Supercurrent diode effect

bachelor's thesis

Submitted to the Faculty of Mathematics, Computer Science and Natural Sciences at RWTH Aachen University

presented by

Yves Rottstaedt

under the supervision of

Prof. Dr. Fabian Hassler
and
Prof. Dr. Markus Müller

07/2022

Abstract

This thesis investigates the supercurrent diode effect. In particular, we interpret a supercurrent as a Galilean boost on the Schrödinger equation and investigate the effect on the energy gap. We look at time symmetry and parity and obtain symmetry conditions for the supercurrent diode effect. Both previous mentioned symmetries have to be broken in order to get the supercurrent diode effect. With the helical superconductor as an example, we calculate the eigenenergies for the single particle Hamiltonian analytically and obtain the band structure for the superconducting system including a magnetic field and a supercurrent with first order perturbation theory improving previous approximations.

Contents

Chapter 1

Introduction

The subject of this is thesis is the supercurrent diode effect (SDE). The effect was introduced in 2021 and examined by e.g. Liang Fu and Noah F. Q. Yuan [1], which will be a reference used later in this thesis. The SDE can be described as a direction dependend critical current for superconductors. This means that if the criteria for the SDE are fulfilled, the value of a critical current under which the superconducting phase collapses is not the same for all directions, which should be the case for regular superconductors. In particluar, we will investigate the requirements for the SDE, especially looking at the symmetries and mathmetically understanding a supercurrent. Moreover, we will move on to the example of a two dimensional helical superconductor with an in plane magnetic field and analyze the band structure with and without a supercurrent.

We will start in Ch. 2 with a short introduction in the microscopic theory of superconductivity and the concept of the Bogoliubov-de Gennes Hamiltonian. In Ch. 3 we will interpret a supercurrent as a Galilean boost on the Schrödinger equation and use this information to understand the impact on the Bogoliubov-de Gennes Hamiltonian. Furthermore, we will look at the parity and time symmmetry and how breaking them could be necessary for the SDE. After that, we will move on in Ch. 4 to the example of the helical superconductor. The first part of this chapter will be the analysis of the band structure and looking at the single particle Hamiltonian and then the Bogoliubov-de Genne Hamiltonian. The effects of the external magnetic field and the supercurrent will be calculated with perturbation theory. Finally, we will return to the symmetry considerations but in the context of the helical superconductor.

Chapter 2

Bardeen-Cooper-Schrieffer theory of superconductivity

Superconductivity was discovered in 1911 by Heike Kamerlingh Onnes. He took a mixture of helium and mercury and cooled it down to $4\,K$ and found an abrupt drop of the specific resistance of the mercury. A microscopic theory has been developed by Bardeen, Cooper and Schrieffer and is therefore called BCS theory. The basis lies in the observation of Cooper, that any weak attracting potential between two electrons can lead to a bounded state of them. This chapter includes a closer look on the theory and introduces the concept of Bogoliubov-de Gennes Hamiltonians, which will be essential for later calculations. The chapter is an adaption of [2].

2.1 BCS theory

In order to analyze the supercurrent diode effect, we have to understand the theory of superconductivity. Since a supercurrent has no resistance, it can be described as a superfluid. An analogon in quantum theory is the Bose Einstein condensate, where we observe a phase transition of an ideal bose gas for small temperatures. It stands to reason to find a bosonic characterization where two electrons in the superconductor form a bosonic state called cooper pair. The BCS theory describes how an effective attracting pair interaction between two electrons leads to these bosonic states which have, in opposite to a superfluid, a charge. First of all, we look on a Hamiltonian of fermions with an effective attracting potential V_{eff} in second quantization

$$H_{\text{pair}} = \sum_{\boldsymbol{k}\sigma} (\varepsilon_{\boldsymbol{k}} - \mu) c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \sum_{\boldsymbol{kl}} V_{\text{eff}}(\boldsymbol{k} - \boldsymbol{l}) c^{\dagger}_{\boldsymbol{k}\uparrow} c^{\dagger}_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{l}\downarrow} c_{\boldsymbol{l}\uparrow} \,. \tag{2.1}$$

The first term describes the electrons with energy $\xi_{\boldsymbol{k}} = \varepsilon_{\boldsymbol{k}} - \mu$ without interaction, where $\varepsilon_{\boldsymbol{k}}$ is the single particle energy and μ is the chemical potential. The second term describes the electron-electron interaction as a scattering process between cooper pairs. When the cooper pairs condense they form coherent states and thus the operator $c_{-\boldsymbol{k}\downarrow}c_{\boldsymbol{k}\uparrow}$ gets a finite expected value $b_{\boldsymbol{k}}$. Because of the macroscopic occupation number, we expect low fluctuations and can use the Mean-field ansatz

$$c_{-\boldsymbol{k}\downarrow}c_{\boldsymbol{k}\uparrow} = b_{\boldsymbol{k}} + c_{-\boldsymbol{k}\downarrow}c_{\boldsymbol{k}\uparrow} - b_{\boldsymbol{k}} = b_{\boldsymbol{k}} + \delta\,, \tag{2.2}$$

where $\delta = c_{-\boldsymbol{k}\downarrow}c_{\boldsymbol{k}\uparrow} - b_{\boldsymbol{k}} \ll b_{\boldsymbol{k}}$. Using the Mean-field ansatz in Eq. (2.1) and neglecting quadratic terms of δ we gain the BCS Hamiltonian

$$H_{\text{BCS}} = \sum_{\boldsymbol{k}\sigma} \xi_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \sum_{\boldsymbol{kl}} V_{\text{eff}}(\boldsymbol{k}-\boldsymbol{l})(c^{\dagger}_{\boldsymbol{k}\uparrow}c^{\dagger}_{-\boldsymbol{k}\downarrow} b_{\boldsymbol{l}} + b^{\star}_{\boldsymbol{k}} c_{-\boldsymbol{l}\downarrow}c_{\boldsymbol{l}\uparrow} - b^{\star}_{\boldsymbol{k}} b_{\boldsymbol{l}}) \tag{2.3}$$

with the self consistency $b_{\boldsymbol{k}} = \langle c_{-\boldsymbol{k}\downarrow}c_{\boldsymbol{k}\uparrow}\rangle$. In this representation, we no longer have particle number conservation, but for a big number of particles μ specifies the number of particles and fluctuations can be neglected like in the grand canonical ensemble. Furthermore, we define the energy gap

$$\Delta_{\boldsymbol{k}} = -\sum_{\boldsymbol{l}} V_{\text{eff}}(\boldsymbol{k}-\boldsymbol{l}) b_{\boldsymbol{l}} \tag{2.4}$$

and obtain another representation of the BCS Hamiltonian

$$H_{\text{BCS}} = \sum_{\boldsymbol{k}\sigma} \xi_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} - \sum_{\boldsymbol{k}} (\Delta_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\uparrow}c^{\dagger}_{-\boldsymbol{k}\downarrow} + \Delta^{\star}_{\boldsymbol{k}} c_{-\boldsymbol{k}\downarrow}c_{\boldsymbol{k}\uparrow} - b^{\star}_{\boldsymbol{k}}\Delta_{\boldsymbol{k}})\,. \tag{2.5}$$

However, this representation does not allow us to directly get an idea of the spectrum. Therefore, we can define a new set of fermionic operators $\beta_{\boldsymbol{k}\sigma}$ diagonalizing the Hamiltonian. These operators can be found by doing a Bogoliubov-Valentin transformation and looking at the Bogoliubov-de Gennes(BdG) Hamiltonian (see Sec. 2.2). The diagonalized form reads as follows

$$H = \sum_{\boldsymbol{k}} (\xi_{\boldsymbol{k}} - E_{\boldsymbol{k}} + \Delta_{\boldsymbol{k}} b^{\star}_{\boldsymbol{k}}) + \sum_{\boldsymbol{k}\sigma} E_{\boldsymbol{k}} \beta^{\dagger}_{\boldsymbol{k}\sigma}\beta_{\boldsymbol{k}\sigma}\,, \tag{2.6}$$

with the Energy $E_{\boldsymbol{k}} = \sqrt{\xi^2_{\boldsymbol{k}} + |\Delta_{\boldsymbol{k}}|^2}$. The first term describes the base state energy and the second term the excitation levels, in this case as creation and annihilation of quasi particles by the operators $\beta_{\boldsymbol{k}\sigma}$. We can observe a gap in the band structure if the square of the $|\Delta_{\boldsymbol{k}}|^2$ is positive.

2.2 Bogoliubov-de Gennes Hamiltonian

General quadratic Hamiltonians without particle number conservation can be written as

$$H = \sum_{i=1}^{n}\sum_{j=1}^{n}(h_{ij}c_i^\dagger c_j + \frac{1}{2}\Delta_{ij}c_i^\dagger c_j^\dagger + \frac{1}{2}\Delta_{ij}^\star c_j c_i) = \boldsymbol{c}^\dagger h \boldsymbol{c} + \frac{1}{2}(\boldsymbol{c}^\dagger \Delta \boldsymbol{c}^\dagger + \boldsymbol{c}\Delta^\dagger \boldsymbol{c}) \,, \quad (2.7)$$

with n being the number of fermionic modes, h the particle Hamiltonian and the terms with Δ describing the interaction. In order to diagonalize the Hamiltonian, it is customary to write it in a quadratic form by using vectors of fermionic operators as seen in Eq. (2.7) and a matrix called BdG Hamiltonian H_{BdG}. Because it is also used later, we will choose a basis, $\boldsymbol{\Psi}_{\boldsymbol{k}} = \left(c_{\boldsymbol{k}\uparrow}, c_{\boldsymbol{k}\downarrow}, -c^\dagger_{-\boldsymbol{k}\uparrow}, c^\dagger_{-\boldsymbol{k}\downarrow}\right)^T$, where the BdG Hamiltonian contains the time inverted version of the particle Hamiltonian. So we obtain

$$H = \frac{1}{2}\boldsymbol{\Psi}_{\boldsymbol{k}}^\dagger \begin{pmatrix} h & \Delta \\ \Delta^\star & -T^{-1}hT \end{pmatrix} \boldsymbol{\Psi}_{\boldsymbol{k}} + \frac{1}{2}\mathrm{tr}\, h \,. \quad (2.8)$$

We see that the BdG Hamiltonian seems to have double the amount of degrees of freedom than the Hamiltonian we started with. This issue is dealt with by taking a look at the symmetries. We have a particle-hole symmetry in our superconducting system. For this case the associated operator is $C = \tau_y iK$ where K is the operator of complex conjugation and τ_y is acting on the block structure. C anticommutates with the Hamiltonian $\{H_{\mathrm{BdG}}, C\} = 0$. Therefore it is not a real symmetry, since the operators anticommuate but ensures, that we do not have an issue with the degrees of freedom[1].

When diagonalizing H_{BdG} we obtain a transformation for the Hamiltonian $H_{\mathrm{BdG}} = U^\dagger D U$, where D is the diagonalized Hamiltonian and U the transformation. If we let the transformation act on the fermionic operators instead of the Hamiltonian, we get new operators $\boldsymbol{\Phi}_{\boldsymbol{k}} = U\boldsymbol{\Psi}_{\boldsymbol{k}}$ being a linear combination of the previous ones.

[1]Unlike in semiconductor pyhsics, particles and holes are here redudant copies from each other

Chapter 3

Supercurrent and supercurrent diode effect

A supercurrent with a static flow rate can be realized as a Galilean boost on the Schrödinger equation. If we consider a system which is large enough to neglect edge effects, the Schrödinger equation should be invariant under the transformation. The associated generator and effects on the BdG Hamiltonian will be discussed further in this chapter. Another interesting aspect are the conservation and break of symmetries that go along with the SDE. Especially parity and time symmetry will be needed later, so we will take a look on them in this the chapter aswell.

3.1 Galilean transformation

In this section we will use the Hamiltonian $H = T$, only containing the term of kinetic energy[1] for our electrons in the superconductor, since an additional external potential would give no extra information due to the translation symmetry. First, we transform the Schrödinger equation for the electrons and holes and then take a look on the superconducting system with a transformation on H_{BdG}. The generator for a Galilean transformation, where we have no shift in time and location, but a uniform motion with velocity $\boldsymbol{v}$ can be found in literature [3] as

$$G_{\text{lit}} = \exp\left[i\boldsymbol{v} \cdot (m\boldsymbol{r} - \boldsymbol{p}t)/\hbar\right] . \tag{3.1}$$

We demand, that the Schrödinger equation has to be invariant under the transformation $\psi = G_{\text{lit}}\psi'$. However, when transforming the Schrödinger equation,

[1]The additional term would not be affected by the generator in the location representation and stay invariant under the transformation

we obtain another term

$$\begin{aligned} G_{\text{lit}}^{\dagger}\left(i\hbar\partial_t + \frac{\hbar^2\Delta}{2m}\right)G_{\text{lit}}\psi' &= 0 \\ \frac{1}{2}m\boldsymbol{v}^2\psi' + \left(i\hbar\partial_t + \frac{\hbar^2\Delta}{2m}\right)\psi' &= 0\,, \end{aligned} \tag{3.2}$$

such that the generator must contain an additional phase which is given by

$$G_{el} = \exp\left\{i\left[\boldsymbol{v}\cdot(m\boldsymbol{r} - \boldsymbol{p}t) - \frac{1}{2}m\boldsymbol{v}^2 t\right]/\hbar\right\}. \tag{3.3}$$

This is the final generator for our Galilean transformation and we can interpret the different parts. The first part contains the change of the position, since the system is in motion while the second term describes the additional momentum of the system. Moreover, we have additional kinetic energy because of the motion, which is described in the last term.

When performing a transformation for the holes, it stands near to derive the associated generator from Eq. (3.3). The simplest way to transform from electrons to holes is by negating charge and time. As the charge does not occur in the generator, we negate the time and obtain the ansatz

$$G_{ho} = \exp\left\{-i\left[\boldsymbol{v}\cdot(m\boldsymbol{r} + \boldsymbol{p}t) - \frac{1}{2}m\boldsymbol{v}^2 t\right]/\hbar\right\} \tag{3.4}$$

for the generator. It is noticable that velocity, momentum and time change their sign. To verify Eq. (3.4) we have to perform a transformation $\psi = G_{\text{ho}}\psi'$ and demand invariance. Since we consider a time inverted system, the Schrödinger equation has to be time inverted too, resulting in

$$\begin{aligned} G_{\text{ho}}^{\dagger}\left(i\hbar\partial_t - \frac{\hbar^2\Delta}{2m}\right)G_{\text{ho}}\psi' &= 0 \\ \left(i\hbar\partial_t - \frac{\hbar^2\Delta}{2m}\right)\psi' &= 0 \end{aligned} \tag{3.5}$$

and verifying our ansatz.

With the generators for the electrons and holes we can now formulate our problem in the context of the BdG Hamiltonian and then add the superconduction to observe the resulting effects. As already introduced in Ch. 1, we can write the Schrödinger equation as

$$i\hbar\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & -T^{-1}HT \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.6}$$

where u and v are the wave functions and the BdG Hamiltonian contains the aforementioned single particle Hamiltonian for the electrons respectively holes. The time inversion of the single particle Hamiltonian has no effect because we only consider the kinetic energy, which is invariant under the time inversion. To perform a Galilean transformation in this formulation, we have to build the generator which is simply given by

$$G = \begin{pmatrix} G_{el} & 0 \\ 0 & G_{ho} \end{pmatrix}, \tag{3.7}$$

with the generators for electrons and holes as calculated before. Now adding the superconduction and performing the transformation $\boldsymbol{\Psi} = G\boldsymbol{\Psi'}$, where $\boldsymbol{\Psi}$ is the vector containing u and v. This yields

$$G^\dagger \left(i\hbar\partial_t - H_{\mathrm{BdG}}\right) G\boldsymbol{\Psi'} = 0\,. \tag{3.8}$$

Now H_{BdG} also includes the energy gap terms like in Eq. (2.8). We can see that the transformation leads to a location dependency of the energy gap terms

$$\begin{aligned} \Delta &\to G_{el}^\dagger \Delta G_{ho} = \Delta \exp\left[i(-2m\boldsymbol{v}\cdot\boldsymbol{r} + m\boldsymbol{v}^2 t)/\hbar\right] \\ \Delta^\star &\to G_{ho}^\dagger \Delta^\star G_{el} = \Delta^\star \exp\left[i(2m\boldsymbol{v}\cdot\boldsymbol{r} - m\boldsymbol{v}^2 t)/\hbar\right]\,. \end{aligned} \tag{3.9}$$

The phase with the time dependency is just a general shift in the energy and will vanish under a gauge leaving us with a periodic modulation with the velocity of the Galilean transformation. This result was also observed in [1].

3.2 Symmetry and supercurrent diode effect

When analyzing an effect such as the SDE, it is important to look at it from a more general aspect in order to make predictions about the system without having to perform detailed calculations. In this case, we look at the conservation and break of time symmetry and parity. When looking at the symmetries from a band structure $E(\boldsymbol{k})$ perspective, both time inversion and parity would mirror the band structure at the energy axis because $\boldsymbol{k}$ would change its sign. When applying a supercurrent to collapse the superconduction, the energy gap will be closed. As the supercurrent has a direction, the band structure will close only in the direction of the applied current. To get an SDE the energy should not be the same for every direction and thus the band structure would not be symmetric around the energy axis. Hence, we have to analyze the symmetry conditions resulting in the SDE.

Considering the basis $\boldsymbol{\Psi}_{\boldsymbol{k}}$ introduced in Sec. 2.2, a spin interaction can either conserve time symmetry or parity. We can assure this by taking a look on the angular momentum being analogous to a spin. Time inversion flips an angular momentum, because the momentum changes its sign but the location vector does not. However, parity does not flip an angular momentum, since both momentum and location change their sign. Thus, a spin interaction can either conserve time symmetry if the interacting term also conserves time symmetry, or conserves parity if the interacting term breaks the time symmetry. A Hamiltonian conserving both time symmetry and parity would have degenerated eigenenergies. Breaking one of the aforementioned symmetries would cancel the degeneracy, but since one of the symmetries will still be conserved, the band structure will be symmetric around the energy axis. Only breaking both symmetries will result in an asymmetric band structure and therefore only then the SDE will be possible. A more mathematical approach will be discussed later when looking at the example of the helical superconductor in Ch. 4.

Chapter 4

Helical superconductivity

We will start this chapter by going further into the details working with the helical superconductor as an example for the SDE. The helical superconductor is a two dimensional material with a spin-orbit coupling(SOC). In order to analyze the SDE we will start by diagonalizing the single particle Hamiltonian and the BdG Hamiltonian. When having the band structure, we will add the magnetic field and the supercurrent as a perturbation and finally conisder the SDE from a symmetry perspective.

4.1 Single particle Hamiltonian

As mentioned before, the helical superconductor is an electron system with spin-orbit coupling and in order to have a SDE, also an additional magnetic field (see Sec. 4.3). The Hamiltonian is in first quantization given by

$$H = \underbrace{\frac{\boldsymbol{k}^2}{2m} - \mu}_{\xi_{\boldsymbol{k}}} + \alpha_R \left(\boldsymbol{e}_z \times \boldsymbol{k}\right) \cdot \boldsymbol{\sigma} + \boldsymbol{B} \cdot \boldsymbol{\sigma} \,, \tag{4.1}$$

with the chemical potential μ and the spin vector ***sigma*** with pauli matrices in each row. The second term is the Rashba spin-orbit coupling with parameter α_R. Because the material is two dimensional and we need an in plane magnetic field for the SDE, $\boldsymbol{k}$ and $\boldsymbol{B}$ only have non-zero components in $x-$ and $y-$direction.$\xi_{\boldsymbol{k}}$ also contains an identity matrix, which is not written due to clarity.

We see that the spinor basis with $\boldsymbol{k}$ and σ_z as quantum numbers should diagonalize the Hamiltonian. Therefore, our eigenvalue problem can be written as

$$\det\begin{pmatrix} \xi_{\boldsymbol{k}} - \lambda & (B_x - iB_y) - \alpha_R(k_y + ik_x) \\ (B_x + iB_y) - \alpha_R(k_y - ik_x) & \xi_{\boldsymbol{k}} - \lambda \end{pmatrix} = 0\,, \tag{4.2}$$

where λ is the parameter to be solved for and giving the eigenenergies.
Since the matrix is just quadratic, we can easily figure out the eigenenergies

$$E_\pm = \xi_{\boldsymbol{k}} \pm \sqrt{(B_x - \alpha_R k_y)^2 + (B_y + \alpha_R k_x)^2}\,. \tag{4.3}$$

The first term contains the band structure for free electrons and the chemical potential. The second term describes the additional effects due to the spin-orbit coupling and the magnetic field, where we observe a $\boldsymbol{k}$-dependency only for the spin-orbit terms.

However, the form of Eq. (4.3) is not really intuitive to see the effect of each term (SOC and magnetic field). Since the momentum and magnetic field are two dimensional, it stands near to use polar coordinates. We write $\boldsymbol{k} = ke^{i\varphi}$ and $\boldsymbol{B} = Be^{i\vartheta}$. This gives us

$$E_\pm = \xi_{\boldsymbol{k}} \pm \sqrt{B^2 + \alpha_R^2 k^2 + 2B\alpha_R k \sin(\vartheta - \varphi)}\,. \tag{4.4}$$

Now we have three terms under the root. The first term is just the split of the energy bands due to the magnetic field. The second term adds a linearity to the band structure depending on the value of α_R. However, most interesting is the third term depending on the phase difference between momentum and magnetic field. When being parallel or antiparallel the term has no effect, but otherwise it affects the band structure. We get a first glance at what will lead us to the SDE and considering Sec. 3.2 we notice, that only if time symmetry and parity are broken, we obtain the last term leading to asymmetry in the band structure.

Aditionally, we can calculate the eigenfunctions as

$$(H - E_\pm)\psi_\pm = 0\,,\ \psi_\pm = \begin{pmatrix} u \\ v \end{pmatrix} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\,, \tag{4.5}$$

resulting in

$$\begin{aligned} \psi_\pm &= \frac{1}{N} \begin{pmatrix} Be^{-i\vartheta} - i\alpha_R k e^{-i\varphi} \\ \mp\sqrt{B^2 + \alpha_R^2 k^2 + 2B\alpha_R k \sin(\vartheta - \varphi)} \end{pmatrix} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \,, \\ N &= \sqrt{2} \cdot \sqrt{B^2 + \alpha_R^2 k^2 + 2B\alpha_R k \sin(\vartheta - \varphi)} \,. \end{aligned} \tag{4.6}$$

We obtain periodically modulated eigenvectors of the matrix diagonalized before. The normalization with the factor N is actually not necessary since the oscillating factor cannot be normalized.

4.2 BdG Hamiltonian

After solving the single particle problem, we are now moving on to the superconducting system. Therefore, we have to look at the BdG Hamiltonian

$$H_{\text{BdG}} = \begin{pmatrix} H & \Delta \\ \Delta^\star & T^{-1}HT \end{pmatrix} \,, \tag{4.7}$$

where H is the single particle Hamiltonian given in Eq. (4.1) and the transformation T is the time inversion. The associated basis was introduced in Sec. 2.2. Because we are dealing with a 4x4 matrix, the whole calculation of the diagonalization can be found in appendix A. When trying to find the roots of the characteristic polynomial analytically, we obtain a non biquadratic degree 4 polynomial. However, when leaving out the magnetic field ($\boldsymbol{B} = 0$) the characteristic polynomial becomes biquadratic. Thus, we will solve the problem first without the magnetic field and then add the magnetic field as a perturbation. Similar behaviour can be observed when adding a supercurrent to the system. In order to keep the notation clear, we will refer to the Hamiltonian H as the single particle Hamiltonian Eq. (4.1) without the magnetic field. The resulting eigenenergies for H_{BdG} are

$$E_{\pm,\pm} = \pm\sqrt{(\xi_{\boldsymbol{k}} \pm |\boldsymbol{g}|)^2 + |\Delta|^2} = \pm\Lambda_\pm \,, \tag{4.8}$$

with $\boldsymbol{g} = \alpha_R \, \boldsymbol{e}_z \times \boldsymbol{k}$. The first term under the square root is the square of the eigenenergy Eq. (4.3) with $\boldsymbol{B} = 0$ and the other term is the square of the energy gap. Therfore, the band structure is similar, but with an extra energy shift coming from the gap.

In order to calculate the corrections, we have to examine the perturbations to decide whether we must use degenerated or non degenerated perturbation theory. The perturbations are given as

$$W^B = \begin{pmatrix} \boldsymbol{B}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{B}\cdot\boldsymbol{\sigma} \end{pmatrix} \tag{4.9}$$

for the magnetic field and

$$W^q = \begin{pmatrix} \frac{\boldsymbol{q}\cdot(\boldsymbol{k}+\frac{\boldsymbol{q}}{4})}{2m} + \frac{\alpha_R}{2}(\boldsymbol{e}_z \times \boldsymbol{q})\cdot\boldsymbol{\sigma} & 0 \\ 0 & \frac{\boldsymbol{q}\cdot(\boldsymbol{k}-\frac{\boldsymbol{q}}{4})}{2m} + \frac{\alpha_R}{2}(\boldsymbol{e}_z \times \boldsymbol{q})\cdot\boldsymbol{\sigma} \end{pmatrix} \tag{4.10}$$

for a supercurrent $\frac{\boldsymbol{q}}{2}$. The supercurrent is an external effect like the magnetic field and therefore not affected by the time inversion. Magnetic field and supercurrent have to be small in comparison to the matrix elements of H_{BdG}. Moreover, we can use non degenerated perturbation theory if $\boldsymbol{k}$ and σ_z are still suitable quantum numbers. Because we assume an infinitely extended material, we have translation symmetry and $\boldsymbol{k}$ is a good quantum number. If the translation symmetry is not broken we can still use $\boldsymbol{k}$. Since we assume a homogenic magnetic field, it does not break the translation symmetry. The supercurrent does not break the symmtery as the Schrödinger equation is invariant under the Galilean transformation and the material is infinitely extended. Thus, we can use non degenerated perturbation theory.

Perturbation theory requires the eigenfunctions of Eq. (4.7) and because of its structure, we will only discuss the case $E_{+,\pm} = \Lambda_\pm$ further. Thus, we have to solve

$$(H - \Lambda_\pm)\boldsymbol{\Psi}_\pm = 0\,,\; \boldsymbol{\Psi}_\pm = (a_\pm, b_\pm, c_\pm, d_\pm)^T\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\,. \tag{4.11}$$

When calculating the vector elements we obtain two fractions

$$\begin{aligned} \frac{b_\pm}{c_\pm} &= \frac{2i\xi_{\boldsymbol{k}} k\Delta e^{i\varphi}}{(\xi_{\boldsymbol{k}} - \Lambda_\pm)(\xi_{\boldsymbol{k}}^2 - \Lambda_\pm^2) - (\xi_{\boldsymbol{k}} + \Lambda_\pm)k^2 + (\xi_{\boldsymbol{k}} - \Lambda_\pm)|\Delta|^2}\,, \\ \frac{a_\pm}{d_\pm} &= \frac{-2i\xi_{\boldsymbol{k}} k\Delta e^{-i\varphi}}{(\xi_{\boldsymbol{k}} - \Lambda_\pm)(\xi_{\boldsymbol{k}}^2 - \Lambda_\pm^2) - (\xi_{\boldsymbol{k}} + \Lambda_\pm)k^2 + (\xi_{\boldsymbol{k}} - \Lambda_\pm)|\Delta|^2}\,. \end{aligned} \tag{4.12}$$

Since the fractions are not explicit, we have to define the components in a way that they solve Eq. (4.11). Furthermore, $\boldsymbol{\Psi}_\pm$ should be normalized. This leads us to

$$\mathbf{\Psi}_\pm = \frac{e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}}{N_\pm} \begin{pmatrix} 2\xi_{\boldsymbol{k}} k\Delta e^{-i\varphi/2} \\ \pm 2i\xi_{\boldsymbol{k}} k\Delta e^{i\varphi/2} \\ \pm\left[(\xi_{\boldsymbol{k}}-\Lambda_\pm)(\xi_{\boldsymbol{k}}^2-\Lambda_\pm^2)-(\xi_{\boldsymbol{k}}+\Lambda_\pm)k^2+(\xi_{\boldsymbol{k}}-\Lambda_\pm)|\Delta|^2\right] e^{-i\varphi/2} \\ i\left[(\xi_{\boldsymbol{k}}-\Lambda_\pm)(\xi_{\boldsymbol{k}}^2-\Lambda_\pm^2)-(\xi_{\boldsymbol{k}}+\Lambda_\pm)k^2+(\xi_{\boldsymbol{k}}-\Lambda_\pm)|\Delta|^2\right] e^{i\varphi/2} \end{pmatrix}, \tag{4.13}$$

with $N_\pm = 4\xi_{\boldsymbol{k}} k\sqrt{\Lambda_\pm(\Lambda_\pm - \xi_{\boldsymbol{k}} \mp k)}$.

Now moving on to the calculation of the correction, perturbation theory states, that the energy is in first order given by

$$\varepsilon_\pm^{B/q} = \Lambda_\pm + \langle \Lambda_\pm \mid W^{B/q} \mid \Lambda_\pm \rangle \,, \tag{4.14}$$

where the states $|\Lambda_\pm\rangle$ are the eigenstates to the energies $\Lambda_\pm$ and in our representation given by Eq. (4.13).

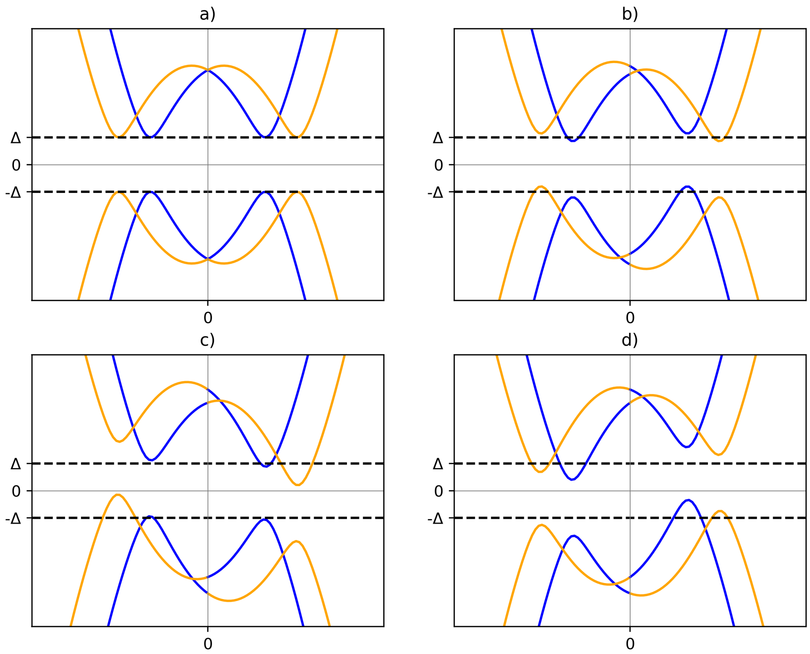

Figure 4.1: Band structure for the helical superconductor approximated in first order perturbation theory: a) helical superconductor without magnetic field and without supercurrent, b) helical superconductor with magnetic field and without supercurrent, c) helical superconductor with magnetic field and with positive supercurrent, d) helical superconductor with magnetic field and with negative supercurrent. We use m = 1, $\mu = 1$, $\alpha_R = 1$, $\Delta = 3$, $\boldsymbol{k} = k_x \boldsymbol{e}_x$, $\boldsymbol{q} = q_x \boldsymbol{e}_x$, $\boldsymbol{B} = B_y \boldsymbol{e}_y$, $B_y = 0.6$ and $q_x = \pm 0.6$.

The exact calculations can be found in appendix B and lead to the final band structure

$$E_{+,\pm} = \sqrt{(\xi_{\boldsymbol{k}} \pm |\boldsymbol{g}|)^2 + |\Delta|^2} \pm \hat{\boldsymbol{g}} \cdot \boldsymbol{B} + \frac{1}{2} \boldsymbol{q} \cdot \boldsymbol{v}_{\boldsymbol{k}} \left(1 \pm \frac{\alpha_R}{v_{\boldsymbol{k}}}\right) , \tag{4.15}$$

whith $\hat{\boldsymbol{g}} = \frac{\boldsymbol{g}}{|\boldsymbol{g}|}$ and the electron velocity $\boldsymbol{v}_{\boldsymbol{k}} = \partial_{\boldsymbol{k}} \xi_{\boldsymbol{k}}$. A quadratic correction in $\boldsymbol{q}$ is left out, because we assume it is small compared to the momentum. Both corrections contain normalized terms in $\boldsymbol{k}$ leading to an asymmetry in the band structure and thus, to an asymmetric band structure, resulting from the magnetic field, the supercurrent closes the energy gap for different values of $\boldsymbol{q}$, being the supercurrent diode effect. Directory restrictions are a parallel or antiparallel supercurrent and a magnetic field perpendicular to the momentum. Further illustrations of the band structure can be found in Fig. 4.1. When

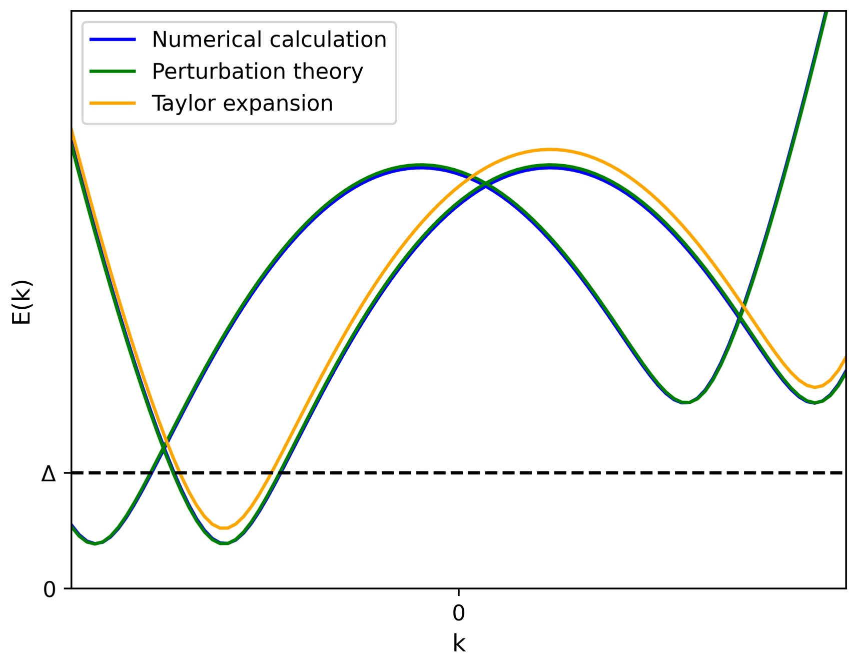

Figure 4.2: Comparison of band structure of the helical superconductor in presence of a supercurrent appoximated with first order perturbation theory and with first order Taylor expansion: Not only is the solution with Taylor expansion the same for both energy bands, but the solution with perturbation theory is much closer to the numerical solution. We use m = 1, $\mu = 1$, $\alpha_R = 1$, $\Delta = 3$, $\boldsymbol{k} = k_x \boldsymbol{e}_x$, $\boldsymbol{q} = q_x \boldsymbol{e}_x$, $\boldsymbol{B} = 0$ and $q_x = 0.8$

comparing Eq. (4.15) to the results of [1] we observe an additional term in our correction for the boost, probably coming from the different approches to calculate the corrections. We used perturbation theory instead of first order Taylor expansion. In order to examine which result is closer to the real solution, Fig. 4.2 shows both results and a numerical solution of the eigenvalue problem. We can see that our result is closer to the numerical solution than the result of [1].

4.3 Symmetry considerations

This chapter investigates the symmetries of the helical superconductor with and without magnetic field and supercurrent. We can write the BdG Hamiltonian as

$$H_{\text{BdG}} = \xi_{\boldsymbol{k}} \tau_z \otimes \sigma_0 + \tau_z \otimes \alpha_R (\boldsymbol{e}_z \times \boldsymbol{k}) \cdot \boldsymbol{\sigma} + \Delta \tau_x \otimes \sigma_0 \, , \tag{4.16}$$

with the Pauli matrices σ_i and the matrices τ_i acting on the block structure. With H_{BdG} in this form, we can calculate the commutators with the time inversion and parity operators allowing us to see which symmetries are conserved. The operator for time inversion is $T = \tau_0 \otimes \sigma_y iK$ and the parity operator is $P = \tau_0 \otimes \sigma_x K$.

Starting with time inversion, we can calculate the commutator

$$\begin{aligned}[H_{\text{BdG}}, T] = \tau_z \otimes [\sigma_0, \sigma_y]\xi_{\boldsymbol{k}} iK - \alpha_R k_y \tau_z \otimes \{\sigma_x, \sigma_y\} iK \\ + \alpha_R k_x \tau_z \otimes [\sigma_y, \sigma_y] iK + \Delta \tau_x \otimes [\sigma_0, \sigma_y] iK = 0\,,\end{aligned} \tag{4.17}$$

noticing that the symmetry is conserved. However, when introducing an external magnetic field the Hamiltonian becomes $H^B = H_{\text{BdG}} + \tau_0 \otimes \boldsymbol{B} \cdot \boldsymbol{\sigma}$ and the commutator is

$$[H^B, T] = [H_{\text{BdG}}, T] + [\tau_o \otimes \boldsymbol{B} \cdot \boldsymbol{\sigma}, T] = \tau_0 \otimes 2B_x iK\sigma_z\,. \tag{4.18}$$

Therefore, the magnetic field breaks the time symmetry, as we already suspected from the considerations of Sec. 3.2. A similar result should be achieved for a supercurrent $\boldsymbol{q}$. In this case, the Hamiltonian cannot be written in such a compact form like for the magnetic field and is given by

$$H^q = \left(\frac{\boldsymbol{k}^2 + \boldsymbol{q}^2}{2m} - \mu\right) \tau_z \otimes \sigma_0 + \tau_z \otimes \alpha_R \left[\boldsymbol{e}_z \times (\boldsymbol{k} + \boldsymbol{q})\right] \cdot \boldsymbol{\sigma} + \Delta \tau_x \otimes \sigma_0 + \frac{\boldsymbol{k} \cdot \boldsymbol{q}}{m} \tau_0 \otimes \sigma_0. \tag{4.19}$$

Because we assume the supercurrent as an external factor like the magnetic field, it is not affected by time inversion. With a closer look we can identify the spin-orbit term containing $\boldsymbol{q}$ to break the time symmetry. Because the system with a supercurrent is not a requirement for the SDE, but the tool to achieve it, we want to find a description for the system with the supercurrent conserving time symmetry. We can obtain this by introducing an effective time inversion operator T_{eff}, treating the supercurrent as an internal factor. Since the time inversion operator also appears in the Hamiltonian, we have to define an effective Hamiltonian

$$H^q_{\text{eff}} = \left[\frac{(\boldsymbol{k} + \boldsymbol{q})^2}{2m} - \mu\right] \tau_z \otimes \sigma_0 + \tau_z \otimes \alpha_R \left[\boldsymbol{e}_z \times (\boldsymbol{k} + \boldsymbol{q})\right] \cdot \boldsymbol{\sigma} + \Delta \tau_x \otimes \sigma_0\,. \tag{4.20}$$

The difference to Eq. (4.19) is the block structure of the term coupling $\boldsymbol{k}$ and $\boldsymbol{q}$ resulting from the first term in Eq. (4.20). Now, the term breaking the symmetry for the non effective case has the same structure as the spin-orbit term containing $\boldsymbol{k}$. Because T_{eff} affects momentum and boost equally, time symmmetry is conserved. However, with an additional magnetic field the symmetry is still broken, as we can deduce from the calculations above.

The other symmetry that should be considered is parity. When calulating the effect on the BdG Hamiltonian

$$PH_{\text{BdG}} = \xi_{\boldsymbol{k}}\tau_z \otimes \sigma_0 P + \tau_z \otimes \alpha_R(\boldsymbol{e}_z \times (-\boldsymbol{k})) \cdot \boldsymbol{\sigma} P + \Delta\tau_x \otimes \sigma_0 P \neq H_{\text{BdG}}P\,, \tag{4.21}$$

we see that the SOC breaks parity. Therefore, with the magnetic field, both symmetries are broken and we get the SDE. Nevertheless, we can look at parity in the boosted system H^q. Again, calculating the effect of the parity operator

$$\begin{aligned} PH^q = &\left(\frac{\boldsymbol{k}^2 + \boldsymbol{q}^2}{2m}P - \mu\right)\tau_z \otimes \sigma_0 P - \tau_z \otimes \alpha_R\left[\boldsymbol{e}_z \times (\boldsymbol{k} - \boldsymbol{q})\right] \cdot \boldsymbol{\sigma} P \\ &+ \Delta\tau_x \otimes \sigma_0 P - \frac{\boldsymbol{k}\cdot\boldsymbol{q}}{m}\tau_0 \otimes \sigma_0 P \neq H^q P\,, \end{aligned} \tag{4.22}$$

we can also see, that the supercurrent breaks parity. Contrary to time inversion, we are not able to introduce an effective parity operator, leading to a conservation of parity.

Chapter 5

Conclusion and Outlook

This thesis dealt with the supercurrent diode effect. In Ch. 3, we discussed the supercurrent as a Galilean boost on the Schrödinger equation and found the generator for the transformation. Moreover, we derived the periodic modulation of the energy gap terms in the BdG Hamiltonian. Additionally, we discussed the break of time symmetry and parity and found out, that when our system breaks both, it leads to the supercurrent diode effect.

In Ch. 4 we discussed the supercurrent diode effect on the example of a helical superconductor. We derived the band structure of the single particle Hamiltonian with an external magnetic field Eq. (4.4). Furthermore, we calculated the band structure of the superconducting system without an external magnetic field and additionally calculated corrections for a magnetic field and a supercurrent improving the calculations of Noah F. Q. Yuan and Liang Fu [1]. Finally, we discussed the break of the aforementioned symmetries in our example approving the discussion of Ch. 3.

Of course, there are some additional questions to be answered. For example, could we calculate the critical currents leading to the collapse of the superconductivity. Additionally, we can use the knowledge about symmetries to find other materials allowing the supercurrent diode effect.

Appendix A

BdG diagonalization

In order to obtain the eigenenergies we have to diagonalize Eq. (4.7). We do that by finding the roots of the characteristic polynomial

$$
\begin{aligned}
0 &= \det(H_{\mathrm{BdG}} - \lambda) \\
&= (\xi_{\boldsymbol{k}} - \lambda)\big\{(\xi_{\boldsymbol{k}} - \lambda) \\
&\underbrace{\big[(-\xi_{\boldsymbol{k}} - \lambda)^2 - \{(B_x - iB_y) + \alpha_R(k_y + ik_x)\}\{(B_x + iB_y) + \alpha_R(k_y - ik_x)\}\big]}_{\gamma} \\
&- |\Delta|^2(-\xi_{\boldsymbol{k}} - \lambda)\big\} - [(B_x + iB_y) - \alpha_R(k_y - ik_x)] \\
&\Big\{ [(B_x - iB_y) - \alpha_R(k_y + ik_x)]\, \gamma + |\Delta|^2\, [(B_x - iB_y) + \alpha_R(k_y + ik_x)] \Big\} \\
&+ \Delta^\star\big\{ - [(B_x - iB_y) - \alpha_R(k_y + ik_x)]\, \Delta(B_x + iB_y) + \alpha_R(k_y - ik_x) \\
&- (\xi_{\boldsymbol{k}} - \lambda)\Delta(-\xi_{\boldsymbol{k}} - \lambda) + \Delta^\star\Delta^2\big\} \\
&= \gamma\big\{(\xi_{\boldsymbol{k}} - \lambda)^2 - [(B_x + iB_y) - \alpha_R(k_y - ik_x)]\, [(B_x - iB_y) - \alpha_R(k_y + ik_x)]\, \big\} \\
&+ |\Delta|^2\big\{|\Delta|^2 - [(B_x - iB_y) - \alpha_R(k_y + ik_x)]\, [(B_x + iB_y) + \alpha_R(k_y - ik_x)] \\
&- [(B_x + iB_y) - \alpha_R(k_y - ik_x)]\, [(B_x - iB_y) + \alpha_R(k_y + ik_x)] \\
&- 2(\xi_{\boldsymbol{k}} - \lambda)(-\xi_{\boldsymbol{k}} - \lambda)\big\} \\
&= \gamma\left[(\xi_{\boldsymbol{k}} - \lambda)^2 - \boldsymbol{B}^2 - \alpha_R^2\boldsymbol{k}^2 + 2\alpha_R(B_x k_y - B_y k_x)\right] \\
&+ |\Delta|^2\left[|\Delta|^2 - 2(\boldsymbol{B}^2 - \alpha_R^2\boldsymbol{k}^2 + \lambda^2 - -\xi_{\boldsymbol{k}}^2)\right] \\
&= (\xi_{\boldsymbol{k}}^2 + \lambda^2 - \boldsymbol{B}^2 - \boldsymbol{g}^2)^2 - (2\boldsymbol{g}\cdot\boldsymbol{B} + 2\lambda\xi_{\boldsymbol{k}})^2 - 2^2|\Delta|^2 \\
&+ \underbrace{|\Delta|^2(|\Delta|^2 - 2\boldsymbol{B}^2 + 2\boldsymbol{g}^2 + 2\xi_{\boldsymbol{k}}^2)}_{R_1} \\
&= \lambda^4 + 2\lambda^2(\xi_{\boldsymbol{k}}^2 - \boldsymbol{B}^2 - \boldsymbol{g}^2) + \underbrace{(\xi_{\boldsymbol{k}}^2 - \boldsymbol{B}^2 - \boldsymbol{g}^2)^2 - 4\big[(\boldsymbol{g}\cdot\boldsymbol{B})^2}_{R_2} \\
&+ 2\lambda\boldsymbol{g}\cdot\boldsymbol{B}\xi_{\boldsymbol{k}} + \lambda^2\xi_{\boldsymbol{k}}^2\big] - 2\lambda^2|\Delta|^2 + R_1 \\
&= \lambda^4 - 2\lambda^2(\xi_{\boldsymbol{k}}^2 + |\Delta|^2 + \boldsymbol{B}^2 + 2\boldsymbol{g}^2) - 8\lambda\xi_{\boldsymbol{k}}\boldsymbol{g}\cdot\boldsymbol{B} + R_1 + R_2\,.
\end{aligned}
\tag{A.1}
$$

We get a fourth degree polynomial which can be solved analytically. However, the calculations are long and the resulting terms will be complicated and not intuitive. Moreover, we see that by leaving out the magnetic field the characteristic polynomial becomes biquadratic and thus, easy to solve. Effects of the magnetic field can be calculated with perturbation theory as long as the magnetic field is small compared to the energies. With this assumptions, we must solve

$$0 = \lambda^4 - 2\lambda^2 \underbrace{(\xi_{\boldsymbol{k}}^2 + |\Delta|^2 + 2\boldsymbol{g}^2)}_{f} + R_1^0 + R_2^0 \,, \tag{A.2}$$

whith $R_i^0 = R_i|_{\boldsymbol{B}=0}$. Because of the biquadratic form, the roots are given by

$$\begin{aligned}
&\longrightarrow E_{\pm,\pm} = \pm\sqrt{f \pm \sqrt{f^2 - R_1^0 - R_2^0}} \\
&R_1^0 = |\Delta|^2(|\Delta|^2 + 2\boldsymbol{g}^2 + 2\xi_{\boldsymbol{k}}^2) \\
&R_2^0 = (\xi_{\boldsymbol{k}}^2 - \boldsymbol{g}^2)^2 \\
&f^2 = (\xi_{\boldsymbol{k}}^2 + \boldsymbol{g}^2)^2 + |\Delta|^4 + 2|\Delta|^2(\xi_{\boldsymbol{k}}^2 + \boldsymbol{g}^2) \\
&f^2 - R_1^0 - R_2^0 = 4\xi_{\boldsymbol{k}}^2\boldsymbol{g}^2 \\
&\longrightarrow E_{\pm,\pm} = \pm\sqrt{\xi_{\boldsymbol{k}}^2 + |\Delta|^2 + \boldsymbol{g}^2 \pm 2\xi_{\boldsymbol{k}}|\boldsymbol{g}|} \\
&= \pm\sqrt{(\xi_{\boldsymbol{k}} \pm |\boldsymbol{g}|)^2 + |\Delta|^2} = \pm\Lambda_{\pm} \,.
\end{aligned} \tag{A.3}$$

Appendix B

Perturbation theory

We want to derive the corrections for the band structure in first order non degenerated perturbation theory. As we see in Eq. (4.14), we need to calculate the mean of the perturbation in the eigenstates. The perturbations are given by Eq. (4.9) and Eq. (4.10). Since we already established that both pertubations do not break the translation symmetry, we can calculate each correction for itself and then add both to the eigenenergies $\Lambda_{\pm}$. Starting with the magnetic field, we calculate the energy as

$$\begin{aligned}
\varepsilon_{\pm}^{B} &= \Lambda_{\pm} + \langle \Lambda_{\pm} \mid W^{B} \mid \Lambda_{\pm} \rangle \\
&= \Lambda_{\pm} + \frac{1}{N_{\pm}^{2}} \begin{pmatrix} a_{\pm} \\ b_{\pm} \\ c_{\pm} \\ d_{\pm} \end{pmatrix}^{\star} \begin{pmatrix} 0 & Be^{-i\vartheta} & 0 & 0 \\ Be^{i\vartheta} & 0 & 0 & 0 \\ 0 & 0 & 0 & Be^{-i\vartheta} \\ 0 & 0 & Be^{i\vartheta} & 0 \end{pmatrix} \begin{pmatrix} a_{\pm} \\ b_{\pm} \\ c_{\pm} \\ d_{\pm} \end{pmatrix} \\
&= \Lambda_{\pm} + \frac{B}{N_{\pm}^{2}} \left[e^{i\vartheta} \left(b_{\pm}^{\star} a_{\pm} + d_{\pm}^{\star} c_{\pm} \right) + e^{-i\vartheta} \left(a_{\pm}^{\star} b_{\pm} + c_{\pm}^{\star} d_{\pm} \right) \right] \\
&= \Lambda_{\pm} \pm \frac{B}{N_{\pm}^{2}} \left(4\xi_{\boldsymbol{k}}^{2} k^{2} |\Delta|^{2} + |c_{\pm}|^{2} \right) \left(e^{i\left(\vartheta - \varphi - \frac{\pi}{2}\right)} + e^{-i\left(\vartheta - \varphi - \frac{\pi}{2}\right)} \right) \\
&= \Lambda_{\pm} \pm B \cos\left(\vartheta - \varphi - \frac{\pi}{2} \right) = \Lambda_{\pm} \pm B \sin\left(\vartheta - \varphi \right) \\
&= \Lambda_{\pm} \pm \hat{\boldsymbol{g}} \cdot \boldsymbol{B} \,,
\end{aligned} \tag{B.1}$$

with $\hat{\boldsymbol{g}}$ the normalized vector of $\boldsymbol{g} = \alpha_{R} \boldsymbol{e}_{z} \times \boldsymbol{k}$.

Moving on to the boost, representing the supercurrent, we can calculate the energy as

$$\begin{aligned}
\varepsilon_{\pm}^{q} &= \Lambda_{\pm} + \langle \Lambda_{\pm} \mid W^{q} \mid \Lambda_{\pm} \rangle \\
&= \Lambda_{\pm} + \frac{1}{N_{\pm}^{2}} \begin{pmatrix} a_{\pm} \\ b_{\pm} \\ c_{\pm} \\ d_{\pm} \end{pmatrix}^{\star} \begin{pmatrix} \frac{\boldsymbol{q}\cdot(\boldsymbol{k}+\frac{\boldsymbol{q}}{4})}{2m} & -i\frac{q\alpha_R}{2}e^{-i\delta} & 0 & 0 \\ i\frac{q\alpha_R}{2}e^{i\delta} & \frac{\boldsymbol{q}\cdot(\boldsymbol{k}+\frac{\boldsymbol{q}}{4})}{2m} & 0 & 0 \\ 0 & 0 & \frac{\boldsymbol{q}\cdot(\boldsymbol{k}-\frac{\boldsymbol{q}}{4})}{2m} & -i\frac{q\alpha_R}{2}e^{-i\delta} \\ 0 & 0 & i\frac{q\alpha_R}{2}e^{i\delta} & \frac{\boldsymbol{q}\cdot(\boldsymbol{k}-\frac{\boldsymbol{q}}{4})}{2m} \end{pmatrix} \begin{pmatrix} a_{\pm} \\ b_{\pm} \\ c_{\pm} \\ d_{\pm} \end{pmatrix} \\
&= \Lambda_{\pm} + \frac{1}{N_{\pm}^{2}} \Bigg[N_{\pm}^{2} \frac{\boldsymbol{q}\cdot\boldsymbol{k}}{2m} + \frac{\boldsymbol{q}^{2}}{8m} \left(|a_{\pm}|^2 + |b_{\pm}|^2 - |c_{\pm}|^2 - |d_{\pm}|^2 \right) \\
&- i\frac{q\alpha_R}{2}e^{-i\delta} \left(a_{\pm}^{\star} b_{\pm} + c_{\pm}^{\star} d_{\pm} \right) + i\frac{q\alpha_R}{2}e^{i\delta} \left(b_{\pm}^{\star} a_{\pm} + d_{\pm}^{\star} c_{\pm} \right) \Bigg] \\
&= \Lambda_{\pm} + \frac{1}{N_{\pm}^{2}} \Bigg[N_{\pm}^{2} \frac{\boldsymbol{q}\cdot\boldsymbol{k}}{2m} + \frac{\boldsymbol{q}^{2}}{8m} \left(8\xi_{\boldsymbol{k}}^{2} k^{2} |\Delta|^{2} - 2|c_{\pm}|^{2} \right) \\
&\pm \left(4\xi_{\boldsymbol{k}}^{2} k^{2} |\Delta|^{2} + |c_{\pm}|^{2} \right) \frac{q\alpha_R}{2} \left(e^{-i(\varphi-\delta)} + e^{i(\varphi-\delta)} \right) \Bigg] \\
&= \Lambda_{\pm} + \frac{1}{N_{\pm}^{2}} \Bigg[N_{\pm}^{2} \frac{\boldsymbol{q}\cdot\boldsymbol{k}}{2m} + \frac{\boldsymbol{q}^{2}}{8m} \left(8\xi_{\boldsymbol{k}}^{2} k^{2} |\Delta|^{2} - 2|c_{\pm}|^{2} \right) \pm N_{\pm}^{2} \frac{q\alpha_R}{2} \cos(\varphi-\delta) \Bigg] \\
&= \Lambda_{\pm} + \frac{\boldsymbol{q}\cdot\boldsymbol{k}}{2m} \pm \frac{q\alpha_R}{2} \cos(\varphi-\delta) + \frac{\boldsymbol{q}^{2}}{8m} \frac{8\xi_{\boldsymbol{k}}^{2} k^{2} |\Delta|^{2} - 2|c_{\pm}|^{2}}{8\xi_{\boldsymbol{k}}^{2} k^{2} |\Delta|^{2} + 2|c_{\pm}|^{2}} \\
&= \Lambda_{\pm} + \frac{\boldsymbol{q}\cdot\boldsymbol{k}}{2m} \pm \frac{\alpha_R}{2k} \boldsymbol{q}\cdot\boldsymbol{k} + \frac{\boldsymbol{q}^{2}}{8m} \frac{8\xi_{\boldsymbol{k}}^{2} k^{2} \left[|\Delta|^{2} - (k \pm \xi_{\boldsymbol{k}} \mp \Lambda_{\pm})^{2} \right]}{16\xi_{\boldsymbol{k}}^{2} k^{2} \Lambda_{\pm} \left(\Lambda_{\pm} - \xi_{\boldsymbol{k}} \mp k \right)} \\
&= \Lambda_{\pm} + \frac{1}{2} \boldsymbol{q}\cdot\boldsymbol{k} \left(\frac{1}{m} \pm \frac{\alpha_R}{k} \right) + \frac{\boldsymbol{q}^{2}}{8m} \frac{|\Delta|^{2} - \Lambda_{\pm} \left(\Lambda_{\pm} + \xi_{\boldsymbol{k}} \pm k \right)}{\Lambda_{\pm} \left(\Lambda_{\pm} - \xi_{\boldsymbol{k}} \mp k \right)} \\
&= \Lambda_{\pm} + \frac{1}{2} \boldsymbol{q}\cdot\boldsymbol{v}_{\boldsymbol{k}} \left(1 \pm \frac{\alpha_R}{v_{\boldsymbol{k}}} \right) + \frac{\boldsymbol{q}^{2}}{8m} \frac{|\Delta|^{2} - \Lambda_{\pm} \left(\Lambda_{\pm} + \xi_{\boldsymbol{k}} \pm k \right)}{\Lambda_{\pm} \left(\Lambda_{\pm} - \xi_{\boldsymbol{k}} \mp k \right)} \\
&\approx \frac{1}{2} \boldsymbol{q}\cdot\boldsymbol{v}_{\boldsymbol{k}} \left(1 \pm \frac{\alpha_R}{v_{\boldsymbol{k}}} \right),
\end{aligned} \tag{B.2}$$

with the electron velocity $\boldsymbol{v}_{\boldsymbol{k}} = \partial_{\boldsymbol{k}} \xi_{\boldsymbol{k}}$.

With both calculations we obtain the band structure with corrections given by Eq. (4.15).

Acknowledgements

Last but not least, I would like to thank all the people without whom I would not have been able to achieve this. First of all, I would like to thank my supervisor Fabian Hassler for the close supervision and allowing me to work on this interesting topic.

Second, I want to thank Alexander Ziesen for countless Zoom calls discussing physics and introducing me in scientific working and writing.

I want to thank all members of the IQI for welcoming me and making the time at the office a great experience.

Finally, I want to thank my family and friends for all their support during the last months.

Bibliography

[1] Noah F. Q. Yuan and Liang Fu. Supercurrent diode effect and finite momentum superconductivity. 2021.

[2] Fabian Hassler. *Quantentheorie der Vielteilchensysteme*. 2021. Scriptum.

[3] Albert Messiah. *Quantenmechanik, Bd. 1*. de Gruyter, 1991.