \tau\Gamma(1 - \sqrt{1 - 4i\chi\eta\bar{n}})/2$. The probability to detect n-'photons' is then given by, see Eq. (3.2),

$$P(n) = \int \frac{d\chi}{2\pi} e^{-in\chi + \lambda(\chi)}. \tag{3.24}$$

Note that n is now a real number as the photons are not quantized any more.

In this limit, we want to calculate the probability of large deviations of the photon current $I = n/\tau$ from its average value $\langle I \rangle = \langle n \rangle/\tau$. To this end, we evaluate the probability in Eq. (3.24) at $n = I\tau$. Using the saddle-point approximation, we obtain $P(I) \propto e^{L(I)}$ with $L(I) = \min_x[-xI\tau + \lambda(-ix)]$ to exponential accuracy. The final result is given by

$$P(I) \propto \exp\left[-\tau\Gamma \frac{(I - \langle I \rangle)^2}{4I\langle I \rangle}\right]. \tag{3.25}$$

A Gaussian approximation around $I = \langle I \rangle$ yields the expected result $P(I) \propto \exp[-(I - \langle I \rangle)^2/2\sigma_I^2]$ with $\sigma_I^2 = \langle\langle n^2 \rangle\rangle/\tau^2$.

3.5 Homodyne detection

In Sec. 3.3, we have introduced a photon detector that counts the number of photons emitted at the detector but does not retain any information on the phase of the radiation. In Sec. 5.1, we are going to see that a laser breaks the symmetry in phase and is characterized by a fast relaxation to a state with a finite number of photons, followed by a slow dephasing. In order to detect this physics experimentally, it is important to obtain information on the phase of the light field.

Homodyne detection is a method to extract additional information about the phase properties of the radiation. To achieve this, the original signal — characterized by a complex amplitude α — is send through a beam splitter, where it interferes with an external, coherent reference signal at the same frequency as the source signal — characterized by the complex amplitude $Ae^{i\varphi_a}$ producing the amplitude $(\alpha^{\pm} + Ae^{i\varphi_a})/\sqrt{2}$ at its output.

The method is illustrated in Fig. 3.3. The phase φ_a of the reference signal can be controlled externally by a phase shifter. The mixed signal is measured by a photon detector. As the properties of the mixed signal depend on the adjustable phase-difference between the source and the reference signal, the detected photon statistics retains information on the phase of the radiation. As introduced in Eq. (3.17), the term

$$S_\chi = -i\eta\Gamma \int dt\, (e^{i\chi(t)} - 1)\overline{\alpha^-}\alpha^+ \tag{3.26}$$

in the action corresponds to a zero temperature photon detector that is coupled to a system with a coupling rate γ_d such that $\Gamma = \gamma + \gamma_d$ corresponds to the total dissipation of the system and $\eta = \gamma_d/\Gamma$ corresponds to the detection efficiency of the detector. For homodyne detection with a 50:50 beam splitter, we need to couple the counting field to the mixed signal $(\alpha + Ae^{i\varphi_a})/\sqrt{2}$, instead of the pure source signal characterized by α. If the remaining parameters remain unchanged and A is scaled in accordance with the system parameters, this change can be incorporated in the action by the mapping $\alpha^\pm \mapsto (\alpha^\pm + Ae^{i\varphi_a})/\sqrt{2}$. Furthermore, we are interested in a large reference signal with $|\alpha| \ll A$ such that the counting terms that are independent of A can be neglected. We obtain the homodyne counting action

$$S_h = -\frac{i\eta\Gamma}{2}\int dt\,(e^{i\chi(t)} - 1)(A^2 + \overline{\alpha^-}Ae^{i\varphi_a} + \alpha^+ Ae^{-i\varphi_a}). \tag{3.27}$$

One particular advantage of homodyne detection is the opportunity to obtain information on the first-order correlation function

$$g^{(1)}(t) \propto \langle \hat{a}^\dagger(t)\hat{a}(0)\rangle \equiv \langle\overline{\alpha^-(t)}\alpha^+(0)\rangle \tag{3.28}$$

of the source signal. A pure photon detector can only ever measure the previously introduced second-order coherence of $\hat{n}(t) = \hat{a}^\dagger(t)\hat{a}(t)$ with

$$g^{(2)}(t) \propto \langle{:}\,\hat{n}(t)\hat{n}(0)\,{:}\rangle \equiv \langle\overline{\alpha^-(t)}\alpha^+(t)\overline{\alpha^-(0)}\alpha^+(0)\rangle. \tag{3.29}$$

However, such a $g^{(2)}(t)$ measurement of the mixed signal where $\overline{\alpha^-}\alpha^+ \mapsto \frac{1}{2}(A^2 + \overline{\alpha^-}Ae^{i\varphi_a} + \alpha^+ Ae^{-i\varphi_a})$ can be used to obtain information about the first-order correlation $g^{(1)}(t)$ of the source signal.

In particular, the constant amplitude $\frac{1}{2}A^2$ of the reference signal can be subtracted from the measured signal obtaining $\delta\hat{n} = \hat{n} - \frac{1}{2}A^2$. The second-order correlation of the remaining signal is then given by

$$\begin{aligned}\langle{:}\,\delta\hat{n}(t)\delta\hat{n}(0)\,{:}\rangle \equiv \tfrac{1}{4}A^2\big[&e^{2i\varphi_a}\langle\overline{\alpha^-}(t)\overline{\alpha^-}(0)\rangle + e^{-2i\varphi_a}\langle\alpha^+(t)\alpha^+(0)\rangle \\ &+ \langle\overline{\alpha^-}(t)\alpha^+(0)\rangle + \langle\overline{\alpha^-}(0)\alpha^+(t)\rangle\big].\end{aligned} \tag{3.30}$$

Without squeezing, the term $\langle\hat{a}(t)\hat{a}(0)\rangle \equiv \langle\alpha^+(t)\alpha^+(0)\rangle$ vanishes such that only the two last terms remain. More generally, we can average the measured correlation with regards to the phase of the reference signal yielding

$$\overline{\langle{:}\,\delta\hat{n}(t)\delta\hat{n}(0)\,{:}\rangle} \propto \big[g^{(1)}(t) + g^{(1)}(-t)\big]. \tag{3.31}$$

The second-order coherence $\overline{\langle{:}\,\delta\hat{n}(t)\delta\hat{n}(0)\,{:}\rangle}$ of the mixed signal is thus directly related to the first-order coherence of the system under test.

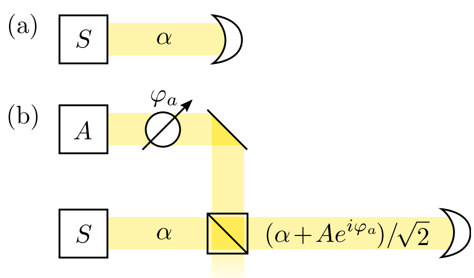

Figure 3.3: (a) Schematic illustrating the pure photon detection of a source signal characterized by a complex amplitude α. (b) Schematic illustrating homodyne detection. Before detection, the original signal is send through a 50:50 beam splitter, where it interferes with an external, coherent reference signal at the same frequency as the source signal — characterized by the complex amplitude $Ae^{i\varphi_a}$, where the phase φ_a of the reference signal can be controlled.

Example: As an example, we apply the formalism to the example of a coherently driven harmonic oscillator with the total dissipation rate Γ. The classical dynamics of the system is described by (cf. Eqs. (2.33) and following)

$$m\ddot{x} + m\Gamma\dot{x} + m\omega_0^2 x + \sqrt{2m\omega_0}\, f\sin[(\omega_0 - \Delta)t = 0, \tag{3.32}$$

with a finite detuning $\Delta \ll \omega_0$. In the rotating frame, we parameterize $x(t) = \sqrt{2/m\omega_0}\,\mathrm{Re}[\alpha(t)e^{-i(\omega_0-\Delta)t}]$ via the slow degrees $\alpha(t)$. The corresponding quasi-classical action of the system coupled to a homodyne detector is given by

$$\begin{aligned} S = S_h + \int dt\Big[& i\big(\overline{\alpha^q}\dot{\alpha^c} - \alpha^q\dot{\overline{\alpha^c}}\big) - \Delta\big(\overline{\alpha^q}\alpha^c + \alpha^q\overline{\alpha^c}\big) \\ & + \tfrac{i}{2}f\big(\alpha^q - \overline{\alpha^q}\big) + \tfrac{i}{2}\Gamma\big(\overline{\alpha^q}\alpha^c - \alpha^q\overline{\alpha^c}\big) + \tfrac{i}{2}\Gamma|\alpha^q|^2\Big], \end{aligned} \tag{3.33}$$

valid for $\Gamma, k_BT \ll \omega_0$. Note that for homodyne detection with $|\alpha| \ll A$, only the saddle-point of the counting action contributes to the steady-state counting statistics independent of the particular system dynamics. In order to calculate the counting statistics, we, thus, have to solve the linear steady-state equations $\partial S/\partial\alpha^\pm = 0$ and $\partial S/\partial\overline{\alpha^\pm} = 0$ at a finite counting field for $\dot{\alpha}^\pm = 0$. By inserting the obtained steady-state solutions $\alpha_0^\pm$ and $\overline{\alpha_0^\pm}$ back into the action in Eq. (3.33), we obtain the cumulant generating function $\lambda_h = iS_0$ which is extensive $S_0 \propto \tau$ due to the integration over time. It is useful to write the final result in the form $\lambda_h = \lambda_A + \lambda_{\varphi_a}$ with

$$\lambda_A = \tfrac{1}{2}A^2\eta\Gamma\tau(e^{i\chi} - 1) \quad \text{and} \quad \lambda_{\varphi_a} = A\frac{f[\Gamma\cos(\varphi_a) - 2\Delta\sin(\varphi_a)]}{\Gamma^2 + 4\Delta^2}\eta\Gamma\tau(e^{i\chi} - 1)\,; \tag{3.34}$$

Both counting statistics describe Poissonian photon statistics. The first term λ_A corresponds to the counting statistics of the reference signal and does not contain any information about the source signal. The second term λ_{φ_a} is of much more interest as it carries information about both the phase and the amplitude of the source signal.

For reference, the pure photon counting statistics, obtained by directly measuring the photon current of the system, is given by

$$\lambda = \frac{f^2}{\Gamma^2 + 4\Delta^2}\eta\Gamma\tau(e^{i\chi} - 1) = N\eta\Gamma\tau(e^{i\chi} - 1). \tag{3.35}$$

This result was obtained by replacing $S_h \mapsto S_\chi$ in Eq. (3.33) and solving the corresponding steady-state equations. Note that the ratio $N = \langle n\rangle/\eta\Gamma\tau$ corresponds to the photon number in the oscillator N, see below Eq. (3.21) where $\bar{n}$ plays the role of the average occupation of the oscillator in equilibrium.

For the homodyne detection, the signal $\langle n_h\rangle$ depends on the reference phase φ_a. In an experiment, the phase of the reference signal is often adjusted to the value

$$\varphi_a^* = -\arctan(2\Delta/\Gamma) \tag{3.36}$$

which leads to the largest signal. The phase can be understood from the fact that the classical steady-state solution is given by $x(t) = A_0 \cos[(\omega_0 - \Delta)t - \varphi_a^*]$ with $A_0^2 \propto N$. In particular, on resonance with $\Delta = 0$, the oscillator lags $\pi/2$ behind the drive $\propto \sin(\omega_0 t)$, see Sec. 4.1 below. Mixing the source $x(t)$ with a reference oscillator, we get the maximal signal when the phases coincide. For the optimal phase, the homodyne detection is given by the expression

$$\lambda_{\varphi_a^*} = \frac{f}{\sqrt{\Gamma^2 + 4\Delta^2}}\, \eta\Gamma\tau(e^{i\chi} - 1) = \sqrt{N}\eta\Gamma\tau(e^{i\chi} - 1)\,. \tag{3.37}$$

For the homodyne detection, the measured signal $\langle \delta n_n \rangle = \langle n_h \rangle - \frac{1}{2}A^2 = \langle n_{\varphi_a^*} \rangle = \sqrt{N}\eta\Gamma\tau$ is related to the square-root of the photon number in the resonator with $\sqrt{N} = \langle n_{\varphi_a^*} \rangle / \eta\Gamma\tau$.

The measurement of every photon detector is impeded by shot noise. Shot noise is caused by the discrete nature of the photons and dominates when the detected number of photons is sufficiently small so that uncertainties due to the Poisson distribution, which describes the occurrence of independent, random detection events, are of significance. In particular, we obtain for the pure photon counting statistics in Eq. (3.35) that the (signal-to-noise ratio)

$$\mathrm{SNR} = \frac{|\langle n \rangle|^2}{\langle\!\langle n^2 \rangle\!\rangle} = N\eta\Gamma\tau \propto \tau, \tag{3.38}$$

with $\mathrm{SNR} = \langle n \rangle$ that is typical for a purely Poissonian process. For longer measurement times and thus a larger number of detected photons, the signal-to-noise ratio increases. In order to obtain a signal-to-noise ratio of 100, the detector needs to detect 100 photons.

In the homodyne detection method, the photon detector measures the signal $\langle \delta n_h \rangle = \langle n_{\varphi_a} \rangle$, where the trivial contribution due to the reference signal has been subtracted. At the optimal phase of the reference oscillator, the signal corresponds to $\langle n_{\varphi_a^*} \rangle$, while the noise is dominated by the reference signal λ_A with $\langle\!\langle n_A^2 \rangle\!\rangle = \langle n_A \rangle = \frac{1}{2}A^2\eta\Gamma\tau$. The signal to noise ratio is given by

$$\mathrm{SNR}_h = \frac{|\langle n_{\varphi_a^*} \rangle|^2}{\langle\!\langle n_A^2 \rangle\!\rangle} = 2N\eta\Gamma\tau = 2\,\mathrm{SNR}\,. \tag{3.39}$$

The homodyne detection allows to determine the individual quadratures of the oscillator and results in a larger lower signal-to-noise ratio. The discussion in this section might indicate that the homodyne detection method yields nothing but benefits. Note, however, that in a more experimental context, the reference signal and the beam splitter can constitute additional noise sources that further impede the signal-to-noise ratio.

Chapter 4

Degenerate parametric oscillator

4.1 Parametrically driven harmonic oscillator

The externally driven harmonic oscillator is one of the most well known examples for a driven system. On resonance, the classical, noiseless dynamics is described by an equation of motion of the form

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x + 2 f \omega_0 \gamma \sin(\omega_0 t) = 0. \tag{4.1}$$

Here, γ corresponds to the dissipation rate, ω_0 is the resonance frequency of the oscillator, and $2f\omega_0\gamma$ is the amplitude of the resonant drive. For convenience, we have set the mass of the particle $m = 1$. In this chapter, we will be interested in the regime of weak damping with $\gamma \ll \omega_0$.

This equation has the steady state solution $x_0(t) = A_0 \sin(\omega_0 t + \varphi_0)$ with the amplitude $A_0 = 2f$ and the phase-shift $\varphi_0 = \pi/2$ such that the resonant response lags behind the driving signal. Moreover, if the system is at rest at $t = 0$ with $x(0) = \dot{x}(0) = 0$, the amplitude initially increases linearly until it starts to saturate to its steady state value. In total, we obtain $A(t) = A_0(1 - e^{-\gamma t/2})$ as displayed in Fig. 4.1(a).

In contrast, a resonant parametric drive is described by the equation of motion

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x + 2 f \omega_0 \gamma \sin(2\omega_0 t) x = 0. \tag{4.2}$$

The parametric drive corresponds to a periodic variation of the resonance frequency at twice the resonance frequency with $\omega^2(t) = \omega_0^2[1 + (2f\gamma/\omega_0)\sin(2\omega_0 t)]$. This resonance condition can be easily understood by the following handwaving argument. We assume that the system oscillates at its resonance frequency ω_0. Then, we can apply a trigonometric identity to show that the product of the drive $f(t) = 2f\omega_0\gamma\sin(2\omega_0 t)$ and the system variable $x(t) = A\sin(\omega_0 t + \varphi)$ produces two driving signals, one at frequency ω_0 and the other at $3\omega_0$

$$f(t)x(t) = Af\omega_0\gamma[\cos(\omega_0 t - \varphi) - \cos(3\omega_0 t + \varphi)]. \tag{4.3}$$

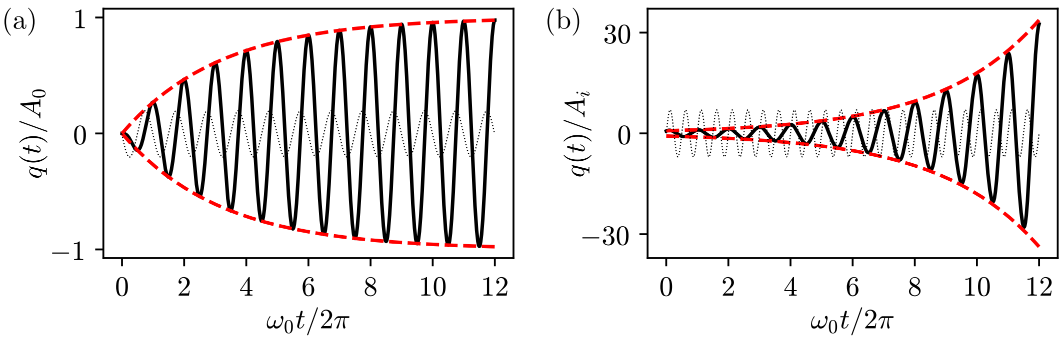

Figure 4.1: (a) Exemplary time-evolution of an externally driven harmonic oscillator with a quality factor $\omega_0/\gamma = 10$ and initial condition $x(0) = \dot{x}(0) = 0$ (black, solid). The amplitude of the oscillation initially increases linearly until it starts to saturate to its steady state value A_0 as indicated by the dashed, red line. For reference, the resonant driving force $-2f\omega_0\gamma\sin(\omega_0 t)$ is indicated by the thin, dotted line. (b) Exemplary time-evolution of a parametrically driven harmonic oscillator with a quality factor $\omega_0/\gamma = 10$, a driving strength $f = 2 > f_*$, and an initial amplitude A_i (black, solid). As the driving strength exceed the bifurcation threshold $f_* = 1$, the amplitude of the oscillation increases exponentially as indicated by the dashed, red line. For reference, the periodic variation of the squared resonance frequency $2f\gamma\omega_0\sin(2\omega_0 t)$ is indicated by the thin, dotted line.

Being off-resonance, the signal at frequency $3\omega_0$ can be neglected for this discussion. In contrast, the remaining term can be interpreted as a resonant external driving force. This argument leads to two important observations: First, the amplitude of the resonant driving signal is proportional to the amplitude A of the oscillation itself. Therefore, it is expected that the amplitude grows exponentially in time unless the initial amplitude is zero. The second observation is more subtle. As we discussed above, the resonant response of an externally driven harmonic oscillator lags behind the driving signal by a phase of $\frac{\pi}{2}$. In our handwaving argument, this condition is only fulfilled for the two phase-shifts $\varphi = \pm\frac{\pi}{2}$. This property is often referred to as phase-sensitive amplification since only certain oscillations of the system are amplified by the parametric drive. This property is further discussed below.

A more detailed calculation, see below, for $\gamma \ll \omega_0$ with an initial amplitude A_i arrives at the phase-locked solutions $x(t) = A(t)\sin(\omega_0 t + \varphi_0)$ with the amplitude $A(t) = A_i \exp[\frac{1}{2}\gamma(f-1)t]$ and the two possible phase-shifts $\varphi_0 = \pm\frac{\pi}{2}$. As for an externally driven oscillator, a finite dissipation leads to an exponential decrease of the amplitude over time. However, in contrast to the externally driven oscillator, the parametric drive adds an additional exponential contribution to the dynamics. As a result, the system dynamics can be divided into two regimes: For $f < 1$, the dissipation outweighs the parametric drive and the initial amplitude of the oscillation decreases exponentially. At the bifurcation threshold $f_* = 1$, the system undergoes qualitative change in dynamics (second-order nonequilibrium phase transition) such that for $f > 1$ the amplitude of the oscillation increases exponentially as displayed in Fig. 4.1(b). Thus, in this regime, the dissipation is no longer sufficient to stabilize the oscillations. As a result, non-linearities are needed to stabilize the system in the regime $f > 1$. This behavior is quite different from the externally driven system, where a finite steady-state amplitude A_0 exists for every non-zero value of f and γ.

4.2 Classical above-threshold dynamics

To stabilize the system above threshold ($f > 1$), we have to include nonlinearities. Here, we assume that the driver that causes the parametric resonance also provides the nonlinearity. The system can be then described by

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x + 2f\omega_0\gamma\sin(2\omega_0 t)x\left(1 - 4\omega_0\epsilon x^2\right) = 0, \tag{4.4}$$

where $\epsilon \ll 1$ measures the strength of the nonlinearity. For weak nonlinearities, we can discard any higher order nonlinear terms and also ensure that the anharmonicity does not influence the system below threshold. As above, we are interested in the limit of weak dissipation $\gamma \ll \omega_0$ such that the quality factor of the oscillator is high, thus, the relevant frequencies of the dynamics are centered around the resonance frequency. This allows to perform a rotating wave approximation. We introduce

$$x(t) = A_1(t)\cos(\omega_0 t) + A_2(t)\sin(\omega_0 t) \tag{4.5}$$

and assume that the amplitudes $A_1(t), A_2(t)$ are slow variables where only frequencies much smaller than ω_0 contribute. Plugging this ansatz in the Eq. (4.4), neglecting $\ddot{A}_i$ (which is smaller than $\omega_0\dot{A}_i$), and collecting the coefficients in front of $\sin(\omega_0 t)$ and $\cos(\omega_0 t)$ which have to vanish, we obtain the coupled equations

$$\dot{A}_1 = -\frac{\gamma}{2}(1-f)A_1 - f\gamma\omega_0\epsilon A_1\left(A_1^2 + 3A_2^2\right), \tag{4.6}$$

$$\dot{A}_2 = -\frac{\gamma}{2}(1+f)A_2 + f\gamma\omega_0\epsilon A_2\left(A_2^2 + 3A_1^2\right) \tag{4.7}$$

for the slow amplitudes A_i. Here, the first term corresponds to the parametric drive while the second term encodes the nonlinear stabilization.

Looking only at the contribution of the parametric drive, we observe that the amplitudes A_1 and A_2 corresponding to the two quadratures decay with different time constants. The mode A_2 decays with a rate $\gamma(1+f)/2$ while the mode A_1 decays with $\gamma(1-f)/2$. Above the threshold with $f = f_* = 1$, the amplitude A_1 even gets amplified, while A_2 is strongly surpressed. This represents the phase-sensitive amplification discussed above.

For $f > 1$, we obtain the steady-state solution $A_2 = 0$ and $A_1 = \pm A_0$, where

$$A_0 = \pm\sqrt{\frac{1}{2\omega_0\epsilon}}\sqrt{\frac{f-1}{f}} \tag{4.8}$$

This is again an indication of the phase-sensitive amplification where only the waves with the phases $\varphi_0 = \pm\frac{\pi}{2}$ corresponding to $\cos(\omega_0 t)$ and $-\cos(\omega_0 t)$ get amplified. Unsurprisingly, the steady state amplitude grows with larger driving strengths f and weaker nonlinearities. Also, the behavior that a stable solution, here at $A = 0$, becomes unstable at $f = f_*$ and two new stable solutions emerge that grow with an inverse power-law $A \sim (f - f_*)^{1/m}$, here with $m = 2$, is the benchmark of a so called pitchfork bifurcation. Since the phase φ_0 results such that $A_2 = 0$, we say that the bifurcation happens in the A_1 quadrature. Note that the steady state amplitude reaches a finite value for large f since the stabilizing potential is also provided by the driver, thus the nonlinearity grows stronger for increasing f. Also, it is possible to show that another bifurcation happens at $f = 2$ splitting the two steady state solutions in a total of four solutions with a phase difference of $\pi/4$. However, we will be interested in the dynamics close to the instability at $f \approx 1$ such that we do not have to concern ourselves with large driving strengths.

4.3 Photon radiation below the instability threshold

While classical, coherent radiation is no longer possible below the threshold, quantum fluctuations enable the emission of photons. In this section, we investigate the resulting counting statistics of photon radiation. We focus on the limit $\epsilon\omega_0 x^2 \ll 1$ where the anharmonic stabilization of the system can be neglected. Employing the action of a general environment in Eq. (2.18), we can write down a quadratic quantum action that corresponds to the classical equation of motion in Eq. (4.2). We obtain

$$S = \int dt \left\{ p^q \left[\dot{x}^c - p^c\right] - x^q \left[\dot{p}^c + \omega_0^2 x^c + 2f\omega_0\gamma \sin(2\omega_0 t) x^c \right.\right.$$
$$\left.\left. + \gamma \dot{x}^c + \frac{i}{2}\int dt' K(t-t') x^q(t') \right] \right\}. \tag{4.9}$$

with $K(t) = \int (d\omega/2\pi)\, \omega\gamma(1 + 2n_\omega)e^{-i\omega t}$. Analogously to the previous sections, we are interested in the limit of small dissipation $\gamma \ll \omega_0$ such that we can perform a rotating-wave approximation. As in Sec. 2.3, we introduce the coherent state basis in the rotating frame via

$$x(t) = \sqrt{2/\omega_0}\,\mathrm{Re}\left[\alpha(t)e^{-i\omega_0 t}\right], \qquad p(t) = \sqrt{2\omega_0}\,\mathrm{Im}\left[\alpha(t)e^{-i\omega_0 t}\right]. \tag{4.10}$$

The real and imaginary part of α correspond to the two quadratures A_1 and A_2. In particular, we have that $\alpha = \sqrt{\frac{\omega_0}{2}}(A_1 + iA_2)$. Moreover, α has been normalized such that $|\alpha|^2$ counts the number of photons in the oscillator.

After neglecting the fast-oscillating terms, we obtain the local-in-time Lindblad-type action

$$S=\int dt\Big[i\big(\overline{\alpha^q}\dot{\alpha}^c-\alpha^q\dot{\overline{\alpha^c}}\big)+\tfrac{i}{2}f\gamma\big(\alpha^q\alpha^c-\overline{\alpha^q}\overline{\alpha^c}\big)+\tfrac{i}{2}\gamma\big(\overline{\alpha^q}\alpha^c-\alpha^q\overline{\alpha^c}\big)+\tfrac{i}{2}\gamma(1+2n_0)|\alpha^q|^2\Big]. \tag{4.11}$$

In frequency space, we introduce the vector $\boldsymbol{\alpha}_\omega=(\alpha_k)^T=(\alpha^c_\omega,\overline{\alpha^c_{-\omega}},\alpha^q_\omega,\overline{\alpha^q_{-\omega}})^T$ and express the action by a 4×4 matrix via

$$S=\frac{1}{2}\int\frac{d\omega}{2\pi}\,\boldsymbol{\alpha}^T_{-\omega}A_\omega\boldsymbol{\alpha}_\omega \text{ with } A_\omega=\begin{bmatrix}0 & M^T_{-\omega}\\ M_\omega & Q_\omega\end{bmatrix}. \tag{4.12}$$

Here, we introduced the two 2×2 matrices

$$M_\omega=\begin{bmatrix}\frac{i}{2}f\gamma & -\omega-\frac{i}{2}\gamma\\ \omega+\frac{i}{2}\gamma & -\frac{i}{2}f\gamma\end{bmatrix} \text{ and } Q_\omega=\tfrac{i}{2}\gamma(2n_0+1)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}. \tag{4.13}$$

As previously discussed, the matrix above can be used to obtain the time-dependent correlators $\langle\alpha_k(\tau)\alpha_l(0)\rangle=i\int(d\omega/2\pi)e^{-i\omega\tau}(A^{-1}_\omega)_{kl}$.

In particular, it is interesting to note the fluctuation strength of the dimensionless quadratures $x^c=\sqrt{\omega_0}A_1$ and $y^c=\sqrt{\omega_0}A_2$ such that $\alpha^c=(x^c+iy^c)/\sqrt{2}$. Here, we obtain $\langle(x^c)^2\rangle=(\frac{1}{2}+n_0)/(1-f)$ and $\langle(y^c)^2\rangle=(\frac{1}{2}+n_0)/(1+f)$. Note that from here on x corresponds to the renormalized amplitude A_1 and note to the full solutions $x(t)$ any more. The probability distribution $P(x^c,y^c)$ in phase space is displayed in Fig. 4.2. For $n_0=f=0$, these results coincide with the expected vacuum fluctuation strength $\langle(x^c)^2\rangle=\langle(y^c)^2\rangle=\frac{1}{2}$. However, for $f>2n_0$ the fluctuation strength of the second quadrature is suppressed below the scale of the vacuum fluctuations and we obtain a squeezed state. At zero temperature where $n_0=0$, this condition is achieved for any driving strength $f>0$ below the instability threshold. As the temperature increases, a stronger and stronger driving strength is necessary to suppress the additional thermal fluctuations until finally, at an average thermal occupation of $n_0=\frac{1}{2}$, a squeezed state can no longer be achieved below the bifurcation threshold. In contrast, the fluctuation strength of the first quadrature is increased by a stronger parametric drive such that the total fluctuation strength $\langle|\alpha^c|^2\rangle=(\frac{1}{2}+n_0)/(1-f^2)$ also increases with increasing f.

In this context, it is important to note that, as mentioned above, these results are only valid for small amplitudes with $\epsilon\omega_0q^2\ll1$ such that the anharmonic stabilization of the system can be neglected. We have $q^2\simeq A_1^2+A_2^2\simeq|\alpha^c|^2/\omega_0\simeq(\frac{1}{2}+n_0)/[\omega_0(1-f^2)]$. Thus, self-consistently, we require $\epsilon(\frac{1}{2}+n_0)\ll(1-f^2)$. Given a fixed but small nonlinearity ϵ, this relation puts a restriction on the driving strength f for which the results are applicable as the inequality is violated for $f\to1$. In particular, the distance $\delta f=1-f$ from the instability threshold has to be larger than $\epsilon(\frac{1}{2}+n_0)$.

In order to obtain the photon counting statistics below the bifurcation threshold, we follow the steps laid out in Sec. 3.3 and couple the system to a zero-temperature

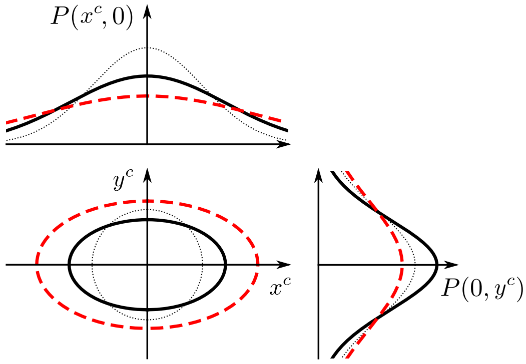

Figure 4.2: Probability distribution $P(x^c, y^c)$ in phase space at driving strength $f = \frac{1}{2}$ for a thermal occupation $n_0 = 0$ (black, solid line) and $n_0 = \frac{1}{2}$ (red, dashed line). The vacuum fluctuation strength is indicated by the thin, dotted line. For $n_0 = 0$, the fluctuation strength of the second quadrature is suppressed below the scale of the vacuum fluctuations such that we obtain a squeezed state. At $n_0 = \frac{1}{2}$, the driving strength is no longer sufficient to suppress the additional thermal fluctuations below the scale of the vacuum fluctuations.

detector at a detection rate γ_d such that the total dissipation rate of the system is given by $\Gamma = \gamma + \gamma_d$. Since the additional dissipation raises the bifurcation threshold, we rescale the driving strength $f \mapsto \bar{f} = \Gamma f/\gamma$ such that the threshold is at $\bar{f} = f_* = 1$. The counting field can be introduced via the mapping (see Eq. (3.19) in the $\pm$ basis)

$$A_\omega \mapsto A_\omega(\chi) = A_\omega + i\eta\Gamma(e^{i\chi(t)} - 1) \begin{bmatrix} 0 & -1 & 0 & -\frac{1}{2} \\ -1 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & 0 \end{bmatrix}. \tag{4.14}$$

As before, we set $\chi(t) = \chi$ during the measurement time τ and $\chi(t) = 0$ at all other times. In the limit of long measurement time, we follow the derivation outlined in Sec. 3.4 to obtain the cumulant generating function (for simplicity, at zero temperature with $n_0 = 0$)

$$\lambda(\chi) = -\frac{\tau}{2} \int \frac{d\omega}{2\pi} \ln \left[\frac{\text{Det } A_\omega(\chi)}{\text{Det } A_\omega(\chi = 0)} \right] = -\frac{\tau}{2} \int \frac{d\omega}{2\pi} \ln \left[1 - \frac{4f^2\Gamma^4 s(\chi)}{\rho(\omega)} \right], \tag{4.15}$$

where we defined the auxiliary functions $s(\chi) = \eta^2(e^{2i\chi} - 1) + 2\eta(1-\eta)(e^{i\chi} - 1)$ and $\rho(\omega) = [4\omega^2 + \Gamma^2(1-\bar{f})^2][4\omega^2 + \Gamma^2(1+\bar{f})^2]$. For $\eta = 1$, we obtain $s(\chi) = e^{2i\chi} - 1$, suggesting that photons are always emitted in pairs. By performing the integral, we obtain

$$\lambda(\chi) = \frac{\Gamma\tau}{2} \left[1 - \sqrt{1 - \tfrac{1}{2}(1-\bar{f}^2) + \sqrt{\tfrac{1}{4}(1-\bar{f}^2)^2 - \bar{f}^2 s(\chi)}} \right]. \tag{4.16}$$

For the first two cumulants, we obtain

$$\langle n \rangle = \frac{\eta\Gamma\tau\bar{f}^2}{2(1-\bar{f}^2)}, \tag{4.17}$$

$$\langle\langle n^2 \rangle\rangle = \frac{\eta\Gamma\tau\bar{f}^2[(1-\bar{f}^2)^2 + \eta(1+3\bar{f}^2)]}{2(1-\bar{f}^2)^3} \tag{4.18}$$

For $\eta = 1$, the Fano factor $F = \langle\langle n^2 \rangle\rangle / \langle n \rangle$ is of the form $F = 2 + \bar{f}^2(5-\bar{f}^2)/(1-\bar{f}^2)^2$. The first term underlines the previous observation that photons are always emitted in pairs. The second term dominates the Fano factor close to the instability threshold. In particular, we find that close to the threshold $F \propto N^2$, where $N = \langle n \rangle / \eta\Gamma\tau$ corresponds to the average photon occupation of the oscillator. As such, the number of photons correlated exceeds by far the number of photons present in the oscillator at any given time. This is a special property of the parametrically driven system as a naive picture of photon bunching would assume that the N photons in the oscillator stimulate the emission of one another such that $F \propto N$ and the photons are correlated over the timescale $1/\Gamma$. In contrast, the photons emitted by a parametrically driven system are correlated over the long, divergent time scale $\tau_* = F/\bar{I} \approx 4/\Gamma(1-\bar{f})$, with $\bar{I} = \langle n \rangle/\tau \approx \Gamma/4(1-\bar{f})$ the average photon current.

4.4 ‘Integrating-out’ modes

In the following section, we want to derive an effective model for the parametric oscillator at threshold where $f \approx 1$. We will identify two modes, a fast and slow decaying mode. In the long time limit, the slow mode dominates the dynamics of the system. Thus, we want to eliminate the fast mode to obtain an effective descirption of the relevant slow mode. Having the possibility of removing unwanted modes and having an effective description of one or a few modes is one of the main powers of the path-integral approach. Removing the modes is called ‘integrating-out’ as it corresponds to performing a subset of the integrals and reinterprating the remaining integrand as an action on the remaining modes.

4.4.1 Without dynamics

As a simple example, let us understand the process of integrating-out modes for simple Gaussian integrals. In the next section, we continue on to explain how this process can be utilized in a physical system with dynamics to separate one or a few modes.

We study the Gaussian integral (for simplicity we treat the ‘real’ case, the generalization to complex-integrals is straightforward)

$$\int d^n z \, \exp(-\tfrac{1}{2}\boldsymbol{z}^T A \boldsymbol{z}) = \frac{(2\pi)^{n/2}}{(\mathrm{Det}\, A)^{1/2}} \tag{4.19}$$

with $\boldsymbol{z} \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ with $A^T = A$. The matrix A has to be positive definite in order that the integral converges. The matrix A also determines the correlation function with $(i, j \in \{1, \dots, n\})$

$$\langle z_i z_j \rangle = \frac{\int d^n z \, z_i z_j \exp(-\frac{1}{2}\boldsymbol{z}^T A \boldsymbol{z})}{\int d^n z \, \exp(-\frac{1}{2}\boldsymbol{z}^T A \boldsymbol{z})} = (A^{-1})_{ij} \,. \tag{4.20}$$

In the next step, we separate $\boldsymbol{z}$ into two subspaces with $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y})$ where $\boldsymbol{x} \in \mathbb{R}^k$ and $\boldsymbol{y} \in \mathbb{R}^m$ with $n = k + m$. The aim is to integrate over $\boldsymbol{y}$ while keeping $\boldsymbol{x}$ fixed. In this spirit, we partition the matrix A into submatrices with

$$A = \begin{bmatrix} B & C^T \\ C & D \end{bmatrix}, \qquad \tfrac{1}{2}\boldsymbol{z}^T A \boldsymbol{z} = \tfrac{1}{2}\boldsymbol{x}^T B \boldsymbol{x} + \boldsymbol{y}^T C \boldsymbol{x} + \tfrac{1}{2}\boldsymbol{y}^T D \boldsymbol{y} \tag{4.21}$$

with $B = B^T \in \mathbb{R}^{k\times k}$, $C \in \mathbb{R}^{m\times k}$, and $D = D^T \in \mathbb{R}^{m\times m}$. The remaining task is to integrate over $\boldsymbol{y}$ to derive an effective description in terms of $\boldsymbol{x}$ alone.

As we already have seen, integrating over $\boldsymbol{y}$ can be simplest achieved by completing the square. We find

$$\boldsymbol{y}^T C \boldsymbol{x} + \tfrac{1}{2}\boldsymbol{y}^T D \boldsymbol{y} = \tfrac{1}{2}(\boldsymbol{y} + D^{-1}C\boldsymbol{x})^T D (\boldsymbol{y} + D^{-1}C\boldsymbol{x}) - \tfrac{1}{2}\boldsymbol{x}^T C^T D^{-1} C \boldsymbol{x} \,. \tag{4.22}$$

After integrating over $\boldsymbol{y}$, we obtain

$$\frac{(2\pi)^{n/2}}{(\mathrm{Det}\, A)^{1/2}} = \int d^n z \, \exp(-\tfrac{1}{2}\boldsymbol{z}^T A \boldsymbol{z}) = \frac{(2\pi)^{m/2}}{(\mathrm{Det}\, D)^{1/2}} \int d^k x \, \exp(-\tfrac{1}{2}\boldsymbol{x}^T \tilde{A} \boldsymbol{x}) \tag{4.23}$$

with the 'effective matrix' (called the Schur complement)

$$\tilde{A} = B - C^T D^{-1} C \,. \tag{4.24}$$

If we integrate over the remaining degrees of freedom, we obtain $(2\pi)^{k/2}(\mathrm{Det}\,\tilde{A})^{-1}$. Thus, we obtain from Eq. (4.23) directly the relation (the Schur determinant identity)

$$\mathrm{Det}\, A = \mathrm{Det}\, D \;\mathrm{Det}\, \tilde{A} = \mathrm{Det}\, D \;\mathrm{Det}(B - C^T D^{-1} C) \tag{4.25}$$

for the determinant of a block-matrix. Of course, we could have as well integrated over $\boldsymbol{x}$ first from which the non-trivial matrix identity

$$\mathrm{Det}\, D \;\mathrm{Det}(B - C^T D^{-1} C) = \mathrm{Det}\, B \;\mathrm{Det}(D - C B^{-1} C^T) \,. \tag{4.26}$$

The effective matrix $\tilde{A}$ is even more useful in order to obtain the correlation between the $\boldsymbol{x}$ degrees of freedom. In particular, we find $(i, j \in \{1, \dots, k\})$

$$\langle x_i x_j \rangle = \langle z_i z_j \rangle = \frac{\int d^n z \, z_i z_j \exp(-\frac{1}{2}\boldsymbol{z}^T A \boldsymbol{z})}{\int d^n z \, \exp(-\frac{1}{2}\boldsymbol{z}^T A \boldsymbol{z})} = \frac{\int d^k x \, x_i x_j \exp(-\frac{1}{2}\boldsymbol{x}^T \tilde{A} \boldsymbol{x})}{\int d^k x \, \exp(-\frac{1}{2}\boldsymbol{x}^T \tilde{A} \boldsymbol{x})} = (\tilde{A}^{-1})_{ij} \,. \tag{4.27}$$

In particular, we have proven the result

$$(A^{-1})_{ij} = (\tilde{A}^{-1})_{ij} \tag{4.28}$$

valid for $i, j \in \{1, \dots, k\}$.

4.4.2 With dynamics

Integrating out the degrees of freedom as we have explained so far is in many ways rather simple. In physical applications, we are faced with the fact that the system shows dynamics that is related to the fact that time derivatives in the form of $\dot{z}$ appear (or equivalently that the matrix A depends on ω in frequency space). In particular, we often want to separate modes into fast and slow modes depending on the time-scale of their dynamics. We have already seen in Eqs. (4.6) and (4.7) that close to the threshold with $f \approx 1$, there is a separation between the fast timescales of the mode A_2 and the slow timescales corresponding to the relaxation of A_1. Here, we want to describe the general procedure and explain how to obtain an effective description of the problem for $f \approx 1$.

In general, the Keldysh description is given in a doubled space where the classical variables $\boldsymbol{z}^c$ are coupled to their quantum partners $\boldsymbol{z}^q$ that serve as their canonical conjugate variables. Because of this, modes correspond to a certain classical variable x^c together with its canonical conjugate partner x^q. These degrees of freedom have to be integrated out together. In particular, a Keldysh action has the form

$$S = \frac{1}{2} \int \frac{d\omega}{2\pi} \begin{bmatrix} (\boldsymbol{z}^c_{-\omega})^T & (\boldsymbol{z}^q_{-\omega})^T \end{bmatrix} A_\omega \begin{bmatrix} \boldsymbol{z}^c_\omega \\ \boldsymbol{z}^q_\omega \end{bmatrix} \tag{4.29}$$

with

$$A_\omega = \begin{bmatrix} 0 & M^T_{-\omega} \\ M_\omega & Q_\omega \end{bmatrix} \tag{4.30}$$

see (5.17). As we have explained before, M (coupling the classical to the quantum component) encodes the dynamics while Q (coupling the quantum to the quantum component) describes the classical and quantum noise.

As the matrix M encodes the dynamics, we will explain in the following, how the modes of M can be identified and then integrated out. Note that the matrix M in general has the form

$$M_\omega = M_0 - i\omega J \tag{4.31}$$

with M_0, J matrices that do not depend on ω. This corresponds to a path integral in phase space where the equation of motion are Hamilton's equation which are first order in time. In particular, J encodes the information about which modes are initially conjugate to each other. This is due to the fact that the action

$$\int dt\, q^q \dot{q}^c = \int \frac{d\omega}{2\pi} q^q_{-\omega}(-i\omega) q^c_\omega \tag{4.32}$$

that encodes the commutator will give a contribution to J. If we choose the variable $\boldsymbol{z}^q$ and $\boldsymbol{z}^c$ that the individual elements are conjugate to each other J is given by the identity matrix I. However, for general variables J can be arbitrary, see Eq. (4.13) for an example.

The modes correspond to the (generalized) eigenvectors of M_ω to the eigenvalues $i\omega_j$. In general, this problem is not symmetric. In the following, we will describe how the modes can be obtained. In a first step, we determine the eigenfrequencies by solving the polynomial equation

$$\mathrm{Det}(M_0 - i\omega J) = 0 \tag{4.33}$$

for $i\omega_j$. The solution (for a stable system) will have in general the form

$$i\omega_j = i\nu_j + \gamma_j \tag{4.34}$$

where $\nu_j \in \mathbb{R}$ corresponds to the frequency and $\gamma_j \geq 0$ to the damping rate of the eigenmode. Given the eigenvalues $i\omega_j$, we can obtain the corresponding (classical) eigenmodes by finding $\boldsymbol{v}_j^c \neq \boldsymbol{0}$ with

$$(M_0 - i\omega_j J)\boldsymbol{v}_j^c = 0\,. \tag{4.35}$$

Having found all the eigenmodes, we obtain the transformation matrix

$$T^c = \begin{bmatrix}\boldsymbol{v}_1^c & \boldsymbol{v}_2^c & \cdots & \boldsymbol{v}_n^c\end{bmatrix}\,. \tag{4.36}$$

Connecting the old modes $\boldsymbol{z}^c$ to the eigenmodes $\tilde{\boldsymbol{z}}^c$ via

$$\boldsymbol{z}^c = T^c\,\tilde{\boldsymbol{z}}^c\,. \tag{4.37}$$

Importantly, the transformation matrix does not depend on ω such that the relation holds both in time as well as in frequency.

What remains to be determined is the correct modes $\tilde{\boldsymbol{z}}^q$ that are conjugate to the eigenmodes $\tilde{\boldsymbol{z}}^c$. One way to determine these modes is to determine the left eigenvalues which are given by

$$(\boldsymbol{v}_j^q)^T(M_0 - i\omega_j J) = 0 \qquad \Leftrightarrow \qquad (M_0^T - i\omega_j J^T)\boldsymbol{v}_j^q = 0\,. \tag{4.38}$$

We 'normalize' the modes such that $(\boldsymbol{v}_j^q)^T J\boldsymbol{v}_j^c = 1$.

As a result, we obtain the diagonalized propagator $D = \mathrm{diag}\{i\omega_1, \cdots, i\omega_n\}$

$$(\boldsymbol{z}_{-\omega}^q)^T M_\omega \boldsymbol{z}_\omega^c = (\tilde{\boldsymbol{z}}_{-\omega}^q)^T(D - i\omega I)\tilde{\boldsymbol{z}}_\omega^c, \tag{4.39}$$

e.g.,

$$\tilde{M_\omega} = (T^q)^T(M_0 - i\omega J)T^c = D - i\omega I\,. \tag{4.40}$$

Having found all the right eigenmodes and thus T_c, we can obtain T_q directly as the conjugate modes with

$$T^q = (JT^c)^{-T}\,. \tag{4.41}$$

Having obtained the eigenmodes, we can proceed by integrating over some mode (together with their canonical conjugate modes). In particular, a separation $\tilde{\boldsymbol{z}} = (\boldsymbol{x}, \boldsymbol{y})$ in fast modes y_j (with γ_j large) and slow modes x_j (with γ_j small) is useful, as we are often only interested at long times when the fast decaying modes have already been damped. Integrating over the fast modes $\boldsymbol{y}$ then leads to an effective action of the slow modes $\boldsymbol{x}$.

After the diagonalization, the action takes the form

$$S = \frac{1}{2} \int \frac{d\omega}{2\pi} \begin{bmatrix} (\tilde{\boldsymbol{z}}^c_{-\omega})^T & (\tilde{\boldsymbol{z}}^q_{-\omega})^T \end{bmatrix} \tilde{A}_\omega \begin{bmatrix} \tilde{\boldsymbol{z}}^c_\omega \\ \tilde{\boldsymbol{z}}^q_\omega \end{bmatrix} \tag{4.42}$$

with $\tilde{\boldsymbol{z}} = (\boldsymbol{x}, \boldsymbol{y})$,

$$\tilde{A}_\omega = \begin{bmatrix} 0 & \tilde{M}^T_{-\omega} \\ \tilde{M}_\omega & \tilde{Q}_\omega \end{bmatrix}, \tag{4.43}$$

and

$$\tilde{M}_\omega = \begin{bmatrix} D_s - i\omega I & 0 \\ 0 & D_f - i\omega I \end{bmatrix}, \qquad \tilde{Q}_\omega = \begin{bmatrix} Q_{s,\omega} & Q^T_{sf,-\omega} \\ Q_{sf,\omega} & Q_{f,\omega} \end{bmatrix}, \tag{4.44}$$

where Q_{sf} determines the remaining coupling between the slow and fast mode.

Now, we want to write the action in the slow-fast basis $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}^c, \boldsymbol{x}^q, \boldsymbol{y}^c, \boldsymbol{y}^q)$ such that we are able to integrate out the fast modes according to Eq. (4.23). Then, the action reads

$$S = \frac{1}{2} \int \frac{d\omega}{2\pi} \begin{bmatrix} (\boldsymbol{x}_{-\omega})^T & (\boldsymbol{y}_{-\omega})^T \end{bmatrix} A^{sf}_\omega \begin{bmatrix} \boldsymbol{x}_\omega \\ \boldsymbol{y}_\omega \end{bmatrix} \tag{4.45}$$

with

$$A^{sf}_\omega = \begin{bmatrix} B_{s,\omega} & C^T_{-\omega} \\ C_\omega & B_{f,\omega} \end{bmatrix}. \tag{4.46}$$

where the blocks are given by

$$B_{x,\omega} = \begin{bmatrix} 0 & D_x + i\omega I \\ D_x - i\omega I & Q_{x,\omega} \end{bmatrix}, \qquad C_\omega = \begin{bmatrix} 0 & 0 \\ 0 & Q_{sf,\omega} \end{bmatrix}. \tag{4.47}$$

Note that (with $*$ arbitrary matrices)

$$B^{-1}_{x,\omega} = \begin{bmatrix} * & * \\ * & 0 \end{bmatrix}, \tag{4.48}$$

due to the fact that B has the structure

$$B_{x,\omega} = \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}. \tag{4.49}$$

As a result, we find $C^T_{-\omega} B^{-1}_{f,\omega} C_\omega = 0$. Thus, the coupling between the modes do not contribute to the dynamics of the relevant slow mode which is given by the Schur-complement $B_{s,\omega} - C^T_{-\omega} B^{-1}_{f,\omega} C_\omega = B_{s,\omega}$. This means that, even tough the noise $\tilde{Q}$ is not block-diagonalized by the transformation onto $(\boldsymbol{x}, \boldsymbol{y})$, only the block Q_s in the slow subspace contributes to the effective action

$$S_s = \frac{1}{2} \int \frac{d\omega}{2\pi} \boldsymbol{x}^T_{-\omega} B_{s,\omega} \boldsymbol{x}_\omega = \frac{1}{2} \int \frac{d\omega}{2\pi} \left[(\boldsymbol{x}^q)^T_{-\omega} Q_{s,\omega} \boldsymbol{x}^q_\omega + 2(\boldsymbol{x}^q)^T_{-\omega} (D_s - i\omega I) \boldsymbol{x}^c_\omega \right] \tag{4.50}$$

The determination of the eigenmodes of the system is particularly useful if the problem at hand is almost quadratic such that the non-linearity is small. An example for this will be given in the next chapter. In this case, the action can be written as $S = S_0 + \epsilon S_{\rm nl}$ with $\epsilon \ll 1$. As before, we split $S_0 = S_s + S_f$ into the contribution S_s of the slow modes $\boldsymbol{x}$ and S_f of the fast modes $\boldsymbol{y}$.[1]

Including the nonlinearity, we can integrate over the fast modes to obtain an effective action $S_{\rm eff}$ by expanding in $S_{\rm nl}$. In particular, we have

$$e^{iS_{\rm eff}} = e^{iS_s} \int \mathcal{D}[\boldsymbol{y}^c]\mathcal{D}[\boldsymbol{y}^q] e^{iS_f + i\epsilon S_{\rm nl}} = e^{iS_s} \sum_{n=0}^{\infty} \frac{i^n \epsilon^n}{n!} \langle S_{\rm nl}^n \rangle_f \tag{4.51}$$

where $\langle \cdot \rangle_f = \int \mathcal{D}[\boldsymbol{y}^c]\mathcal{D}[\boldsymbol{y}^q] e^{iS_f} \cdot$ denotes the averaging over the fast modes with the action S_f. Using the notion of the cumulants, we can find the effective action as

$$S_{\rm eff} = S_s + \sum_{n=0}^{\infty} \frac{i^{n-1}\epsilon^n}{n!} \langle\!\langle S_{\rm nl}^n \rangle\!\rangle_f \approx S_s + \epsilon \langle S_{\rm nl} \rangle_f \, . \tag{4.52}$$

Note though that after the averaging the effective action $S_{\rm eff}$ in general has the frequency denominators $1/(\omega - \omega_j)$ with j ranging over the fast modes. Because of this, the subblock M_ω is not simply of the form (4.31). Because, we are interested at large times τ such that $\tau\gamma_j \gg 1$, we can simply set $\omega \simeq 1/\tau \approx 0$ in the denominators arising from the fast modes. This is the mathematical statement corresponding to the fact that the modes are fast damped and simply follow the slow motion of the fast variables $\boldsymbol{x}$ while staying in their equilibrium position.

An example of this procedure is demonstrated in the next section.

4.5 Photon counting at threshold

In Sec. 4.3, we have investigated the photon counting statistics of the parametric oscillator far below the threshold, where nonlinearities are not important yet. There, we have seen that photon emissions are already possible below threshold, which is classically not possible. This is because quantum fluctuations wash out the transition. Far above threshold, the system reaches a classical (coherent) state. Then, all cumulants of the photon current are equal and given by A_0^2 in Eq. (4.8). Close to threshold however, both contributions have to be taken into account. In general it is not easy to obtain the counting statistics in this regime, but we are able to derive an effective model of the system. As we have seen above, the amplitudes A_1 and A_2 reach the steady state at different rates. Without the nonlinearity, the relaxation times are given by $\tau_1 \sim 1/(1-f)\gamma$ and $\tau_2 \sim 1/(1+f)\gamma$ respectively, see Eq. (4.6) and (4.7). We see that τ_2 becomes smaller and smaller as we approach the threshold $f_* = 1$ while τ_1 increases significantly. This means that A_2 equilibrates much faster

[1] In general, as Q_{sf} is nonzero, we have terms of the form $Q_{sf} x^q y^q$ which we incorporate into S_f.

than A_1 and that we have two timescales in the system corresponding to a fast and a slow decaying mode. Close to threshold, we are thus able to reduce the dynamics to the relevant slow mode of the system and obtain an effective description. To obtain the eigenmodes of the system, we employ the method described in Sec. 4.4.2. We start from the action given in Eq. (4.11) where we identify (in the basis $\alpha_\omega, \overline{\alpha}_{-\omega}$)

$$M_0 = \tfrac{i}{2}\gamma \begin{bmatrix} f & -1 \\ 1 & -f \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{4.53}$$

From this we obtain the transformation matrix T^c by solving the generalized eigenvalue problem

$$M_0 \boldsymbol{v}_j^c = i\omega_j J \boldsymbol{v}_j^c \tag{4.54}$$

and $T^q = (JT^c)^{-T}$. In this case, we have that

$$T^c = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \tag{4.55}$$

with $i\omega_1 = \frac{1}{2}\gamma(1-f)$ and $i\omega_2 = \frac{1}{2}\gamma(1+f)$ which correspond to the slow and fast mode respectively. The transformation matrix then tells us that x^c belongs to the slow and y^c to the fast mode, where we write $\alpha = (x+iy)/\sqrt{2}$ as before. The conjugate variables are obtained by evaluating

$$T^q = (JT^c)^{-T} = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -1 \\ -i & -1 \end{bmatrix}. \tag{4.56}$$

We see that x^c is conjugate to $y^q = -i(\alpha^q - \overline{\alpha^q})/\sqrt{2}$ and y^c is conjugate to $-x^q = -(\alpha^q + \overline{\alpha^q})/\sqrt{2}$.

This insight can be also obtain directly from the action (4.11) by by transforming it to the variables x, y giving

$$S_0 = \int dt\, \Big[y^q \dot{x^c} - x^q \dot{y^c} + i\omega_1 y^q x^c - i\omega_2 x^q y^c + \tfrac{i}{2}\gamma(n_0 + \tfrac{1}{2}) \left(|x^q|^2 + |y^q|^2 \right) \Big]. \tag{4.57}$$

From the action in this form, we see that x^c and y^q as well as y^c and $-x^q$ form conjugate variables pairs. Also the prefactors of the $y^q x^c$ and $x^q y^c$ terms, tell us about the timescales of the system. This is due to the fact that the action corresponds to the equations of motion given in Eqs. (4.6) and (4.7) (without the nonlinearity) as we have discussed above. Thus, we obtain as before from the diagonalization that y^q and x^c belong to the slow mode and x^q and y^c correspond to the fast mode. This means that we can obtain an effective description of the relevant slow dynamics by setting $y^c = 0 = x^q$ or alternatively integrating over the fast variables y^c and x^q.

$$S_s = \int dt\, \Big[y^q \dot{x^c} + i\omega_1 y^q x^c + \tfrac{i}{2}\gamma(n_0 + \tfrac{1}{2}) |y^q|^2 \Big]. \tag{4.58}$$

We continue by adding the nonlinearity to the action[2]

$$S_{\rm nl} = f\omega_0^2\gamma\epsilon \int dt \left[A_2^q A_1^c\big(|A_1^c|^2 + 3|A_2^c|^2\big) + A_1^q A_2^c\big(|A_2^c|^2 + 3|A_1^c|^2\big)\right] \tag{4.59}$$

which we can read off from the classical equations (4.6) and (4.7). The amplitudes A_1, A_2 are connected to the dimensionless quadratures x, y via $x = \sqrt{\omega_0}A_1$ and $y = \sqrt{\omega_0}A_2$, as noted before. The nonlinear part of the action then reads

$$S_{\rm nl} = \epsilon\gamma \int dt \left[y^q x^c\big(|x^c|^2 + 3|y^c|^2\big) - x^q y^c\big(|y^c|^2 + 3|x^c|^2\big)\right]. \tag{4.60}$$

where we have set $f = 1$ since we are interested in the dynamics close to threshold. To find the effective action of the slow mode, we follow the procedure explained in the previous section. Thus, we have to evaluate $\langle S_{\rm nl}\rangle_f$. In particular we have to consider the second and fourth term in more detail as they couple the slow and fast mode. The first term is already a nonlinearity of the slow mode and the third term is only a constant after averaging with S_f. This means that we have to evaluate $\langle (y^c)^2\rangle_f$ and $\langle x^q y^c\rangle_f$. We obtain $\langle (y^c)^2\rangle_f = (n_0 + \frac{1}{2})/(1+f) \approx \frac{1}{2}(n_0 + \frac{1}{2})$ (see above) and $\langle x^q y^c\rangle_f = 0$. This means that the nonlinear part becomes

$$\langle S_{\rm nl}\rangle_f = \epsilon\gamma \int dt \left[y^q (x^c)^3 + 3\langle |y^c|^2\rangle_f y^q x^c\right]. \tag{4.61}$$

The second term contributes to the quadratic action part of the effective action $S_{\rm eff} = S_s + \langle S_{\rm nl}\rangle_f$ such that the total prefactor of $y^q x^c$ in the effective action $S_{\rm eff}$ equals $i\omega_1 + \frac{1}{2}\epsilon\gamma(n_0 + \frac{1}{2}) = \frac{1}{2}\gamma[1 + \epsilon(n_0 + \frac{1}{2}) - f]$. This term leads to a small shift of threshold from $f_* = 1 \mapsto \bar{f}_* = 1 + \epsilon(n_0 + \frac{1}{2})$. Thus, we find the total effective action $S_{\rm eff} = S_s + \langle S_{\rm nl}\rangle_f$ for the slow mode with the result

$$iS_{\rm eff} = \int dt \left[i y^q \dot{x}^c + i\gamma\, y^q \left[\tfrac{1}{2}(\bar{f}_* - f)x^c + \tfrac{i}{2}(n_0 + \tfrac{1}{2})y^q + \epsilon(x^c)^3\right]\right] \tag{4.62}$$

By comparing this with Eq. (2.61), we identify that this action is equivalent to the Fokker-Planck equation

$$\frac{\partial}{\partial t}P = \gamma\frac{\partial}{\partial x}\left[\frac{1}{2}(\bar{f}_* - f)x + \epsilon x^3 + \frac{1}{2}(n_0 + \tfrac{1}{2})\frac{\partial}{\partial x}\right]P \tag{4.63}$$

where we have set $x^c \equiv x$ and $y^q \equiv -i\partial/\partial x$. This model describes the dynamics of the parametric oscillator at threshold and can be compactly described by the Liouvillian $\mathcal{L}$ with $\dot{P} = \mathcal{L}P$. At this point, we do not yet have access to the photon counting statistics as we have not included the detector in the description. To do so, we repeat the procedure from above: The detector introduces a second decay rate γ_d which results in the total decay rate $\Gamma = \gamma + \gamma_d$. This raises the threshold from

[2] The additional factor ω_0 arises due to the fact that A_1 and A_2 are not canonically conjugate but have a 'commutator' ω_0.

$\bar{f}_* = 1$ to $\bar{f}_* = \Gamma/\gamma$. As before, the effect of the detector can be incorporated via the substitutions $\gamma \mapsto \Gamma$, $f \mapsto \bar{f} = \Gamma f/\gamma$, $\epsilon \mapsto \bar{\epsilon} = \Gamma\epsilon/\gamma$, and $n_0 \to \bar{n} = \gamma n_0/\Gamma$ in the equation (4.63) describing the slow dynamics. In addition, the detector introduces a counting term $\mathcal{L}_c = (e^{i\chi} - 1)\eta\Gamma\alpha^+\overline{\alpha^-}$ where we still have to reduce $\alpha^+\overline{\alpha^-}$ to the slow mode. Here, the fast and slow mode do not couple. Thus we can set $x^q = 0 = y^c$ and obtain $\alpha^+\overline{\alpha^-} \approx \frac{1}{2}(x^c + \frac{i}{2}y^q)^2$. The largest contribution of the photon current will be given by x_c^2 which will become clear in the following.

To continue, we rescale $x = x_* q$ with $x_* = \left[(\bar{n} + \frac{1}{2})/\bar{\epsilon}\right]^{1/4}$. With this we are able to bring the Fokker-Planck equation (4.63) into the form

$$\frac{\partial}{\partial t}P = \frac{1}{\tau_*}\frac{\partial}{\partial q}\left[-\beta q + q^3 + \frac{1}{2}\frac{\partial}{\partial q}\right]P \tag{4.64}$$

with the characteristic timescale $\tau_* = x_*^2/(\bar{n} + \frac{1}{2})\Gamma$ that represents the correlation time and the dimensionless distance to threshold $\beta = \Gamma\tau_*(\bar{f} - \bar{f}_*)/2$.

In terms of the rescaled variables, the counting term reads $\mathcal{L}_c = (e^{i\chi} - 1)\eta\Gamma\frac{1}{2}(x_* q + \frac{i}{2}p/x_*)^2$ where we have written $y^q = p/x_*$ such that p and q are conjugate variables with $[q, p] = [x^c, y^q] = i$. For small nonlinearities $\bar{\epsilon}$, the characteristic amplitude x_* becomes large as it scales with $\bar{\epsilon}^{-1/4}$. Thus, the largest contribution of the counting term is given by $\mathcal{L}_c = (e^{i\chi} - 1)\eta\Gamma x_*^2 q^2/2 = (e^{i\chi} - 1)(N_0/\tau_*)q^2 = (e^{i\chi} - 1)I$. Here, we have introduced the number of correlated photons $N_0 = \eta\Gamma\tau_* x_*^2/2$ and the photon current $I = (N_0/\tau_*)q^2$.

As above, we are interested in the photon counting statistics of the system. Before, we have solved the path integral to obtain the cumulant generating function. This has been possible because we looked at quadratic problems that can be solved by Gaussian integrals. Due to the nonlinearity, this is not possible anymore. However, we are able to approach the problem a bit differently. By solving the stationary Fokker-Planck equation where $\dot{P} = 0$, we are able to obtain the probability distribution $P_s(q)$. This enables us to obtain stationary expressions as the average photon current $\langle I \rangle$ and the second-order coherence for zero time delay $g^{(2)}(0)$ by evaluating

$$\langle I \rangle = \int dq\;\, I\, P_s(q) = \frac{N_0}{\tau_*}\int dq\;\, q^2 P_s(q) \tag{4.65}$$

$$g^{(2)}(0) = \frac{\langle I^2 \rangle}{\langle I \rangle^2} = \frac{\int dq\;\, q^4 P_s(q)}{\left[\int dq\;\, q^2 P_s(q)\right]^2}. \tag{4.66}$$

As shown in Sec. 2.5, the stationary probability distribution is given by $P_s(q) \sim \exp\left[\beta q^2 - q^4/2\right]$ (with proper normalization). We see that below threshold, with $\bar{f} < \bar{f}_*$ ($\beta < 0$), the maximum of the distribution is at $q = 0$. For $\bar{f} > \bar{f}_*$ ($\beta > 0$) the single maximum splits into two with $q = \pm\sqrt{\beta}$, which describes the (classical) pitchfork bifurcation.

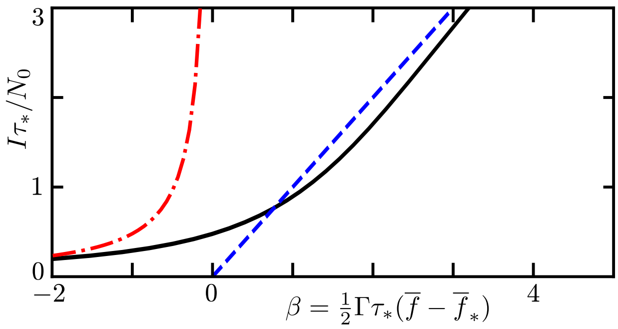

Figure 4.3: Comparison of the average photon current from the effective slow model derived in this section (black) to the expectation below threshold without the nonlinearity (red) and the expectation above threshold (blue). We see that the effective slow model correctly reflect the results from below and above threshold and connects the regimes.

For the average photon current, we obtain

$$\langle I \rangle = \frac{N_0}{\tau_*}\left(\beta + \frac{D_{1/2}(-\beta)}{D_{-1/2}(-\beta)}\right) \tag{4.67}$$

where $D_\nu(x) = \sqrt{2/\pi} e^{x^2/4} \int_0^\infty dt\, e^{-t^2/2} t^\nu \cos(xt - \nu\pi/2)$ is the parabolic cylinder function.

At this point, we want to derive the approximations valid below and above the threshold (with $|\beta| \gg 1$ in both cases). Below threshold, we have obtained that $\langle I \rangle = N_0/2\tau_*|\beta| = \eta\Gamma(1+2\bar{n})/[4(\bar{f}_* - \bar{f})]$. Above threshold, we have the classical expectation $\langle I \rangle = (N_0/\tau_*)\beta = \eta\Gamma(\bar{f} - \bar{f}_*)/\bar{\epsilon}$ corresponding to a coherent state in the symmetry broken state $q = \pm\sqrt{\beta}$. The comparison of all three results can be seen in Fig. 4.3. We see that the result form the effective slow model connects the results from below and above threshold.

Right at threshold ($\beta = 0$), we obtain $\langle I \rangle = 2\pi N_0/[\Gamma(1/4)^2\tau_*] \approx 0.478 N_0/\tau_*$. The second-order coherence at threshold reads $g^{(2)}(0) = 32\Gamma(5/4)^4/\pi^2 \approx 2.19$, meaning that the radiation is bunched and also stronger correlated than the radiation of a thermal oscillator (see above).

Chapter 5

Non-degenerate parametric oscillator

In the previous sections, we discussed a single oscillator that is parametrically driven at twice the resonance frequency. In literature, this system is sometimes referred to as a degenerate parametric oscillator. As previously discussed, a degenerate drive is only capable of phase-sensitive amplification. In contrast, a non-degenerate parametric drive can realize phase-insensitive amplification. A simple model for such a system consists of two oscillators — characterized by the resonance frequencies ω_a and ω_b as well as the dissipation rates γ_a and γ_b — that are coupled by a parametric drive at the sum of their two resonance frequencies. Here, the phase-insensitive amplification is realized for oscillator a, while oscillator b plays an auxiliary role and has a much higher dissipation rate $\gamma_b \gg \gamma_a$. A non-degenerate system with a bifurcation threshold $f_* = 1$ and an analogous anharmonic stabilization as in the previous sections is described by the classical equations of motion

$$\ddot{x}_a + \gamma_a \dot{x}_a + \omega_a^2 x_a + 2f\sqrt{\omega_a\gamma_a\omega_b\gamma_b}\,\sin\left[(\omega_a+\omega_b)t\right] x_b \Big(1 - 4\epsilon\omega_b\sqrt{\gamma_b/\gamma_a}\, x_b^2\Big) = 0, \tag{5.1}$$

$$\ddot{x}_b + \gamma_b \dot{x}_b + \omega_b^2 x_b + 2f\sqrt{\omega_a\gamma_a\omega_b\gamma_b}\,\sin\left[(\omega_a+\omega_b)t\right] x_a \Big(1 - 4\epsilon\omega_a\sqrt{\gamma_a/\gamma_b}\, x_a^2\Big) = 0\,, \tag{5.2}$$

where we have assumed that the stabilizing potential is provided by the parametric drive. As above, we are interested in the limit of small dissipation with $\gamma_a \ll \omega_a$ and $\gamma_b \ll \omega_b$ such that we can perform a rotating-wave approximation. Analogously to the previous section, we introduce

$$x_a(t) = \sqrt{2/\omega_a}\,\mathrm{Re}\left[\alpha(t)e^{-i\omega_a t}\right], \qquad x_b(t) = \sqrt{2/\omega_b}\,\mathrm{Re}\left[\beta(t)e^{-i\omega_b t}\right]. \tag{5.3}$$

After neglecting all fast-oscillating terms, we obtain

$$\dot{\alpha} = -\tfrac{1}{2}\gamma_a\alpha + \tfrac{1}{2}f\sqrt{\gamma_a\gamma_b}\,\overline{\beta} - 3f\epsilon\gamma_b|\beta|^2\overline{\beta}, \tag{5.4}$$

$$\dot{\beta} = -\tfrac{1}{2}\gamma_b\beta + \tfrac{1}{2}f\sqrt{\gamma_a\gamma_b}\,\overline{\alpha} - 3f\epsilon\gamma_a\,|\alpha|^2\overline{\alpha}. \tag{5.5}$$

The (classical) steady-state solutions are given by $\alpha_0 = |\alpha_0|e^{i\varphi_{a0}}$ and $\beta_0 = |\beta_0|e^{i\varphi_{b0}}$ with

$$N_a = |\alpha_0|^2 = \sqrt{\frac{\gamma_b}{\gamma_a}}\frac{f-1}{6f\epsilon}, \quad N_b = |\beta_0|^2 = \frac{\gamma_a}{\gamma_b}|\alpha_0|^2, \quad \text{and} \quad \varphi_{a0} + \varphi_{b0} = 0; \tag{5.6}$$

where N_a, N_b corresponds to the number of photons in the respective oscillator. In the regime $\gamma_b \gg \gamma_a$, we have $N_b \ll N_a$. Note that the steady-state solution for the relative phase $\phi = \varphi_{a0} - \varphi_{b0}$ is arbitrary, demonstrating that the phase-locking between the drive and the oscillators only locks the total phase. This property is qualitatively different to the degenerate parametric oscillator discussed above, where the degeneracy of the phase is only two-fold and the phase can take two values differing by π.

This qualitative difference is also the reason why a non-degenerate parametric drive can implement phase-insensitive amplification. To illustrate this point, we focus on the dynamics below the instability threshold $f < 1$ where the anharmonic terms in Eqs. (5.4) and (5.5) can be neglected. In the limit $\gamma_b \gg \gamma_a$, oscillator b with the complex amplitude β equilibrates much faster than oscillator a with the complex amplitude α. By comparison, we can thus set $\dot{\beta} = 0$ in Eq. (5.5) to obtain the $\beta = f\sqrt{\gamma_a/\gamma_b}\,\overline{\alpha}$. Inserting this relation into Eq. (5.4), we obtain

$$\dot{\alpha} = -\tfrac{1}{2}\gamma_a(1 - f^2)\alpha\,. \tag{5.7}$$

As $f \to f_* = 1$, the time-scale diverges indicating the threshold of parametric amplification. Note that different from the degenerate result in Eqs. (4.6) and (4.7), the state $\alpha = 0$ becomes instable independent of phase. Thus, the non-degenerate parametric drive amplifies the signal of the first oscillator independent of the phase of the signal.

5.1 Phase coherence above the instability threshold: stability of a laser

In this section, we investigate the phase coherence of a non-degenerate parametrically driven system above the instability threshold. This system is a model system for a laser. In particular, we concentrate on the mode a with the signal $x_a = A\cos(\omega_a t - \varphi_a)$. Below the threshold of instability, all the different phases φ_a of the signal are equivalent. Above the threshold and depending on the initial condition, the system 'chooses' one of the phases φ_a^*. This is different from the degenerate system, where there are only two possible solutions that are allowed as stationary solutions. The response $x_a = A\cos(\omega_a t - \varphi_a^*)$ is a lasing response where the system 'spontaneously' breaks the U(1) symmetry corresponding to the phase φ_a. In real systems, the phase information is lost over some time scale, called the dephasing time. In this section, we would like to study what limits the phase coherence of a laser above the threshold of instability.

For our calculations, we focus on the quantum regime with $k_B T \ll \omega_a, \omega_b$. As before, we are interested in the regime $\gamma_b \gg \gamma_a$ where oscillator b plays the role of a fast equilibrating auxiliary mode. In the previous section, we demonstrated that the non-degenerate drive locks only the total phase of both oscillators which results in a phase-insensitive amplification of oscillator a. However, if the system is driven across the bifurcation threshold non-adiabatically, a classical symmetry breaking occurs as the relative phase of both oscillators is spontaneously fixed by the prevalent fluctuations at the bifurcation point when the system relaxes to an essentially classical state on the timescale of $1/\gamma_a$. As the system evolves, quantum fluctuations wash out this initial phase coherence and ensure that the system eventually equilibrates back to a symmetry-conserving state. The aim of this section is to investigate the time-scale on which this process occurs.

The starting point is the quasi-classical action

$$\begin{aligned} S = \int dt \Big[& i\Big(\overline{\alpha^q}\dot{\alpha}^c - \tfrac{1}{2} f\sqrt{\gamma_a\gamma_b}\,\overline{\alpha^q \beta^c} + \tfrac{1}{2}\gamma_a \overline{\alpha^q}\alpha^c + 3f\epsilon\gamma_b|\beta^c|^2\overline{\alpha^q\beta^c} - \text{H.c.}\Big) \\ & + i\Big(\overline{\beta^q}\dot{\beta}^c - \tfrac{1}{2} f\sqrt{\gamma_a\gamma_b}\,\overline{\beta^q \alpha^c} + \tfrac{1}{2}\gamma_b \overline{\beta^q}\beta^c + 3f\epsilon\gamma_a|\alpha^c|^2\overline{\beta^q\alpha^c} - \text{H.c.}\Big) \\ & + \tfrac{i}{2}\gamma_a(1+2n_a)|\alpha^q|^2 + \tfrac{i}{2}\gamma_b(1+2n_b)|\alpha^q|^2 \Big]. \end{aligned} \tag{5.8}$$

that corresponds to the classical equations of motion in Eqs. (5.4) and (5.5), see Eq. (4.11).

As we are interested in the dynamics after the classical symmetry-breaking occurred, we expand the action around a classical above-threshold solution with $f > 1$ for a small nonlinearity $\epsilon \ll 1$ such that $N_a \gg N_b \gg 1$. We want to expand the action around the steady-state solution $\alpha = \alpha_0$, $\beta = \beta_0$. Since in the steady-state solution the relative phase is arbitrary, we have made the convenient choice $\varphi_{a0} = 0$ and $\varphi_{b0} = 0$.

We parameterize the (small) deviations from classical solutions conveniently via

$$\alpha^c = \sqrt{N_a + \delta_a^c}\, e^{i\phi^c/2} \approx \alpha_0\Big(1 + \frac{\delta_a^c}{2N_a} + \tfrac{i}{2}\phi^c\Big), \tag{5.9}$$

$$\beta^c = \sqrt{N_a + \delta_b^c}\, e^{i(\tilde{\phi}^c - \phi^c/2)} \approx \beta_0\Big(1 + \frac{\delta_b^c}{2N_b} + i(\tilde{\phi}^c - \tfrac{1}{2}\phi^c)\Big), \tag{5.10}$$

$$\alpha^q = \alpha_0 \left(\frac{\tilde{\delta}^q + 2\delta^q}{2N_a} + i\phi_a^q \right), \tag{5.11}$$

$$\beta^q = \beta_0 \left(\frac{\tilde{\delta}^q}{2N_b} + i\phi_b^q \right). \tag{5.12}$$

The deviations are parameterized by the deviation of the photons numbers $\delta_a^c, \delta_b^c, \delta^q, \tilde{\delta}^a$ and the phases $\phi_a^q, \phi_b^q, \phi^c, \tilde{\phi}^c$. We have chosen the factors such that the amplitudes and the phases are canonically conjugate, see below. The phases $\phi^c, \tilde{\phi}^c$ can be imagined as 'center of mass' phase $\tilde{\phi}^c$ and the relative phase ϕ^c that is not affected by the parametric amplification, see the discussion above. In the limit $\gamma_a \ll \gamma_b$, ϕ_b absorbs the value of $\tilde{\phi}^c$ such that α_c only depends on ϕ^c.

Up to second order in the amplitudes and phases, the action is given by

$$\begin{aligned} S_0 = \int dt \Big\{ &(\phi_a^q \dot{\delta}_a^c + \phi_b^q \dot{\delta}_b^c - \delta^q \dot{\phi}^c - \tilde{\delta}^q \dot{\tilde{\phi}}^c) + f(\gamma_a \phi_b^q \delta_a^c + \gamma_b \phi_a^q \delta_b^c) \\ &- \frac{\gamma_a}{2} \Big[(3\phi_b^q - \phi_a^q)\delta_a^c + (2\delta^q + \tilde{\delta}^q)\tilde{\phi}^c \Big] - \frac{\gamma_b}{2} \Big[(3\phi_a^q - \phi_b^q)\delta_b^c + \tilde{\delta}^q \tilde{\phi}^c \Big] \\ &+ \frac{i\gamma_a(1+2n_a)}{2} \left[N_a (\phi_a^q)^2 + \frac{(2\delta^q + \tilde{\delta}^q)^2}{4N_a} \right] + \frac{i\gamma_b(1+2n_b)}{2} \left[N_b(\phi_b^q)^2 + \frac{(\tilde{\delta}^q)^2}{4N_b} \right] \Big\}. \end{aligned} \tag{5.13}$$

Note that ϕ^c does not enter the action S_0 which corresponds to the fact that the phase-difference is arbitrary and not affected by the parametric driving.

In general, the correlation times of these variables are of the order of $1/\gamma_a$ or below. However, as one might expect from the previous degeneracy of the system, this is not the case for the phase difference ϕ^c. In fact, the only term of the action that depends on the phase difference is the dynamical term $-\int dt\, \delta^q \dot{\phi}^c = i \int (d\omega/2\pi)\, \omega (\delta^q)_{-\omega} (\phi^c)_\omega$ with the conjugate variable δ^q. As such, the action S_0 does not contain any stabilizing terms for the dynamics of the phase difference. This results in a zero mode of the action in the limit of long times $t \to \infty$ or small frequencies $\omega \to 0$. Which would correspond to an infinite coherence-time of the laser.

This zero mode is only present in the expanded action S_0. In order to figure out how terms beyond second-order give this mode a finite coherence time (which will correspond to the dephasing time of the laser), we consider the non-linear terms in a mean-field approximation. In particular, the most important contribution is a term of the form

$$S_\Gamma = -\Gamma \int dt\, \delta^q \phi^c \tag{5.14}$$

that will lead to a shift of the zero mode with the combined action $-\int dt\, \delta^q(\Gamma + \partial_t)\phi^c$ that corresponds to a dissipative dynamics with the phase information of ϕ^c decaying exponentially with the rate Γ, see below.

In the framework of Sec. 4.4.2, the mode δ^q, ϕ^c corresponds to a slow mode while the remaining modes are fast. To obtain a finite dephasing rate, the nonlinear terms have to involve a term of the form $-\int dt\, \delta^q \phi^c \mathcal{O}$ that couples $\delta^q \phi^c$ to another operator $\mathcal{O}$ with a finite expectation value $\langle \mathcal{O} \rangle_f$. In particular, the dephasing rate Γ is then given by $\langle \mathcal{O} \rangle_f$.

Expanding the action S to fourth order, we find such a term with

$$\mathcal{O} = \frac{3f\epsilon\sqrt{\gamma_a\gamma_b}}{4N_b}\left[(\delta_b^c)^2 + 12N_b^2(\tilde{\phi}^c)^2\right]. \tag{5.15}$$

In the limit $\gamma_b \gg \gamma_a$, $\epsilon \ll 1$, we are interested, we have $N_b \gg 1$. In this case, the second term proportional to $(\tilde{\phi}^c)^2$ dominates and we have

$$\Gamma = 9f\epsilon\sqrt{\gamma_a\gamma_b}N_b\,\langle(\tilde{\phi}^c)^2\rangle_f\,. \tag{5.16}$$

In order to evaluate $\langle(\tilde{\phi}^c)^2\rangle_f$ in the limit $\gamma_a \ll \gamma_b$, we can set $\gamma_a = 0$ and only consider the relevant part of the action in $\tilde{\phi}^c$ and the conjugate variable $\tilde{\delta}^q$,

$$S_f \approx \int dt\left[-\tilde{\delta}^q\dot{\tilde{\phi}}^c - \tfrac{\gamma_b}{2}\tilde{\delta}^q\tilde{\phi}^c + \tfrac{i}{8}\gamma_b(1+2n_b)(\tilde{\delta}^q)^2/N_b\right]. \tag{5.17}$$

Going to frequency space and performing the Gaussian integral, we obtain the result

$$\langle(\tilde{\phi}^c)^2\rangle_f = \frac{\gamma_b(1+2n_b)}{4N_b}\int\frac{d\omega}{2\pi}\frac{1}{\frac{1}{4}\gamma_b^2+\omega^2} = \frac{1+2n_b}{4N_b}\,. \tag{5.18}$$

With this, we obtain the dephasing rate

$$\Gamma = \frac{9f\epsilon(1+2n_b)\sqrt{\gamma_a\gamma_b}}{4} = \frac{3(f-1)(1+2n_b)}{8N_b}\gamma_a \tag{5.19}$$

valid for low temperatures and weak nonlinearities $\epsilon \ll 1$ such that N_b is large enough to guarantee that $\Gamma \ll \gamma_a$. Thus, the above the threshold, the long-time dynamics is dominated by the new effective timescale $1/\Gamma$.

We can include the finite decay rate Γ by adding S_Γ to the effective action S_0. We obtain the long-time correlator

$$\langle\phi^c(t)\phi^c(0)\rangle_{0+\Gamma} \approx \frac{(1+n_a+n_b)\gamma_a}{N_a\Gamma}e^{-\Gamma|t|}, \tag{5.20}$$

valid for times $|t| \gg 1/\gamma_a$ with $\Gamma \ll \gamma_a \ll \gamma_b$. The phase then evolves as

$$\begin{aligned}\langle(\delta\phi^c)^2\rangle = \langle[\phi^c(t)-\phi^c(0)]^2\rangle &= \frac{2(1+n_a+n_b)\gamma_a}{N_a\Gamma}(1-e^{-\Gamma|t|}) \approx \frac{2(1+n_a+n_b)}{N_a}\gamma_a|t| \\ &= \Gamma_\phi|t| \end{aligned}\tag{5.21}$$

with the diffusion rate

$$\Gamma_\phi = \frac{2(1+n_a+n_b)}{N_a}\gamma_a \ll \gamma_a \tag{5.22}$$

relevant for intermediate times $1/\Gamma \gg |t| \gg 1/\gamma_a$, similar to the limit of applicability of Fermi's golden rule.

The dephasing rate Γ_ϕ is connected to the decay of the correlation function of the mode α. If we calculate the first-order correlation function of $\alpha^c \approx \alpha_0 e^{i\phi^c/2}$ in the long time limit, we obtain (by averaging over $S_0 + S_\Gamma$)

$$g^{(1)}(t) \approx \frac{\langle \overline{\alpha^c(t)}\alpha^c(0)\rangle}{\langle |\alpha^c(0)|^2\rangle} = \langle e^{-i\delta\phi^c/2}\rangle = e^{-\langle(\delta\phi^c)^2\rangle/8} = e^{-\Gamma_\phi |t|/8}\,, \tag{5.23}$$

where we used in the third equality the fact that for Gaussian averages, only the first and second cumulants contribute. Analogously, the second-order coherence can be approximated as

$$g^{(2)}(t) \approx \frac{\langle |\alpha^c(t)\alpha^c(0)|^2\rangle}{\langle |\alpha^c(0)|^2\rangle^2} = 1 + |g^{(1)}(t)|^2 = 1 + e^{-\Gamma_\phi |t|/4}. \tag{5.24}$$

Note that the decay of $g^{(1)}$ corresponds to the finite phase-coherence of a realistic laser as an ideal laser has the coherence function $g^{(1)}(t) = 1$ for all times. In particular, the Fourier transform of $g^{(1)}(t)$ corresponds to the spectrum of the emitted radiation. It is give by the Lorentzian form

$$n_\omega \propto \frac{1}{\omega^2 + (\Gamma_l/2)^2} \tag{5.25}$$

with the line-width

$$\Gamma_l = \frac{\Gamma_\phi}{4} = \frac{1 + n_a + n_b}{2N_a}\gamma_a \tag{5.26}$$

of the laser.

It is instructive to compare this result with the classic Schawlow-Townes limit

$$\Gamma_l \geq \Gamma_{\text{ST}} = \frac{\hbar\omega_a\gamma_a^2}{2P_{\text{emit}}} \tag{5.27}$$

for a laser, derived using semiclassical arguments. Here, ω_a is the frequency, γ_a the natural linewidth, P_{emit} the emission power of the laser. For us the emitted photon current (from the a cavity) is given by $I_{\text{emit}} = N_a\gamma_a$. As each photon carries the energy $\hbar\omega_a$, the emitted power assumes the form $P_{\text{emit}} = N_a\gamma_a\hbar\omega_a$. Thus, our linewidth (5.26) assumes the form

$$\Gamma_l = \frac{(1 + n_a + n_b)\hbar\omega_a\gamma_a^2}{2P_{\text{emit}}} = (1 + n_a + n_b)\Gamma_{\text{ST}} \geq \Gamma_{\text{ST}}\,. \tag{5.28}$$

Note that the Schawlow-Townes limit corresponds to low temperatures when only shot noise due to zero-point fluctuations remains.