# Algebraic Methods for the 1D Schrödinger Equation

Master thesis

by

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**Abstract**

We discuss algebraic methods to obtain exact results about the eigenvalue spectrum of the one dimensional Schrödinger equation. We focus on degeneracies in the spectrum and eigenvalues that are exactly solvable. The central object in our discussion is supersymmetry. First, we give an introduction to the description of problems using superpotentials. We present insights that can be obtained from the supersymmetric structure, in particular the degeneracies between bosonic and fermionic states and that the groundstate admits an algebraic solution. We use these insights to study the double sine potential. By tuning the double sine potential away from the point where it can be described using supersymmetry, we find an extension of supersymmetry that is related to a class of problems for which multiple eigenstates admit algebraic solutions. Next, we prove, for a subclass of these problems, that a degenerate partner state exists for almost all eigenstates. Finally, we highlight that the class of periodic problems with more than one exactly solvable eigenvalue is not limited to the double sine potential.

# Contents

# 1. Introduction

When Schrödinger [1] formulated his equation

$$i\hbar\frac{\partial}{\partial t}\Psi(r,t) = \left[-\frac{\hbar^2}{2m}\partial_r^2 + V(r,t)\right]\Psi(r,t) \tag{1.1}$$

in 1926, he founded the mathematical basis for the exploration of quantum mechanics. Given a one dimensional system, in which a particle moves influenced by the potential $V(r,t)$, the equation describes the evolution of the wave function $\Psi(r,t)$, the quantum mechanical description of the state of the system.

Solutions to the equation were analyzed quite extensively. Eigenstates $\Psi_E(r,t)$, which exist given that the potential $V(r)$ is constant in time, are of main interest in the analysis of the equation. These states $\Psi_E$ evolve according to $\Psi_E(r,t) = \exp(-itE/\hbar)\Psi_E(r,0)$, i.e., their time dependence can be described as a rotating phase alone. For most relevant problems, these states exist only for certain values of the parameter $E$ that can be identified with the energy of the configuration. These discrete ("quantized") states led eventually to the name of quantum mechanics. The states $\Psi_E$ are obtained as solutions of the time-independent Schrödinger equation

$$H\psi_E = E\psi_E \quad \text{with the Hamiltonian} \quad H = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V(r). \tag{1.2}$$

For convenience and without loss of generality, we will set $\hbar/(2m)^{1/2} = 1$. The original form of the equation can always be recovered by adequate rescaling of $E$ and $V$. For readability, we will drop the dependence of $\Psi$ and $V$ on $r$ in the notation. The time-independent Schrödinger equation in one dimension is studied in the mathematical literature under the term Sturm-Liouville equation. Some general results concerning its properties are summarized in Chapter 1.

The analysis of this equation was carried out for a vast variety of systems, i.e., potentials $V$. The most popular system is the harmonic oscillator, due to two reasons. On the one hand, it is of high practical significance since it can serve as an approximation in many situations. On the other, it assumes an exact solution that can be obtained purely relaying on the methods of algebra. This thesis will discuss many concepts that appear in the algebraic discussion of the harmonic oscillator and extend them to much more broad applications. We will review the steps that are necessary for the algebraic discussion of the harmonic oscillator below and refer to the chapters of the thesis that cover them in more detail.

Using the abbreviations introduced above, the time-independent Schrödinger equa-

tion for the harmonic oscillator is given by

$$H\psi = \left[-\frac{d^2}{dx^2} + x^2\right]\psi = E\psi. \tag{1.3}$$

Coming up with an algebraic expression that simplifies the problem is hard. But since we are standing on the shoulders of giants, we know that the best way is to introduce the operator

$$a = \frac{1}{\sqrt{2}}\frac{d}{dx} + \frac{x}{\sqrt{2}} \quad \text{and its adjoint} \quad a^\dagger = -\frac{1}{\sqrt{2}}\frac{d}{dx} + \frac{x}{\sqrt{2}}. \tag{1.4}$$

With these operators, we can express $H = 2a^\dagger a + 1$ and determine the commutation relation $[a, a^\dagger] = 1$. The commutator indicates that they form a ladder structure. This ladder structure allows solution of the eigenvalues. The concept of ladder operators is tightly linked to a concept known as shape invariance from quantum mechanical supersymmetry. Most of the potentials for which exact eigenvalue spectra are known belong to that class. For a deeper discussion of supersymmetry and shape invariance, see Chapter 3.

The operators $a$ and $a^\dagger$ are not the only type of ladder operators. Angular momentum operators also form an algebra which has similar properties. We will introduce a method to construct Schrödinger equations that are associated with Hamiltonians built from angular momentum operators. The class of problems that can be constructed this way has interesting properties and includes a lot of relevant problems such as the Morse potential or the double sine potential. A discussion of the properties and dynamics of the double sine potential can be found in Chapter 4 and a discussion of the full class of problems that can be constructed using this method can be found in Chapter 5.

One of the most interesting exact results that we can obtain using algebra is the insight that a system has a degeneracy. Degeneracies played an important role in the foundation of modern physics, since a common approach in physics has been to postulate a model for a system and derive its properties. If this model predicts a degeneracy, it is possible to design an experiment that tries to measure the splitting of this degeneracy and therefore tests the model. This approach was most popular for the splitting of degenerate levels in the hydrogen orbits and the associated discovery of the fine- and hyper-fine-structure.

Degeneracies are also interesting in the context of quantum information systems as they serve as a candidate for qubits. The idea here is to associate a quantum mechanical state with the logical state 0 and the degenerate partner state with logical state 1, so as to use them as realization of a qubit. It is desirable to use a pair of states with equal energy, to reduce the noise in the system. To design and tune such models, we discuss a generalization of supersymmetry, in Chapter 6.

# 2. Sturm–Liouville Theory

To determine the energy spectrum of an one dimensional quantum mechanical system we have to solve for the eigenvalues of a second order linear differential operator. The most general form of such a problem is given by

$$f_2(x)\frac{d^2}{dx^2}y(x) + f_1(x)\frac{d}{dx}y(x) + \left[f_0(x) + \lambda g(x)\right] y(x) = 0. \tag{2.1}$$

in which the functions $f_2(x), f_1(x), f_0(x)$, and $g(x)$ enter as parameter and are determined by the specific problem to be solved. Both $f_2$ and $g$ must be positive. The eigenvalue for which we solve the problem is $\lambda$. This class of problems is the scope of Sturm-Liouville theory. The applications of this equation are incredibly broad, they extend far beyond quantum mechanics. For example, it may be used to derive the sound of a string on a guitar or the eigenmodes of an electro magnetic resonator.

With the Liouville transformation [2] we can bring this general expression in the Liouville normalform, which is equivalent to the time independent Schrödinger equation. Thus, the eigenvalue equation

$$\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x) \tag{2.2}$$

with the potential $V(x)$ and the eigenvalue $E$ is in a certain way a universal object to study the general class of eigenvalue problems described by the Sturm-Liouville theory.

If we fix $E$ in Eq. (2.2) we obtain a usual linear second order differential equation and as such it will always, for every $E$, have two linearly independent solutions. To obtain a meaningful eigenvalue problem in which only certain $E$ are eigenvalues we have to specify conditions for $\psi(x)$ that must be satisfied to consider $\psi$ an eigenfunction and the corresponding $E$ an eigenvalue. The kind of constraints we impose will depend on the range of $x$ for which we consider the equation. There are two major cases that we have to distinguish. On the one hand, there are finite domains $x \in [a, b]$ in which we demand conditions for $\psi$ on the boundary, i.e., some linear constraints on $\psi(a)$, $\psi(b)$, $\psi'(a)$, and $\psi'(b)$. Among all possible linear combinations the most relevant are Dirichlet boundary conditions fixing $\psi(a) = 0$ and $\psi(b) = 0$, Neumann boundary conditions fixing $\psi'(a) = 0$ and $\psi'(b) = 0$, and periodic boundary conditions fixing $\psi(a) = \psi(b)$ and $\psi'(a) = \psi'(b)$. On the other hand, we may consider an infinite domain $x \in \mathbb{R}$ in that case we will demand that $\psi$ stays bounded. Thus, we also include non-normalizable states like plain waves in our discussion.

## 2.1. Spectrum

The spectrum of a system is the set of all its eigenvalues $E$. The properties of the spectrum depend a lot on the domain and the boundary conditions that are chosen. The simplest setup is given by a system on a finite domain with boundary conditions that are local, i.e, there appear no mixed constraints between $\psi$ on opposites ends of the domain. Sturm showed with his oscillation theory that for this setup there will be a discrete set of eigenfunctions $\psi_n$ with corresponding eigenvalues $E_n$. The number of nodes of those eigenfunctions, excluding nodes on the boundary due to Dirichlet boundary conditions, is given by $n$.

We are especially interested in the appearance of degeneracies. To show that there can appear no degeneracies in this setup we want to prove that two eigenstate $\psi_a$ and $\psi_b$ can not satisfy the same boundary conditions at the same energy. To prove that statement we employ the Wronskian determinant between the two eigenfunctions

$$W = \psi_a \psi_b' - \psi_b \psi_a' \tag{2.3}$$

and use that a Wronskian which vanishes identically implies that the functions involved are linearly dependent.

We start our argument by showing that the Wronskian is conserved by computing

$$W' = \psi_a' \psi_b' + \psi_a \psi_b'' - \psi_b' \psi_a' - \psi_b \psi_a'' = \psi_a \psi_b'' - \psi_b \psi_a'' \tag{2.4}$$

and substituting the Schrödinger equation

$$W' = \psi_a (E - V) \psi_b - \psi_b (E - V) \psi_a = 0. \tag{2.5}$$

Next, we can evaluate it at the boundary. For either Neumann or Dirichlet boundary conditions the Wronskian vanishes obviously. We can check that this also holds in the case of mixed boundary conditions by evaluating the most general local constraint

$$\alpha\psi(\xi) + \beta\psi'(\xi) = 0 \implies -\frac{\alpha}{\beta}\psi(\xi) = \psi'(\xi) \implies W = \frac{\alpha}{\beta}\psi_a\psi_b - \frac{\alpha}{\beta}\psi_b\psi_a = 0. \tag{2.6}$$

Since we have shown that the Wronskian is conserved in the domain of the problem we can conclude that it must vanish everywhere and thus the functions $\psi_a$ and $\psi_b$ can not be linearly independent eigenfunctions and must be equivalent. This leads to the conclusion that for every energy $E$ there can be only one independent state $\psi_E$, which is equivalent to the statement that there can be no degeneracies.

According to Sturm's oscillation theorem the eigenvalues are not just all different; they can be ordered by the number of nodes. Therefore, the eigenfunction $\psi_0$ without node will have the lowest eigenenergy $E_0$ and is called the ground state. The higher eigenenergies will be ordered [3]

$$E_0 < E_1 < E_2 < ... \tag{2.7}$$

This ordering can be understood by semiclassical arguments.

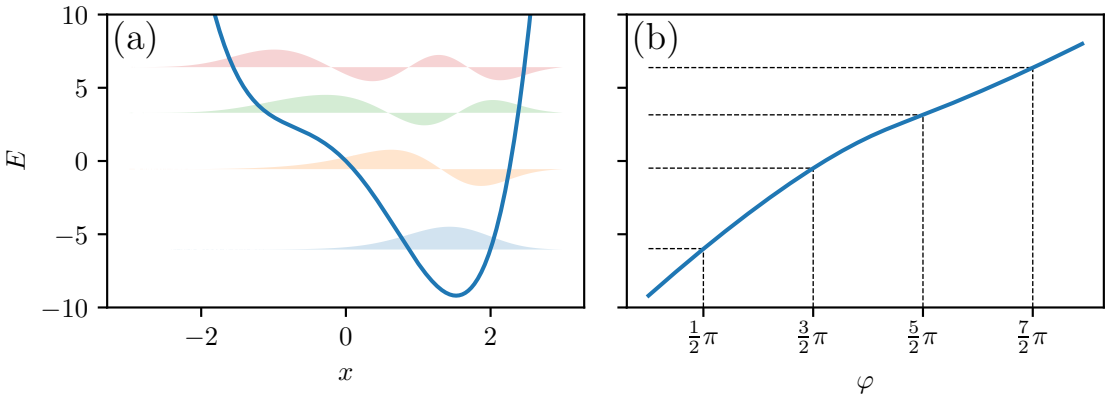


Figure 2.1.: A generic potential around its absolute minimum in (a) with the quantum mechanical states indicated at an offset that equals their eigenenergy. In (b) we show the value of the WKB phase integration and indicate the positions where that phase is $(n + 1/2)\pi$ with $n \in \mathbb{N}$ corresponding to a bound state.

For an introduction to semiclassics and the WKB approximation see [4]. The idea is to make the ansatz

$$\psi(x) = A(x)\cos[\varphi(x) + \phi] \tag{2.8}$$

and find a suitable approximation for $\varphi(x)$. The function $\varphi$ captures the phase of the wavefunction $\psi$ as a function of $x$ and thereby also counts the nodes that are appearing for that eigenstate. It turns out that the first approximation is given by

$$\varphi(x) = \int^{x} dx \sqrt{E - V(x)}. \tag{2.9}$$

The quantization condition turns out to be that $\varphi(x)$ integrated in the classically accessible region must be $(n+1/2)\pi$ with $n \in \mathbb{N}$. Therefore, the energy must increase by a certain amount in order to increase the value of the integration by another $\pi$, this prohibits the existence of states of the same energy. Also, it ensures that the number of nodes is increasing by one for every eigenstate with higher energy. An illustration for an example potential can be found in Figure 2.1.

In the case of an infinite domain the problem gets more involved. Given that the potential satisfies $V(x \to \pm\infty) \to \infty$ we may restrict, as an approximation, the domain and employ on the compact domain around the absolute minimum Dirichlet boundary conditions. If the domain is chosen large enough it will be a reasonable approximation. We will use this approximation for the numerics presented in this thesis.

If the potential does not satisfy $V(x \to \pm\infty) \to \infty$ the kind of states that may appear become more involved since states in the continuum may appear. The continuum is a region in energy in which for every energy $E$ exists a bounded solution to Schrödingers equation.

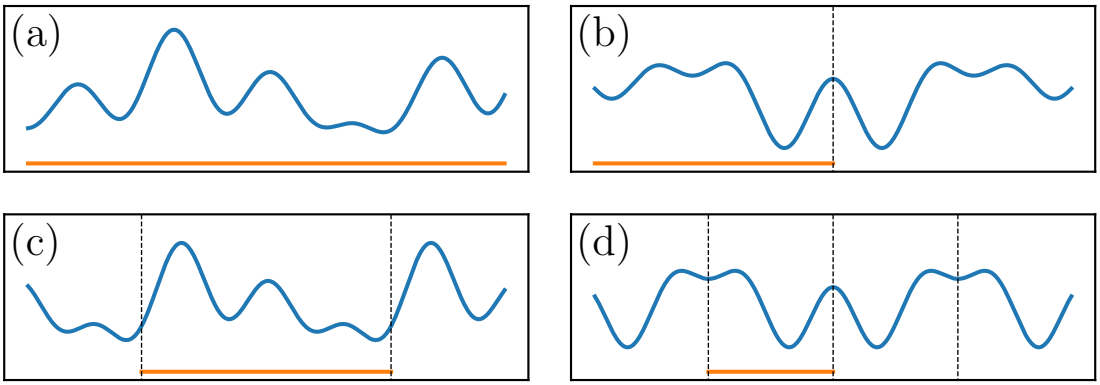


Figure 2.2.: The four possible symmetries and indicated an associated fundamental domain. A system in one dimension can either have no symmetry (a), be invariant under a reflection (b), be invariant under a translation (c), or be invariant under both a reflection and a translation (d).

## 2.2. Symmetry

In the case that the domain of the problem is not limited to an interval, it is of significant interest if the potential obeys a symmetry. A symmetry is in our setup a transformation of the dependent variable $x$ that leaves the system, i.e., the potential $V(x)$, invariant. Since we consider a one dimensional system, there are only two relevant transformations under which the system may be invariant. These are

$$\text{reflections} \quad x \mapsto -x \quad \text{and} \quad \text{translations} \quad x \mapsto x + a. \tag{2.10}$$

A system may have none, any, or both of these symmetries. See Figure 2.2.

Given a symmetry we can identify an associated fundamental domain, an area that is sufficient to describe the whole infinite system. Without a symmetry this is obviously the hole system itself. With a reflection symmetry we can reduce that fundamental domain to half the space. In that case, we can find all states of the problem by searching for odd and even solutions in the fundamental domain alone. It is sufficient to solve two problems on half the domain with the boundary conditions $\psi'(0) = 0$ (even) and $\psi(0) = 0$ (odd).

Something similar is possible in the case that the system posses a translational symmetry. But since the translation symmetry implicitly implies invariance not just for translations by $a$ but also for translations by $ka$ with any $k \in \mathbb{Z}$, the fundamental domain of that problem is finite sized. We can repeat it an infinite times. This will turn out to be useful in the discussion of periodic problems.

It is worth noting that if a system incorporates both symmetries it will have points of reflexion symmetry not just spaced by distance $a$ but every $a/2$ instead. That can be seen by considering a reflection around $a/2$, i.e.,

$$(x + a/2) \mapsto -(x + a/2). \tag{2.11}$$

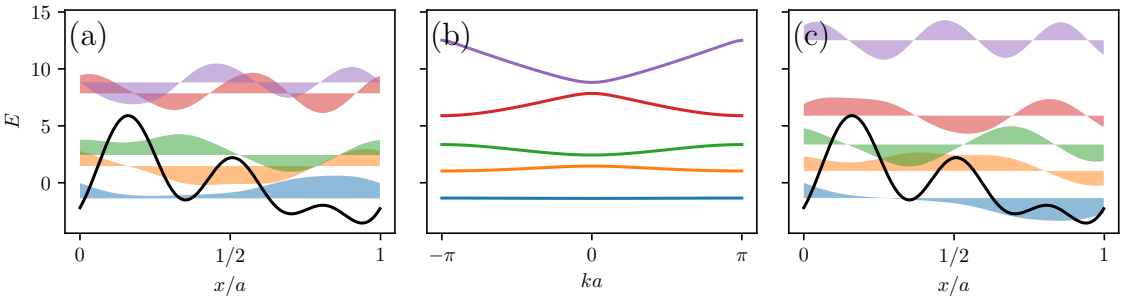


Figure 2.3.: A generic periodic potential with periodic eigenstates (a) and anti periodic eigenstates (c). In between the system in the Bloch band picture (b). Note that the eigenenergies of the periodic states correspond to the point $k = 0$ and the antiperiodic states correspond to $ka = \pi$ and therefore to the borders of the Bloch bands.

We can shift $x$ by $a/2$ to find that the transformation reads

$$x \mapsto -(x + a) \tag{2.12}$$

which is just a shift by $a$ followed by a reflection. Since these are the elementary operations under which the system is invariant by assumption it is also invariant by the reflection around $a/2$.

## 2.3. Periodic systems

If the system has a periodicity, i.e., it satisfies $V(x) = V(x + a)$ for all $a$ the potential can not satisfy $V(\pm \to \infty) \to \infty$ and states in the continuum will appear. We can use Bloch theory to find all eigenvalues of the problem. Bloch theory allows discussing the problem on the fundamental domain $[0, a]$. To obtain the eigenvalues we have to evaluate the problem on the fundamental domain with periodic boundary conditions that are modified with a phase that ranges from $k \in (-\pi/a, \pi/a]$

$$e^{ika}\psi(0) = \psi(a) \quad \text{and} \quad e^{ika}\psi'(0) = \psi'(a). \tag{2.13}$$

The eigenvalue spectrum as a function of $k$ are called Bloch bands. Regions of energy where no states can be found are called band gaps. In Figure 2.3 we perform this method for an example potential.

Among all the possible $k$ there are two points of special interest. These are $k = 0$ with wavefunctions $\psi$ being periodic and $k = \pi/a$ with wavefunctions $\psi$ being anti-periodic. At those $k$-points $\psi$ can be chosen real since the boundary conditions do not introduce a complex phase and the differential equation is real. These special cases are of interest because certain physical systems, such as superconducting circuits [5] or the quantum mechanical description of a pendulum, can be described by such periodic finite systems.

For these finite sized periodic systems exists an oscillation theorem similar to the one for boundary value problems. Yet, there are differences in the ordering and the nodal structure. For the periodic states the theorem implies an ordering

$$E_0 < E_1 \leq E_2 < E_3 \leq E_4 \ldots \tag{2.14}$$

Thus, it is possible for two states to have the same energy, i.e., this system may have a degeneracy. The nodal structure changes, the ground state corresponding to $E_0$ still has zero nodes, but the number of nodes is increasing by two for each of the pairs of states that may have the same energy. Since the two states that may have the same energy are borders of a band gap, the system defined with periodic boundary conditions on the fundamental domain has a degeneracy whenever a band gap in the periodic system closes.

For the anti-periodic states the situation is similar with the exception that the state without any node is missing. Thus, the ordered spectrum reads

$$\tilde{E}_1 \leq \tilde{E}_2 < \tilde{E}_3 \leq \tilde{E}_4 \ldots \tag{2.15}$$

We use $\tilde{E}_n$ to distinguish the energies of the antiperiodic states from those of the periodic ones. Also each eigenfunction has one node less.

While we will concentrate on finite sized systems with periodic boundary conditions, in the mathematical literature a lot work was done to identify conditions under which bandgaps in the infinite domain system close. Since a closing bandgap is associated with a degeneracy of the system reduced to its fundamental domain with periodic boundary conditions, these statements are still of interest for us. Especially Hochstadt proved under the term *inverse problems* various statements for conditions under which band gaps may vanish depending on the analytic properties of the potential. The most relevant statements are:

1. Whenever there is no bandgap opening, the potential must be constant [6].

2. Whenever there is only one bandgap, the potential $V(x)$ must satisfy $V'' = 3V^2 + AV + B$ with some constants $A$ and $B$ [6].

3. Whenever there is only a finite number of bandgaps, the potential must be smooth [6].

4. Whenever all antiperiodic gaps close, the potential has a periodicity not just of $a$ but also of $a/2$ [7].

5. Whenever all periodic gaps close, the potential must be of the form $V(x) = q^2(x) + q'(x)$ with a $q$ that satisfies $q(x + a/2) = -q(x)$ [7].

Especially the last statement is of high interest and we will rediscover that fact using algebraic insides from the formulation of the problem in the scope of supersymmetry in the next chapter.

# 3. Supersymmetry

Supersymmetry (SUSY) is a concept from quantum field theory which was first discovered in 1971 by Gelfand and Likhtman and later by many others. In quantum field theories SUSY relates fermionic and bosonic particles using a so called graded Lie-Algebra, an algebra that does not close under commutation relations alone but also requires anticommutation relations. The idea behind grading is to split the Hilbert space into a bosonic and a fermionic part. The fermion parity of a state determines into which sector it belongs. The fermion parity is given by $(-1)^F$ where $F$ is the number of fermions that are present. In consequence, adding a fermion to a state changes it from bosonic to fermionic and vice versa.

In the same way all operators are grouped in bosonic and fermionic operators. A fermionic operator changes the number of fermions in a system by an odd number, thus mapping a fermionic state onto a bosonic and vice versa. A bosonic operator does not change the fermion parity of a state. The canonic form of the commutator, which has to be evaluated to close the algebra, is given by the nature of the operators involved. If we want to evaluate the relation between two fermionic operators, we have to evaluate the anticommutator otherwise we have to uses the commutator [8]. To build a supersymmetric theory the central object is the fermionic supercharge operator $Q$ from which the Hamiltonian of a theory is derived.

We will introduce the concept of supersymmetry in the context of quantum mechanics and work out the relevant algebraic structures and discuss canonic ways to construct models which resemble 1D Schrödinger equations.

## 3.1. Quantum Mechanics

The ideas that were developed for quantum field theories can also be applied to plain quantum mechanics [9]. As in SUSY field theories we introduce a supercharge $Q$ and its Hermitian conjugate $Q^\dagger$ from which the Hamiltonian $H$ of the supersymmetric quantum mechanics (SUSY QM) system follows by the algebra

$$H = \{Q, Q^\dagger\}, \quad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0. \tag{3.1}$$

In this notation the anticommutator $\{A, B\}$ is a shorthand for $AB + BA$. From the anticommutation structure it is straightforward to derive

$$[H, Q] = 0 \tag{3.2}$$

and showing thereby that the supercharge is a conserved quantity.

We can already learn quite some properties about the spectrum of the Hamiltonian from the algebraic structure. First of all, we can note that the Hamiltonian is a positive semidefinite operator since it is written as the sum of two positive semidefinite operators $Q^\dagger Q$ and $QQ^\dagger$, thus it can not have eigenstates with negative energy. Another aspect and the main reason this setup is of interest to us is the degeneracy structure. Assume we have an eigenstate $|\psi\rangle$ with eigenenergy $E$, then we can evaluate

$$HQ|\psi\rangle = QH|\psi\rangle = EQ|\psi\rangle \quad \text{and} \quad HQ^\dagger|\psi\rangle = Q^\dagger H|\psi\rangle = EQ^\dagger|\psi\rangle\,. \tag{3.3}$$

This implies that the states $Q|\psi\rangle$ and $Q^\dagger|\psi\rangle$ are either vanishing, linear dependent on $|\psi\rangle$, or another independent eigenstate of $H$ with the same eigenenergy. First, we can rule out that $Q|\psi\rangle$ is proportional to $|\psi\rangle$ since $Q|\psi\rangle = \alpha|\psi\rangle$ would imply $Q^2|\psi\rangle = \alpha^2|\psi\rangle$. Given that $\{Q, Q\} = 0$, we could conclude that $\alpha = 0$. Analogous holds for $Q^\dagger$. It is worth noting that this is related to the idea that the fermionic operators $Q$ and $Q^\dagger$ change the nature of a state from bosonic to fermionic and vice versa, so it is not possible for them to map a state onto itself.

By these considerations there are four types of states remaining: There are states that are annihilated, meaning that they are mapped to 0, by both $Q$ and $Q^\dagger$, by $Q$ alone, by $Q^\dagger$ alone, or by neither.

If $|\psi\rangle$ is annihilated by both we can evaluate

$$H|\psi\rangle = QQ^\dagger|\psi\rangle + Q^\dagger Q|\psi\rangle = 0 \tag{3.4}$$

to see that this is a zero energy state. Since we have already showed that $H$ positive semidefinite it must be a ground state.

If $|\psi\rangle$ is annihilated by one of $Q$ or $Q^\dagger$ we introduce a terminology similar to that of quantum field theories and call the state bosonic if it is annihilated by $Q$ and fermionic if it is annihilated by $Q^\dagger$. See Figure 3.1. The state obtained by applying the adjoint of the operator that annihilates it produces a state of the opposite fermion parity. As stated before, this partner must have the same energy.

If $|\psi\rangle$ annihilated by neither $Q$ nor $Q^\dagger$ we can decompose it into its fermionic and bosonic part by introducing

$$|b\rangle = \frac{1}{E}QQ^\dagger|\psi\rangle \quad \text{and} \quad |f\rangle = \frac{1}{E}Q^\dagger Q|\psi\rangle\,. \tag{3.5}$$

Both of which are eigenstates of $H$.

Thus, for every state which is not a ground state there will be a degenerate partner state of the opposite fermion parity.

## 3.2. Witten Model

The most extensively studied class of SUSY QM problems are Witten models. The Witten model was introduced by Witten as a toy model to discuss the dynamic breaking of supersymmetry in quantum field theories [9]. Formally it is a spin 1/2 particle

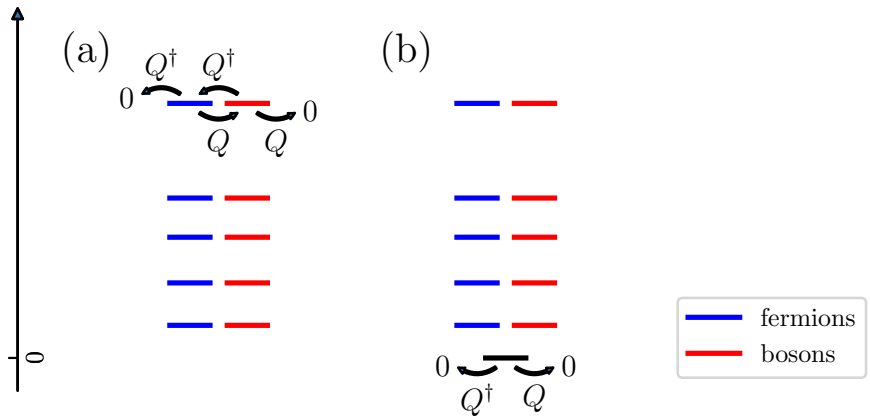


Figure 3.1.: The generic structure of energy levels in the $M = 1$ SUSY QM Witten model. In the Witten model every state comes with a degenerate partner of the other fermion parity and the supercharge operator $Q$ and its hermitian conjugate $Q^\dagger$ map them onto each other. The only exception is a single ground state, that can either be present in a model (b) or not (a), this ground state is annihilated by both $Q$ and $Q^\dagger$.

confined in one dimension and influenced by a spatially dependent magnetic field aligned along an axis. Therefore, its Hamiltonian is of the form

$$H = -\frac{d^2}{dx^2} + V(x) + B(x)\sigma_z. \tag{3.6}$$

The problem becomes supersymmetric when $V(x) = q^2(x)$ and $B(x) = q'(x)$, where we call $q(x)$ the superpotential. In that case, we can express the Hamiltonian using the algebra Eq. (3.1) with the relation

$$Q = \frac{d}{dx} + q(x)\sigma_+, \tag{3.7}$$

where $\sigma_+$ is the operator $|\uparrow\rangle\langle\downarrow|$. It is straightforward to check that this supercharge operator $Q$ generates the Hamiltonian

$$H = \{Q, Q^\dagger\} = -\frac{d^2}{dx^2} + q(x)^2 + q'(x)\sigma_z. \tag{3.8}$$

Since the problem is diagonal in the spin $z$-projection we can treat the problem as two independent spinless systems with the potentials $V^+(x) = q(x)^2 + q'(x)$ and $V^-(x) = q(x)^2 - q'(x)$. These potentials are called pair potentials. Since the general properties of SUSY algebras developed earlier apply to this problem, it gives interesting relations between the two potentials. See Figure 3.2.

By the properties of the algebra we can rule out states with negative energy but we have to investigate if the problem incorporates a ground state with zero energy. To

find a zero energy state we have to find solutions to

$$[q(x)^2 \pm q'(x)]\psi(x) - \psi''(x) = 0. \tag{3.9}$$

Due to the SUSY structure we can factorize the Hamiltonian

$$H\psi(x) = \left[\frac{d}{dx} \pm q(x)\right]\left[-\frac{d}{dx} \pm q(x)\right]\psi(x) = 0 \tag{3.10}$$

into two operators of the form $[d/dx + f(x)]$ allowing to discuss the properties of the system on the basis of first order differential equations.

These first order operators have a continuous spectrum and by integrating the differential equation

$$\frac{d}{dx}\psi(x) + f(x)\psi(x) = 0 \tag{3.11}$$

we can obtain the eigenstate

$$\psi_0(x) = \exp\left[-\int^x dx' f(x')\right] \tag{3.12}$$

with eigenvalue 0. Therefore, the state $\psi$ in Eq. (3.10) is obtained as the state with zero eigenvalue of the first operator expressions $[-d/dx \pm q(x)]$ alone as

$$\psi_0(x) = \exp[\pm Q(x)], \tag{3.13}$$

where $Q(x)$ is the anti-derivative of $q(x)$.

In order to be physical we have to demand that the state $\psi_0$ is normalizable, meaning that the quantity

$$\int dx\, \psi_0^*(x)\psi_0(x) \tag{3.14}$$

exists and is finite. In the following we will assume that the superpotential $q(x)$ is in $C^1$ such that there are no singularities appearing in the relevant expressions. Normalizability in this setup breaks down to the constraint that the wavefunction has to decay fast enough. If we inspect the expression we found for $\psi_0$ in Eq. (3.13) this leads to a constraint on $\pm Q(x)$. In order for $\psi_0$ to vanish for large $|x|$ we require that $\pm Q(x)$ must diverge in both directions to $-\infty$. This implies that the superpotential $\pm q(x)$ must approach $-\infty$ for $x \to -\infty$ and $\infty$ for $x \to \infty$. This is obviously only possible for either $+q(x)$ or $-q(x)$, which leads to the fact that a Witten model has either one or no zero energy state and that there is never a zero energy state in both of the pair potentials simultaneous. Whether or not a Witten model has a zero energy ground state was one the main motivations to study the model in the first place. Conventionally we call SUSY QM broken if there is no zero energy ground state.

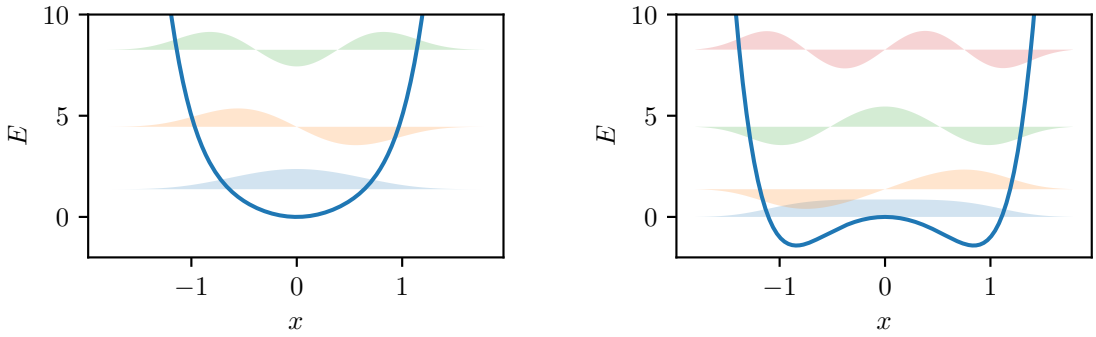


Figure 3.2.: The two pair potentials produced by the superpotential $q(x) = x^3$. With the wavefunctions indicated at the energy level that corresponds to their eigenenergies. Note that the two potentials are isospectral with the exception of one additional zero energy state in the left potential.

## 3.3. Shape Invariance

One of the most important uses of the concept of pair potentials is shape invariance; it is the ingredient to make a system exactly solvable using supersymmetry. The prototypical example for a shape invariant system is the harmonic oscillator. The harmonic oscillator is shifted such that its lowest eigenenergy is vanishing $V(x) = x^2 - 1$. It can be described in the language of the Witten model as $V^-(x) = q(x)^2 - q'(x)$ with $q(x) = x$. The corresponding pair potential $V^+$ is given by $V^+(x) = q(x)^2 + q'(x) = x^2 + 1$ which turns out to have the exact same shape and is just shifted by an offset of 2. Due to supersymmetry we know that both system are isospectral except for the zero energy state in $V^-$. We thus can conclude two statements about the spectra of $V^-$ and $V^+$:

1. The spectrum of $V^-$ is given by 0 and the spectrum of $V^+$.
2. The spectrum of $V^+$ is the spectrum of $V^-$ shifted up by 2.

Using these two statements we can construct the hole spectrum of the system by iterating the statements, i.e., by 1. we know that the spectrum of $V^-$ contains 0, thus we know by 2. that the spectrum of $V^+$ contains 2, thus we know by 1. that the spectrum of $V^-$ contains 2, and starting over. Thus, the full spectrum is given as all the multiples of 2.

This shape invariance in its pure form is given only for the harmonic oscillator. Nevertheless, we can generalize the concept by considering a class of potentials $V^\pm_{\boldsymbol{p}}$ that are parameterized by a vector of parameters $\boldsymbol{p} = (p_1, ...)$. The shape invariance may be between potentials with a different set of parameters $\tilde{\boldsymbol{p}}$, i.e., it may hold

$$V^+_{\boldsymbol{p}} = V^-_{\tilde{\boldsymbol{p}}} + \Delta(\boldsymbol{p}) \tag{3.15}$$

where $\Delta(\boldsymbol{p})$ is the energy shift dependent on the set of parameters.

If for a given class of potentials we are able to derive the map $f$ relating $\boldsymbol{p}$ to $\tilde{\boldsymbol{p}}$ and $\Delta(\boldsymbol{p})$ for a general $\boldsymbol{p}$, we are able to apply the same train of arguments we used to derive the spectrum of the harmonic oscillator. The difference is that the energy we sum up is not given by a constant but by $\Delta$ with the parameter $\boldsymbol{p}$ iterated using $f$. Thus, the eigenvalues are given by

$$\left\{ \sum_{i=0}^{n} \Delta(f^{\circ i}(\boldsymbol{p})) \middle| n \in \mathbb{N} \right\} \cup \{0\} \tag{3.16}$$

where $f^{\circ i}$ denotes the application of $f$ repeated $i$ times.

The potential well

$$V(x) = \begin{cases} 0, & \text{if } -\pi/2 < x < \pi/2 \\ \infty & \text{else} \end{cases} \tag{3.17}$$

should serve as an example. In order to treat it using SUSY QM we need to find its expression as superpotential first. In the discussion of pair potentials there is one exceptional state and that is the zero energy ground state $\psi_0$; it is related to the superpotential $q(x)$ via

$$\psi_0(x) = \exp\left[ -\int^{x} dx' q(x') \right] \tag{3.18}$$

as discussed above. To obtain a superpotential that contains a state $\psi_0$ we can invert the relation and solve for $q(x)$ by taking the logarithm followed by differentiation

$$q(x) = \frac{d}{dx} \ln \psi(x) = \frac{\psi'(x)}{\psi(x)}. \tag{3.19}$$

This method allows to construct potentials that contain an arbitrary ground state.

Since we know the ground state function $\psi_0(x) = \cos(x)$ of the potential well, we can obtain the superpotential $q(x) = \tan(x)$. Evaluating the potential which is described by this superpotential, neglecting singularities that we can interpret as the boundary of the well, we obtain $q^2(x) - q'(x) = -1$. This offset of $-1$ is due to the property of supersymmetric systems to have a zero energy ground state. We have to compensate for this offset later when we sum up the spectrum.

To solve the problem in the context of shape invariance we will introduce the parameter $\alpha$ and discuss the parameterized superpotential $q_\alpha(x) = \alpha \tan(x)$. The analysis boils down to relating $V_\alpha^+(x) = q^2(x;\alpha) + q'(x;\alpha)$ and $V_{\tilde{\alpha}}^-(x) = q^2(x;\tilde{\alpha}) - q'(x;\tilde{\alpha})$, i.e., determining the energy offset $E(\alpha)$ and the transformed $\tilde{\alpha}$ to satisfy the equation

$$V_\alpha^+(x) = \alpha^2 \tan^2(x) + \alpha \sec^2(x) = \tilde{\alpha}^2 \tan^2(x) - \tilde{\alpha} \sec^2(x) + E(\alpha) = V_{\tilde{\alpha}}^-(x). \tag{3.20}$$

The central trigonometric relation to solve that problem is

$$\tan^2(x) = \frac{\sin^2(x)}{\cos^2(x)} = \frac{1 - \cos^2(x)}{\cos^2(x)} = \sec^2(x) - 1. \tag{3.21}$$

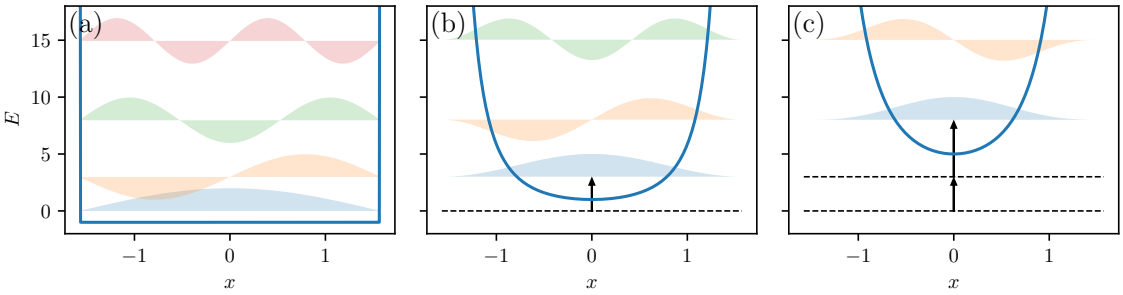


Figure 3.3.: The potential well with an energy offset of 1 to make it conform with the supersymmetric algebra (a) shown together with the eigenenergies. Using the method of shape invariance we remove subsequently the lowest energy state. The potentials obtained are shown in (b) and (c).

Using this relation we can convert $\tan^2$ to $\sec^2$ and apply it on the right hand term of Eq. (3.20)

$$\alpha^2 \tan^2(x) + \alpha \sec^2(x) = (\alpha^2 + \beta)\tan^2(x) + (\alpha - \beta)\sec^2(x) + \beta. \tag{3.22}$$

Finally, we can compare Eq. (3.20) and Eq. (3.22) and read of the relations $\beta = E(\alpha)$, $\tilde{\alpha} = \beta - \alpha$, and $\tilde{\alpha}^2 = \alpha^2 + \beta$. These are resolving to $\tilde{\alpha} = \alpha + 1$ and $E(\alpha) = 1 + 2\alpha$.

Summing up the the energies as described in Eq. (3.16) we obtain

$$E_n = n^2 + 2n. \tag{3.23}$$

If we correct for the shift of $-1$ that we introduced to cast the problem in supersymmetric form, we end up with the spectrum

$$E_n = n^2 + 2n + 1 = (n+1)^2, \tag{3.24}$$

the known result for the potential well.

Applying above procedure of transforming the potential to its partner potential produces a series of potentials that, if shifted by the necessary offset energy, are almost isospectral, except for the low lying states that were removed. See Figure 3.3.

The class of problems that can be discussed using shape invariance includes many of the important potentials in physics like the Morse potential, the Coulomb potential, and many more. For a list of potentials that have this property see [10].

## 3.4. Classification

So far we have used the algebra Eq. (3.1) to characterize supersymmetric systems. While this is convenient to describe Witten models we will soon proceed to models that do not nicely fit in this formulation. There is a hole class of algebras of which most have very similar properties to the algebra we introduced above. We want to give other possible formulations of SUSY QM and a way to classify such algebras [11]. The

first important property of an algebra is the number of conserved supercharges. While in our examples we will only deal with systems containing one type of supercharge, in general one might have multiple such supercharges

$$Q_i, \quad i \in \{1...N\}. \tag{3.25}$$

Different supercharges must anticommute

$$\{Q_i, Q_j\} = 0 \quad \text{for} \quad i \neq j. \tag{3.26}$$

Given this set of supercharges there are two possibilities to construct the Hamiltonian: Either the supercharges are Hermitian in that case the conventional way is to construct

$$H = Q_i^2 \tag{3.27}$$

or they are not in that case we demand

$$\{Q_i, Q_i\} = \{Q_i^\dagger, Q_i^\dagger\} = 0 \tag{3.28}$$

and construct the Hamiltonian as introduced earlier via

$$H = \{Q_i, Q_i^\dagger\}. \tag{3.29}$$

If there are multiple supercharges, each supercharge must produce the same Hamiltonian.

An additional object that may be present in a supersymmetric algebra is an operator $K$ that tells apart bosonic and fermionic states. We demand that this operator is an involution, i.e., $K^2 = 1$ and that the eigenspace with eigenvalue 1 contains all bosonic states of the system and the eigenspace with eigenvalue -1 contains all fermionic states. This can also be captured in the anticommutation relation

$$\{K, Q\} = 0, \quad \text{which also implies} \quad [K, Q] = 0. \tag{3.30}$$

The conventional notation for the class of the algebra captures theses properties. The declaration contains the letter $n$ in the case that the supercharges are Hermitian and the letter $m$ in the case they are not. The letter is used as a capital in the case that the algebra does not contain an involution, otherwise a lowercase is used. Finally, that letter is set equal to the number of different supercharges appearing. For example, the algebra Eq. (3.1) introduced earlier is called '$M = 1$ SUSY QM'.

The most trivial case of $N = 1$ SUSY QM does not actually come with any of the properties introduces earlier, because it does not contain a single anticommutation relation. The algebras $N = 2$, $M = 1$, $n = 1$, $n = 2$, and $m = 1$ are all equivalent as shown in [11], but to transform one formulation into another requires sometimes operations that can in general not be performed analytical such as building the inverse of a supercharge. Therefore, it can still be advantageous to use different formulations knowing that they all come with the properties discussed for $M = 1$ SUSY QM.

## 3.5. Periodic Systems

While the Witten model provides an easy way to generate Hamiltonians with degeneracies, the structure of the Hamiltonian is rather constructed since the degeneracy always lies in between two completely disjoint problems. If one is interested in the consequences of SUSY QM on the dynamics of a system, it might be a lot more interesting to have a problem with a single Hilbert space containing degenerate states. This is not possible for bound states in a one dimensional system with trivial topology because Sturm-Liouville theory forbids that such a system could have a degeneracy. Nevertheless, it can be realized in a system with periodic boundary conditions since these can contain degeneracies as discussed above. Therefore, we want to focus now on one dimensional systems with the configuration space $\phi \in [0, 2\pi)$, periodic boundary conditions, and without any spin degrees of freedom. Systems exhibiting SUSY QM in this configuration can be obtained by superpotentials which are antisymmetric under the operator $\Pi = \exp[i\pi\, d/d\phi]$, i.e., a translation by half the periodicity of the system.

The algebraic properties of $\Pi$ with respect to the other relevant operators are given by

$$[\Pi, d/d\phi] = 0, \quad [\Pi, \phi] = \phi \pm \pi. \tag{3.31}$$

The sign is irrelevant since $\phi$ is $2\pi$-periodic and both expression are equivalent. The operator $\Pi$ is Hermitian since $\Pi^\dagger = \exp[-i\pi\,(-d/d\phi)] = \exp[i\pi\, d/d\phi]$ and also unitary since $\Pi^\dagger\Pi = \Pi^2$ which is a translation by the periodicity $2\pi$ and thus acts trivial. Therefore, it is an involution, i.e., its spectrum is constrained to $\pm 1$.

Given a superpotential

$$\Pi q(x)\Pi = -q(x), \tag{3.32}$$

we can build a $N = 1$ SUSY QM algebra by setting

$$Q = \Pi\,[i\partial_\phi + iq(\phi)] \tag{3.33}$$

and obtaining the Hamiltonian

$$H = Q^2 = -\partial_\phi^2 + q(\phi)^2 - q'(\phi). \tag{3.34}$$

It is worth noting that the form of the Hamiltonian is similar to the Witten model discussed earlier. In contrast to the Witten models which where defined on the full $\mathbb{R}$, the equation

$$\frac{d}{d\phi}\psi(\phi) + q(\phi)\psi(\phi) = 0 \tag{3.35}$$

is defined on $\phi \in [0, 2\pi)$. Therefore, it has always normalizable solutions. That way all of the models of type Eq. (3.34) have a zero energy ground state.

The major drawback is that $N = 1$ actually does not come with the nice properties discussed earlier for $M = 1$ SUSY QM [11]. One possible cure to that problem is introducing a rather ad hoc reflectional antisymmetry on the superpotential $q(-\phi) = -q(\phi)$ that could act as an involution to leverage the model to $n = 1$ SUSY QM, this significantly reduces the space of possible models to be discussed in this setup.

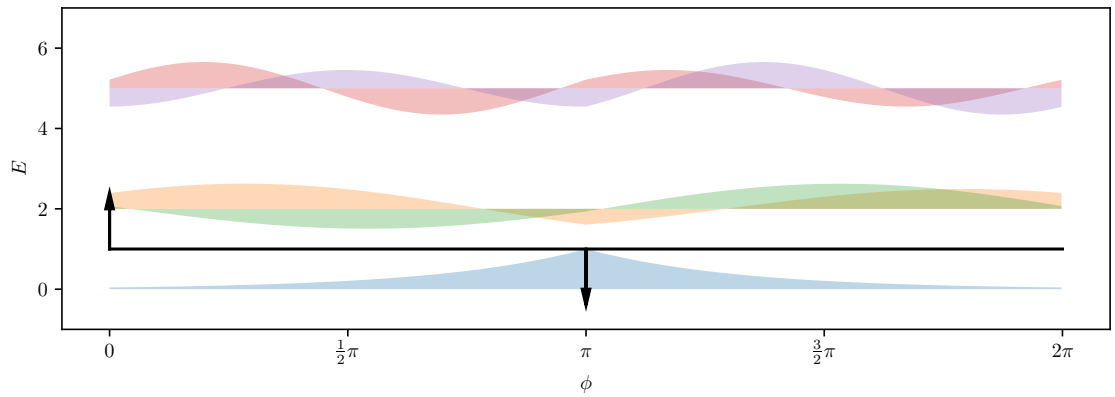


Figure 3.4.: The alternating Kronig-Penney model with an offset to make it conform with the supersymmetric algebra. The arrows indicate the $\delta$-distributive potential peaks.

Instead we will show that we can obtain a $M = 1$ formulation by algebra alone. First, we introduce the projection operators on the space symmetric $\pi_s$ and antisymmetric $\pi_a$ under $\Pi$

$$\pi_s = (1 + \Pi)/2, \quad \pi_a = (1 - \Pi)/2. \tag{3.36}$$

Defining the supercharge as

$$Q = -\pi_a \left[ i\partial_\phi + iq(\phi) \right] \pi_s \tag{3.37}$$

yields the desired properties that

$$Q^2 = Q^{\dagger 2} = 0 \text{ and } \{Q, Q^\dagger\} = H, \tag{3.38}$$

giving a $M = 1$ SUSY QM model with all implications discussed earlier.

As a short example to demonstrate how much information we actually obtain about a system once we can infer the SUSY structure of it, we will discuss the model implied by the superpotential

$$q(\phi) = \alpha \begin{cases} 1/2 & \text{if } 0 \leq \phi < \pi, \\ -1/2 & \text{else.} \end{cases} \tag{3.39}$$

It is straightforward to check that this model satisfies the antisymmetry constraint Eq. (3.32) and thus forms a valid supersymmetric potential. Evaluating the Hamiltonian, the associated potential is given by

$$V(\phi) = \alpha^2/4 + \alpha\delta(\phi) - \alpha\delta(\phi - \pi) \tag{3.40}$$

which is an alternating attractive and repulsive Kronig-Penney model with an additional offset energy $\alpha^2$. The potential is shown in Figure 3.4. Knowing that the SUSY QM properties of that model enforces a vanishing ground state energy, we directly conclude that the ground state energy of the alternating Kronig-Penney model alone is given by $-\alpha^2/4$. Also, the ground state wavefunction is trivial to evaluate using expression Eq. (3.12). In addition, we learn that the higher energy levels are all degenerate.

# 4. Double Sine Potential

Among the periodic potentials that exhibit supersymmetry the potential that has attracted the most interest is the double sine potential. In this chapter we want to asses some of the properties of this potential. The Hamiltonian is given by

$$H_\gamma = -\frac{d^2}{d\phi^2} + V_\gamma(\phi); \quad V_\gamma(\phi) = \alpha^2 \sin^2\phi + \alpha\gamma\cos\phi \tag{4.1}$$

with $\phi \in [0, 2\pi)$ and periodic boundary conditions. The system with $\gamma = 1$ can be obtained by setting $q = \alpha \sin\phi$ in the periodic supersymmetry Eq. (3.34). The shape of the potential is shown in Figure 4.1.

The double sine potential occurs naturally in the description of the Majorana-Cooper-Pairbox [5]. Its analysis dates back to an article by Razavy [12] in 1981 in which he showed that the spectrum of this potential, it was used to model the torsion in the $C - C$ bond of ethylene, contains analytic values in the case that $\gamma$ is an odd integer.

We observed in numerics that, in case that $\gamma$ is an odd integer, the part of the spectrum which does not admit an analytic solutions shows an infinite number of degenerate pairs of states similar to supersymmetry. This property is explored in this chapter in the limit $\alpha \gg 1$, including semiclassic corrections. For an algebraic discussion of that property see Chapter 5.

## 4.1. Strong Potential Limit

In the regime where the potential dominates the dynamics of the system, i.e., in cases where $\alpha \gg 1$ we can approximate both of the wells by harmonic oscillators as shown in Figure 4.1. As a first approximation we neglect the tunnel coupling by the finite barriers in between. In the next section we will include the coupling using an instanton approach.

First, we expand the potential around the local minima of the potential $\phi = 0$ and $\phi = \pi$. We employ a matrix notation in which states lying in the valley close to $\phi = 0$ are noted in the upper component and states which lie close to $\phi = \pi$ are noted in the lower component. We introduce the relative coordinate $\phi = x$ in the upper component and $\phi - \pi = x$ in the lower component. To obtain the Hamiltonian in this notation and to show that the approximation does not break the supersymmetry, we will proceed by bringing subsequently all operators that are involved in the supersymmetric algebra into this notation.

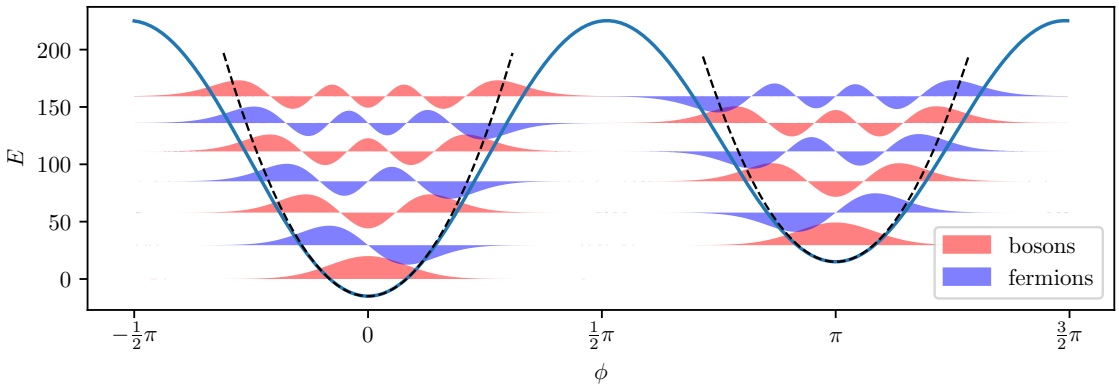


Figure 4.1.: The double sine potential in the strong potential limit in which we can approximate the wells arround $\phi = 0$ and $\phi = \pi$ with harmonic oscilators.

The first relevant operator is $\Pi$, introduced in Section 3.5; it transfers states from 0 to $\pi$ and vice versa. Therefore, it is given by

$$\Pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4.2}$$

in the matrix notation. We obtain the supercharge operator

$$Q = \Pi \left( i\frac{d}{dx} + iq(\phi) \right) \quad \text{with} \quad q(\phi) = \alpha \sin \phi \tag{4.3}$$

in the limit $\alpha \gg 1$ by first order expansion of the superpotential $q(\phi)$ in $\phi$, since in this limit only the dynamics around small excitations from $\phi = 0$ and $\phi = \pi$ will play a role. Together with the representation of $\Pi$ it is given as

$$Q \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i\frac{d}{dx} + i\alpha x & 0 \\ 0 & i\frac{d}{dx} - i\alpha x \end{bmatrix} = \begin{bmatrix} 0 & i\frac{d}{dx} - i\alpha x \\ i\frac{d}{dx} + i\alpha x & 0 \end{bmatrix}. \tag{4.4}$$

Next, we reformulate this expression using the ladder operators for a harmonic oscillator

$$a = \sqrt{\frac{\alpha}{2}} x + \frac{1}{\sqrt{2\alpha}} \frac{d}{dx} \quad \text{satisfying} \quad [a, a^\dagger] = 1 \tag{4.5}$$

to obtain

$$Q = \sqrt{2\alpha} i \begin{bmatrix} 0 & -a^\dagger \\ a & 0 \end{bmatrix}, \quad H_1 = Q^2 = 2\alpha \begin{bmatrix} a^\dagger a & 0 \\ 0 & aa^\dagger \end{bmatrix}. \tag{4.6}$$

The Hamiltonian coincides with that of the textbook example of the SUSY QM harmonic oscillator, but it differs in the classification of fermionic and bosonic states. In the SUSY QM harmonic oscillator we conventionally introduce $K = \sigma_z$ (See Section 3.4) as involution operator, i.e., classifying states in one well as fermions and in the other as bosons. However, sticking to the convention set in [5], the involution is given by the parity operation $\phi \mapsto -\phi$. Because the reflection operator acts due to

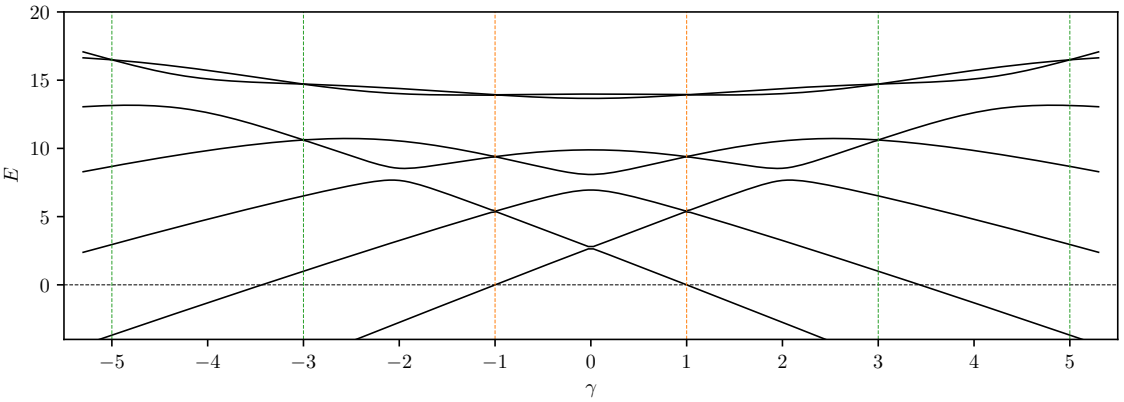


Figure 4.2.: The spectrum of the periodic Hamiltonian with the double sine potential $H = -d^2/d\phi^2 + \alpha^2 \sin\phi + \gamma\alpha\cos\phi$ for $\alpha = 3$ fixed as a function of $\gamma$ obtained numerically. We highlight the two points of supersymmetry at $\gamma = \pm 1$ characterized by 0 groundstate energy and all other levels are degenerate. The other points showing degeneracy appear at $\gamma = \pm 3$ and $\gamma = \pm 5$.

the periodicity simultaneous as reflection around $\phi = 0$ and $\phi = \pi$ the involution acts in both components of the problem in the same way $x \mapsto -x$. Thereby the fermion parity is given by the parity in each of the wells. In the notation of creation and annihilation operators that expression is given as $K = \exp(i\pi a^\dagger a)$.

Tuning the potential away from the points $\gamma = \pm 1$, we obtain in numerical simulations a non-degenerate spectrum unless $\gamma$ is an odd integer. If $\gamma$ is an odd integer we find that there are $|\gamma|$ low lying states that are non degenerate, while all higher excited states are degenerate. As can be found in Figure 4.2. In the following we want to prove that in the limit $\alpha \gg 1$ this statement holds not just in numerics, but it is exact.

So far we only brought the supersymmetric version of the model, i.e., with $\gamma = 1$ fixed, in the matrix notation. In order to tune the model away from that point we have to evaluate $(\gamma - 1)\cos\phi$ in the $\alpha \gg 1$ approximation. In that limit the expression $\cos\phi$ will only contribute as an offset for the states in the wells and thus the correction is given by

$$\alpha(\gamma - 1)\cos\phi \sim \alpha(\gamma - 1)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4.7}$$

Adding that term to the Hamiltonian yields the expression

$$H_\gamma = 2\alpha \begin{bmatrix} a^\dagger a + \frac{\gamma-1}{2} & 0 \\ 0 & aa^\dagger - \frac{\gamma-1}{2} \end{bmatrix}. \tag{4.8}$$

While it is straightforward to solve the model exactly and show that for integer values of $|\gamma|$ the Hamiltonian shows degeneracies, we want to pursue a different way and use an algebraic approach. To do so we go one step back and and try to express $H_\gamma$ in terms of the algebraic objects that we have available: $\Pi$ and $Q$. Since we can

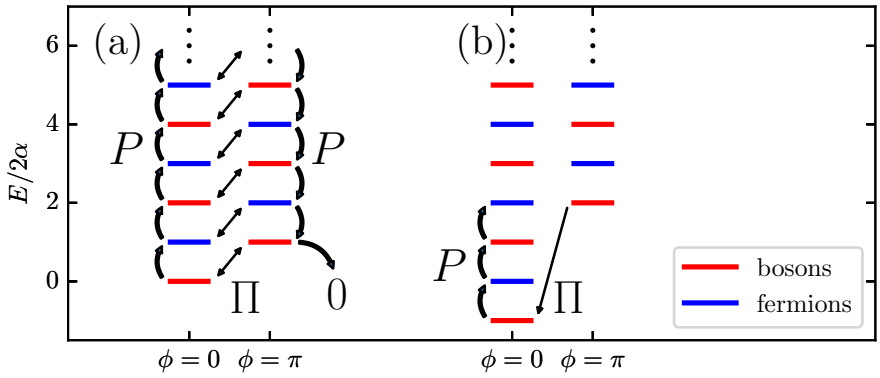


Figure 4.3.: The levels of the double sine potential in the limit $\alpha \to \infty$ given in (a) for $\gamma = 1$ and in (b) for $\gamma = 3$. In (a) we sketch the action of the operators $P$ and $\Pi$. Note that the supercharge operator $Q$ that maps fermions on bosons of the same energy can be obtained by sequentially applying $\Pi$ and $P$. In (b) we only sketch the modified supercharge operator $Q_3$ that does the same thing.

construct the supersymmetric part of the problem, as shown above, we are left with constructing a term proportional to $\cos\phi$. First, we define the operator

$$P = Q\Pi = i\frac{d}{d\phi} - i\sin\phi \tag{4.9}$$

noting that

$$P^\dagger = i\frac{d}{d\phi} + i\sin\phi = \Pi Q. \tag{4.10}$$

Evaluating the commutator yields $[P^\dagger, P] = 2\cos\phi$, just the expression that is required. The Hamiltonian can thus be written as

$$H_\gamma = Q^2 + \frac{\gamma - 1}{2}[\Pi Q, Q\Pi] = PP^\dagger + \frac{\gamma - 1}{2}[P, P^\dagger]. \tag{4.11}$$

In the following we will introduce $n = (\gamma - 1)/2$ to simplify the notation.

Bringing those expressions back in the strong potential limit and the matrix notation we obtain

$$P = \sqrt{\alpha} i \begin{bmatrix} a & 0 \\ 0 & -a^\dagger \end{bmatrix} \tag{4.12}$$

and

$$[P, P^\dagger] = -\alpha \begin{bmatrix} [a, -a^\dagger] & 0 \\ 0 & [-a^\dagger, a] \end{bmatrix} = \alpha \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4.13}$$

For the Hamiltonian in the strong potential limit $H_n$ with $n \neq 0$ is the supercharge $Q$ no longer conserved. To prove that the states are degenerate we need a different fermionic operator $Q_n$ that commutes with the Hamiltonian $H_n$. Because given such

an operator we can argue about the degeneracy structure in the same way as done in Section 3.1 with the exception that being annihilated by $Q_n$ does neither imply that it has zero energy nor that it is a groundstate but only that this conserved charge does not imply the existence of a degenerate partner state.

In the following we are constructing such a charge $Q_n$ by creating an operator that maps the lowest pair of degenerate states onto each other. In order to construct the operator, we study the spectrum of $H_n$ and the action of $P$ and $\Pi$ as shown in Figure 4.3. The operator $P$ acts as creation operator in the $\phi = 0$ well and as annihilation operator in the $\phi = \pi$ well. The $\Pi$ operator moves states from one well to the other. To map the lowest state in one well on the degenerate partner in the other well we first use $\Pi$ to map the lowest lying state in the $\phi = \pi$ well to the lowest lying state in the $\phi = 0$ well. To reach the degenerate partnerstate we have to climb up $2n + 1$ states using $P^{2n+1}$. Combining both operators we obtain $Q_n = P^{2n+1}\Pi$. It is straightforward to check that this operator is actually fermionic, i.e., $\{K, Q_n\} = 0$ and that it commutes with the Hamiltonian. However, this charge $Q_n$ is only conserved in the limit $\alpha \gg 1$, the correction to the charge that are required outside that regime will be introduced in Chapter 5.

## 4.2. Instanton

We have seen so far that in the limit of strong potentials the supersymmetry and the degeneracies appearing for odd $\gamma$ are conserved. In a next step, we want to give an indication that this stays the case even when including a tunnel coupling between the two wells. To study the effect of tunneling we will employ the method of saddlepoint approximation of the path integral in the form of the instanton. For an introduction to the path integral approach to tunnel splitting see [13].

To give an introduction in the method of the instanton we start by evaluating the splitting between the two wells of a pure sinusoidal model, a system that is equivalent to the parameter choice $\gamma = 0$ in the double sine model. In order to determine the tunnel splitting between the states lying in each of the wells the first step is to invert the potential $V_{\text{inv}}(x) = -V(x)$. The splitting can be derived from the action $S$ of a trajectory from one maximum of the potential to the other. The trajectory is shown in Figure 4.4.

The action $S$ is given by time integration of the Lagrangian $L$ of the system

$$S = \int_{t_2}^{t_1} dt\, L = \int_{t_1}^{t_2} dt\, \left[\dot{x}^2 - V_{\text{inv}}(x)\right]. \tag{4.14}$$

Using the conservation of Energy $E = \dot{x}^2 + V_{\text{inv}}(x)$ we can reformulate the expression to

$$S = \int_{t_1}^{t_2} dt\, \left[2\dot{x}^2 - E\right]. \tag{4.15}$$

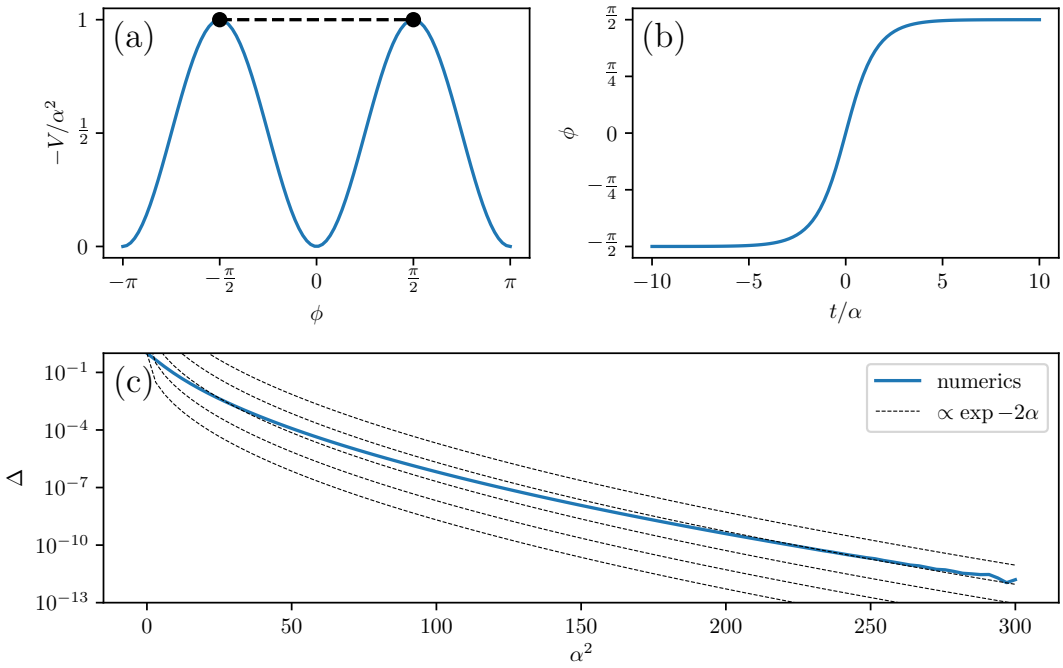


Figure 4.4.: The instanton contributing in the tunnelsplitting between two adjacent wells of the double sine potential with $\gamma = 0$, i.e., a purely sinusoidal model. In (a) we show the inverse potential and indicate the classic trajectory we have to evaluate to determine the tunnelsplitting. In (b) we show the trajectory that is sketched in (a) as a function of time. In (c) we show the approximation of the tunnelsplitting obtained by the instanton approach together with numeric results.

For the splitting of the energylevels only the kinetic part of the action contributes; we have to gauge the energy such that it is vanishes at the starting- and endpoint. Therefore, we can set $E = 0$ in above relation. The first approximation for the splitting $\Delta$ is then given as

$$\Delta = \exp[-S]. \tag{4.16}$$

To evaluate the expression we have to compute the trajectory $x(t)$ of a particle starting in the inverted potential from a maximum and ending up at the next maximum. Using that

$$v^2 = E_0 - V_{\text{inv}}(x) \implies v = \sqrt{E_0 - V_{\text{inv}}(x)} \tag{4.17}$$

we can write down an integral equation for the time which the particle requires to travel from point $x_1$ to $x_2$ given that its total energy is $E_0$. This expression can be obtained by integrating up the inverse of the velocity

$$t = \int dx \frac{dt}{dx} = \int dx v^{-1}(x) = \int dx \frac{1}{\sqrt{E_0 - V_{\text{inv}}(x)}}. \tag{4.18}$$

This is possible in general for one dimensional systems with conserved energy.

We want to solve this integral for the potential with $\gamma = 0$, i.e., $V(x) = \alpha^2 \sin(x)^2$. The motion we are interested in is starting from the maxima of the inverted potential $x = \pm\pi$. Therefore, the Energy of the motion is given by $E = V(\pi) = \alpha^2$ and we face the integration

$$t(x) = \int^x dx' \frac{1}{\sqrt{\alpha^2 - \alpha^2 \sin^2 x'}} = \frac{1}{\alpha} \int^x dx' \frac{1}{\cos x'} = \frac{2}{\alpha} \operatorname{artanh} \tan \frac{x}{2}. \tag{4.19}$$

This function $t(x)$ gives the amount of time passed once the trajectory arrives at $x$. We can invert it to obtain the trajectory as a function of time

$$x(t) = 2 \arctan \tanh \frac{\alpha}{2} t. \tag{4.20}$$

Using this trajectory we can evaluate Eq. (4.15).

$$\int dt\, \dot{x}^2 = \int dt\, \frac{\beta}{\cosh^2 \alpha t} = \alpha \tanh \alpha t. \tag{4.21}$$

Studying the asymptotic behavior of the trajectory, shown in Figure 4.4, it becomes apparent that we have to integrate the action all the way from $t = -\infty$ to $t = \infty$. This gives a total action $S = 2\alpha$ and thus an asymptotic splitting of the levels that is given for $\alpha \gg 1$ by

$$\Delta \approx \exp\left[-2\alpha\right]. \tag{4.22}$$

Including a $\gamma \neq 0$ in this formalism is difficult since there is only one absolute maximum and we can no longer connect two maxima with one trajectory. However, recently a new approach emerged for these systems. A formalism that includes trajectories that are not constrained to the real axis, but allows for excursions in the imaginary plane. The original paper [14] studies the double sine potential around the supersymmetric point $\gamma = 1$ since they want to resolve the issue that the groundstate energy derived in the path integral formalism is not 0 as the supersymmetry suggests.

They proceed the same way as introduced above: First, the equation of motion is solved and afterwards the action is integrated. We want to proceed in a different way and compute the action given by the path alone. Therefore, we take Eq. (4.15) and apply the transformation from time to space relating $dt \mapsto \dot{x}^{-1} dx$ obtaining the expression

$$S = \int dx \dot{x}. \tag{4.23}$$

This is, in our unit choice, equivalent to an integration of the momentum that may be used in other places. By considering energy conservation and choosing an energy offset such that $\dot{x} = 0$ implies that $V_{\text{inv}}$ is vanishing we can restate that expression as

$$S = \int dx \sqrt{V_{\text{inv}}(x)}. \tag{4.24}$$

The approach presented in [14] is equivalent to considering $V_{\text{inv}}^{1/2}(x)$ as a function $\mathbb{C} \to \mathbb{C}$ with a non trivial Riemann surface in which we have to search for all paths

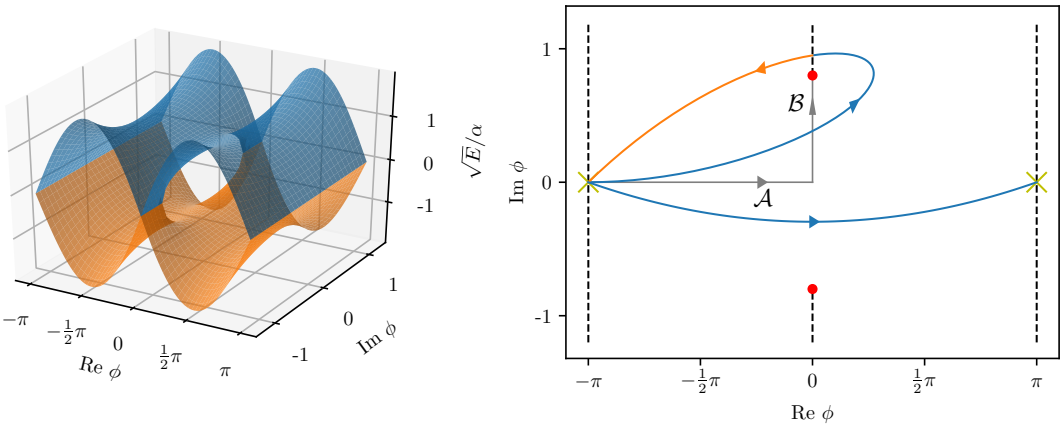


Figure 4.5.: On the left a 3D representation of the Rieman surface associated to the real part of the map $f : \phi \mapsto V_{\text{inv}}^{1/2}$. In the 3D representation the nontrivial topology of the surface becomes apparent. On the left we show a sketch of the same map. We indicate with dashed lines the branch cuts of the functions and mark the branch points with red dots. The yellow crosses denote the points which must be connected by a path for an instanton contribution. The two topologically different trajectories contributing to the groundstate energy are shown in the color corresponding to the Riemansheet they are on (compare to the left plot). In gray we show the paths $\mathcal{A}$ and $\mathcal{B}$ that are used for the evaluation of the integrals (see text).

that connect the maximum of $V_{\text{inv}}$ back to itself in order to get the correct asymptotic expression. The obvious one is going from $x = -\pi$ along the real axis all the way to $x = \pi$, which is equivalent to $x = -\pi$ by periodicity.

If we study the Rieman surface associated with the map $f : x \mapsto (\alpha^2 \sin^2 x - \gamma\alpha \cos x)^{1/2}$ shown in Figure 4.5, we realize that this function has a branch cut at $x = iy$ for $|y| > \operatorname{arcosh} \alpha + \gamma/\alpha = b$ with branchpoints at $x = \pm ib$. We can close a trajectory from $x = -\pi$ back to itself by considering a trajectory starting at $x = -\pi$ encircling $x = ib$ and returning to $x = -\pi$. To obtain the correct groundstate contribution we have to include this path in our calculation.

Since integrations over analytic functions are invariant under deformation of the path, we can choose an integration path which is convenient to evaluate. In total we require only two integrals: We have to evaluate the integration along $\mathcal{A}$ from $x = -\pi$ to $x = 0$ along the real line and we have to evaluate the integration along $\mathcal{B}$ from $x = 0$ to $x = i(\operatorname{arcosh} \alpha + \gamma/\alpha)$. Both paths are shown in Figure 4.5. From these two expressions we can stitch together both paths. The path along the real line is given by $\mathcal{A}$ followed by a contribution that has by inversion symmetry around $x = 0$ the same value as the integration along $\mathcal{A}$. The path including the excursion in the complex direction can be composed from $\mathcal{A}$ followed by $\mathcal{B}$ on the upper Riemann sheet followed by the same path in reverse direction on the lower Riemann sheet; by symmetry of the two branches the value of the integration is given as two times the contribution of

$\mathcal{A}$ and two times the contribution of $\mathcal{B}$. The contribution of the integral along path $\mathcal{A}$ given by

$$\begin{aligned} S_{\mathcal{A}} &= \int_{-\pi}^{0} dx \sqrt{\alpha\gamma + \alpha^2 \sin^2 x + \alpha\gamma \cos x} = \\ &= \sqrt{2\alpha(2\alpha+\gamma)} + \gamma \tanh^{-1}\left(\sqrt{\frac{2\alpha}{2\alpha+\gamma}}\right) \end{aligned} \tag{4.25}$$

and along path $\mathcal{B}$ given by

$$S_{\mathcal{B}} = \int_{0}^{\operatorname{arcosh}\frac{\alpha+\gamma}{\alpha}} i\sqrt{\alpha\gamma - \alpha^2 \sinh^2(x) + \alpha\gamma\cosh(x)} = \frac{i\pi\gamma}{2}. \tag{4.26}$$

Adding up the contribution for the action $S_r$ of the path along the real line and $S_i$ for the path containing complex excursions, we obtain

$$S_r = 2S_{\mathcal{A}} \quad \text{and} \quad S_i = 2S_{\mathcal{A}} + 2B_{\mathcal{B}}. \tag{4.27}$$

To calculate the instanton contribution to the groundstate energy we have to add up the contributions of both paths

$$E_{\text{gs}} \sim -e^{-S_r} - e^{-S_i} = -e^{2S_{\mathcal{A}}}(1 + e^{2S_{\mathcal{B}}}) = -e^{2S_{\mathcal{A}}}(1 + e^{i\pi\gamma}). \tag{4.28}$$

It turns out that for $\gamma = \pm 1$ the expression vanishes. Nevertheless, this is also true in the case that $\gamma = 2k + 1$ for any $k \in \mathbb{Z}$. This hints that at those points a structure similar to supersymmetry is present and that the numerically observed degeneracies are also present outside the $\alpha \gg 1$ regime.

# 5. Partial Algebraization

While the double sine potential exhibits a point of supersymmetry, it becomes apparent that tuning the parameter $\gamma$ away from 1 the system has other points with huge degeneracies. See Figure 4.2. Besides the degeneracies these points are special because they allow to solve exactly for all the states that are nondegenerate as it was first pointed out by Razavy [12]. This property is referred to as partial algebraization. Later it was worked out [15] that the double sine potential and many other examples, see Tab. 5.1, can be studied using a differential operator representations of the $\mathfrak{sl}(2)$ Lie-Algebra. Using these algebraic properties, the Hilbert space of the problem can be separated in one infinite dimensional problem and one decoupled finite dimensional problem which corresponds to the states that can be exactly solved for.

## 5.1. Perturbation Theory

To find the finite dimensional part of the Hilbertspace which is split away from the problem, it is enlightening to discuss the double sine potential in the limit of a weak potential, i.e., $\alpha \ll 1$. In that regime the states can be approximated by perturbation around the eigenstates of a free periodic problem $|k\rangle = \exp ik\phi$. The relevant operators are $d/d\phi$, $\cos\phi$, and $\sin\phi$. All of them are easy expressible in the $k$ basis as

$$\frac{d}{d\phi} \mapsto \sum_k ik \left|k\right\rangle \left\langle k\right| = iK, \tag{5.1}$$

$$\cos\phi \mapsto \frac{1}{2} \sum_k \left|k\right\rangle \left\langle k+1\right| + h.c. = C, \tag{5.2}$$

$$\sin\phi \mapsto \frac{1}{2} \sum_k i \left|k\right\rangle \left\langle k+1\right| + h.c = S. \tag{5.3}$$

To illustrate the perturbation, we will use diagrammatics as shown in Figure 5.1. In this diagram each momentum eigenstate state will be represented by a site and the potential will be represented as a hopping. The term $\cos\phi$ represents a next neighbor hopping and the $\sin^2\phi$ term is represented by a hopping to the second next neighbor. In order to treat both $\sin\phi$ and $\cos\phi$ by the means of a common operator, we introduce a hopping operator that hops to the left

$$G = \sum_k \left|k\right\rangle \left\langle k+1\right|. \tag{5.4}$$

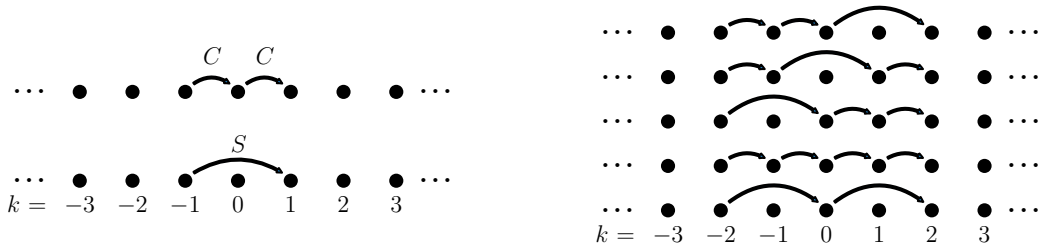


Figure 5.1.: Perturbation of the double sine potential in the weak potential limit. In the first plot we show the two contributing couplings: $S$ corresponding to the $\sin^2\phi$ term and $C$ corresponding to the $\cos\phi$ term. The two diagrams contribute to the splitting between $|k=-1\rangle$ and $|k=1\rangle$. In the second sketch we show all the contributions for the splitting of the states $|k=2\rangle$ and $|k=-2\rangle$ in lowest order in $\alpha$.

using this expression we can reformulate

$$C = \frac{G}{2} + \frac{G^\dagger}{2}, \quad S = \frac{iG}{2} - \frac{iG^\dagger}{2} \tag{5.5}$$

and express the Hamiltonian as

$$H = K^2 + \alpha^2 S^2 + \alpha\gamma C = K^2 + \frac{\alpha^2}{4}(2 - G^2 - G^{\dagger 2}) + \frac{\alpha\gamma}{2}(G + G^\dagger). \tag{5.6}$$

Since our main interest is the degeneracy, we will compute the splitting of the states $|k\rangle$ and $|-k\rangle$ to the lowest order in $\alpha$. By perturbation theory the splitting between the two is given by

$$\Delta = 2\,\langle\psi_a|\, H_{\text{pert}} \prod_{j=0} \left[\frac{1}{E_0 - H_0} H_{\text{pert}}\right]^j |\psi_b\rangle\,. \tag{5.7}$$

Using that our unperturbed Hamiltonian is $K^2$ and the Energy $E_0$ of the unperturbed state is given by $k^2$ all we have to do is to translate the diagrams from Figure 5.1 into equation. For each arrow corresponding to $H_{\text{pert}}$ we have to multiply with $\alpha\gamma$ in the case it is a hop by one site or $\alpha^2$ in the case it hops by two sites. In addition, we have to multiply witch $1/(k^2 - l^2)$ for each intermediate state $l$. Adding up all contribution will give the splitting.

For the splitting between $k = -1$ and $k = 1$ this evaluates to

$$\Delta = \frac{1}{2}\left(\gamma^2\alpha^2 - \alpha^2\right) = \frac{\alpha^2}{2}(\gamma + 1)(\gamma - 1). \tag{5.8}$$

For the splitting between $k = 2, -2$ we have to include already 5 diagrams, shown in Figure 5.1, and we obtain

$$\Delta = 2\frac{\alpha^4}{16}\left(\frac{1}{4} + \frac{\gamma^4}{3\cdot 4\cdot 3} + 2\frac{\gamma^2}{4\cdot 3} + \frac{\gamma^2}{3\cdot 3}\right) = \frac{\alpha^4}{288}(\gamma + 3)(\gamma + 1)(\gamma - 1)(\gamma - 3). \tag{5.9}$$

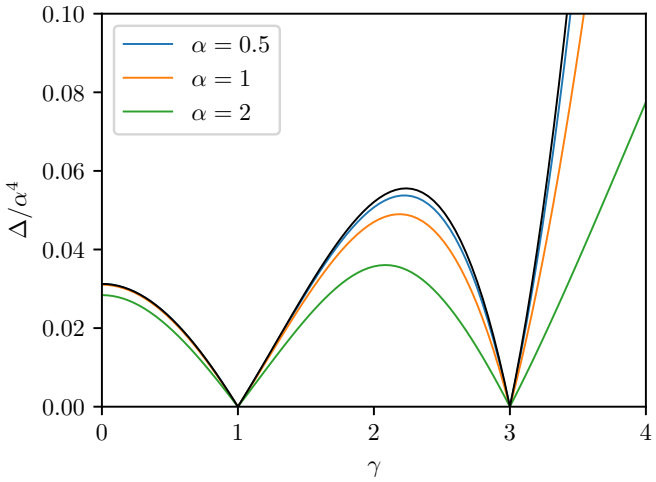


Figure 5.2.: The splitting $\Delta$ between the 4th and 5th eigenstate in the double sine potential as a function of $\gamma$. In black we show the first approximation for $\alpha \ll 1$ in comparison to numeric values for finite $\alpha$.

It turns out that the splitting for the second states does not only vanish at the points $\gamma = \pm 1$ but also at $\gamma = \pm 3$, this hints toward the structure observed in Figure 4.2. We compare the approximation with numerics for different $\alpha$ shown in Figure 5.1.

While it is possible to continue this expansion, the number of diagrams that are required to calculate the splitting between $k = n, -n$ grows exponentially in $n$ and the combinatorics are hard to track. This is because in this expansion we require two kinds of operators that are of different order in the hopping. However, we can cure the problem by introducing a transformation of the Hamiltonian that shifts a part of the perturbation $H_{\text{pert}}$ in the unperturbed Hamiltonian $H_0$ leaving us with a perturbation scheme which does not contain expressions of order $\alpha^2$. We can achieve that by transforming the Hamiltonian using the generator $O = -i\alpha \cos\phi$. Since this generator is not Hermitian we will end up with a non-Hermitian Hamiltonian. However, it is a proper linear transformation thus it will leave the eigenvalues of the system unchanged.

Under this transformation the only operator of interest which changes is the derivative

$$\frac{d}{d\phi} \mapsto \exp[-iO] \frac{d}{d\phi} \exp[iO] = \frac{d}{d\phi} - \alpha \sin\phi \tag{5.10}$$

since both $\cos\phi$ and $\sin\phi$ commute with $O$. Therefore, the transformation results in the Hamiltonian

$$H = -\left(\frac{d}{d\phi} - \alpha \sin\phi\right)^2 + \alpha^2 \sin^2\phi + \gamma\alpha\cos\phi \tag{5.11}$$

$$= -\frac{d^2}{d\phi^2} + \alpha\left(-2\sin\phi\,\frac{d}{d\phi} + (\gamma - 1)\cos\phi\right). \tag{5.12}$$


Figure 5.3.: Perturbation in the transformed Hamiltonian. After the nonunitary transformation there can be sites on which no hopping is possible. In this example we chose $\gamma = 3$ which leads to a vanishing hopping from site $k = 1$ to $k = 2$ and $k = -1$ to $k = -2$. All perturbative calculations starting from the states $k = -1, 0, 1$ will stay confined in that range in all orders of perturbation.

We can discuss this Hamiltonian again using perturbation theory in $\alpha$ around $H_0 = -d^2/d\phi^2$. We have the desired property that the perturbation is linear in $\alpha$. Bringing this operator back into the hopping formulation introduced above it reads

$$H_{\text{pert}} = \alpha G \left( \frac{\gamma - 1}{2} + K \right) + \alpha G^\dagger \left( \frac{\gamma - 1}{2} - K \right). \tag{5.13}$$

Using this expression, it is straightforward to evaluate the splitting between $k$ and $-k$. All that is required, is to evaluate the expression which is obtained by hopping all the way from $-k$ to $k$ by application of $\alpha G^\dagger(\gamma - 1 - 2K)/2$ since that is the only part of the perturbation containing $G^\dagger$. Multiplying up the the contribution, we obtain

$$\Delta = 2 \prod_{n=-k}^{k-1} \frac{\alpha}{2} \frac{2n + \gamma - 1}{k^2 - n^2} = \frac{\alpha^{2k}}{2^{2k-1}} \prod_{n=1}^{k} [\gamma^2 - (2n-1)^2] \Big/ \prod_{n=0}^{k} [k^2 - n^2]. \tag{5.14}$$

This is consistent with the expressions we obtained by the untransformed perturbation theory in the cases $k = 1$ and $k = 2$, but it is possible to evaluate it for arbitrary $k$. The splitting indeed vanishes in first order whenever $\gamma$ is an odd integer smaller or equal to $2k - 1$ which is consistent to the observations in numerics.

A careful examination of the hopping term shows that it no longer involves hopping in both directions for all sites. Usually Hermeticity would enforce that for every hopping in one direction there must be a hopping with the conjugate weight in the other direction, but since we performed a nonunitary transformation this does no longer hold. It turns out that in the case that $\gamma = 2n + 1$ with $n \in \mathbb{Z}$ the expression $G^\dagger(\gamma - 1 - 2K)$ vanishes when attempting to hop from $n$ to $n + 1$ and the expression $G(\gamma - 1 + 2K)$ vanishes when attempting to hop from $-n$ to $-n - 1$. This leaves us with the space between $k = n$ and $k = -n$, containing $\gamma$ states, that can not be left using perturbation theory. See Figure 5.3. This has the consequence that perturbative mixing from one of these states stays in that region. Thus, we obtained precisely what we wanted: a part of the Hilbertspace that is decoupled and finite sized allowing to compute analytic expressions.

## 5.2. Construction

The class of problems that allow this separation of the Hilbert space contains a lot more examples [15]. This class of Hamiltonians is constructed using the one dimensional differential operator representation of the $\mathfrak{sl}(2)$ Lie algebra given by

$$L_+ = 2J\xi - \xi^2\frac{d}{d\xi},\ L_z = -J + \xi\frac{d}{d\xi},\ L_- = \frac{d}{d\xi}. \tag{5.15}$$

One can easily check that $L_+, L_z, L_-$ obey the commutation relation of the $\mathfrak{sl}(2)$ given by

$$[L_z, L_+] = L_+,\ [L_z, L_-] = -L_-,\ [L_+, L_-] = 2L_z. \tag{5.16}$$

It is apparent that in the case that $2J \in \mathbb{N}$ these operators map $2J$th order polynomials of $\xi$ back onto the same class of functions. The space spanned by those polynomials is the part of the Hilbertspace that becomes decoupled. By building a Hamiltonian from these operators we can ensure that it will contain this sector.

To obtain a Hamiltonian $H$ which contains a second order kinetic term one can in general consider combining any term quadratic or linear in the operators

$$H = C_{ij}L_iL_j + c_kL_k. \tag{5.17}$$

Hamiltonians of this form have the disadvantage that they are differential operator representations of a particle in one dimension with nonflat space and with a position dependent mass, i.e., its general form reads

$$H = -P_4(\xi)\frac{d^2}{d\xi^2} + P_3(\xi)\frac{d}{d\xi} + P_2(\xi). \tag{5.18}$$

The term $P_4$ is accountable for the not constant mass and the term $P_3$ is accountable for the nonflat space. By construction the functions $P_j$ are polynomials of $j$-th order. Because we are interested in systems that can be described by the Schrödinger equation of a particle in one dimension these problems have to be cured. Using the Liouville transformation we can subsequently transform $P_4$ and $P_3$ into a deformation of the potential $P_2$.

At first, we can make a transformation flattening out the coefficient of the kinetic term. This transformation is given by

$$x(\xi) = \int d\xi \frac{1}{\sqrt{P_4(\xi)}}. \tag{5.19}$$

One has to invert the relation to obtain $\xi(x)$. By applying this transformation, $H$

changes:

$$H = -P_4(\xi(x))\frac{dx}{d\xi}\left[\left(\frac{d}{dx}\frac{dx}{d\xi}\right)\frac{d}{dx} + \frac{dx}{d\xi}\frac{d^2}{dx^2}\right] + P_3(\xi(x))\frac{dx}{d\xi}\frac{d}{dx} + P_2(\xi(x)) \tag{5.20}$$

$$= -\frac{d^2}{dx^2} - \sqrt{P_4(\xi(x))}\left[\left(\frac{d}{dx}\frac{1}{\sqrt{P_4(\xi(x))}}\right)\frac{d}{dx}\right] + \frac{P_3(\xi(x))}{\sqrt{P_4(\xi(x))}}\frac{d}{dx} + P_2(\xi(x)) \tag{5.21}$$

$$= -\frac{d^2}{dx^2} + \frac{P_4'(\xi(x))}{2\sqrt{P_4(\xi(x))}}\frac{d}{dx} + \frac{P_3(\xi(x))}{\sqrt{P_4(\xi(x))}}\frac{d}{dx} + P_2(\xi(x)). \tag{5.22}$$

Shortening the notation the Hamiltonian is given by

$$H = -\frac{d^2}{dx^2} + \underbrace{\frac{P_4'(x)/2 + P_3(x)}{\sqrt{P_4(x)}}}_{G(x)}\frac{d}{dx} + P_2(x). \tag{5.23}$$

In a next step, we have to gauge away the term linear in $d/dx$. It can be done by the nonunitary transformation $d/dx \mapsto d/dx + G(x)/2$ which can be expressed as $T = \exp[\int dx G(x)/2]$. While the transformation is nonunitary, it is a valid linear transformation leaving the spectrum of the problem invariant. We can apply the transformation

$$T^{-1}HT = -\left(\frac{d}{dx} + G(x)/2\right)^2 + G(x)\left(\frac{d}{dx} + G(x)/2\right) + P_2(x) \tag{5.24}$$

$$= -\frac{d^2}{dx^2} - \left(G'(x)/2 + G(x)\frac{d}{dx}\right) - G^2(x)/4 + G^2(x)/2 + G(x)\frac{d}{dx} + P_2(x) \tag{5.25}$$

$$= -\frac{d^2}{dx^2} + \underbrace{-G'(x)/2 + G^2(x)/4 + P_2(x)}_{V(x)} \tag{5.26}$$

and read off the expression for the potential $V(x)$. We obtained a regular Schrödinger equation in one dimension.

The space of problems that can be generated that way is rather versatile it includes the harmonic oscillator, the Morse potential, the Pöschl-Teller potential and certain exponential, polynomial and hyperbolic potentials. For a summary of interesting potentials see Tab. 5.1. The double sine potential is the special case generated by choosing

$$H = L_z^2 - L_-^2 + 2JL_z + 2aL_- + 2aL_+ \tag{5.27}$$

$$= (1-\xi^2)\frac{d^2}{d\xi^2} + 2\left[\xi + a(1-\xi^2)\right]\frac{d}{d\xi} + 4Ja\xi. \tag{5.28}$$

First, we have to determine $x(\xi)$ via

$$x(\xi) = \int d\xi \frac{1}{\sqrt{1-\xi^2}} = \arcsin(\xi). \tag{5.29}$$

| | | |
|---|---|---|
| Harmonic oscillator | $\omega^2 x^2$ | $C_{--} = 1,\ c_z = 2\omega$ |
| Morse potential | $A(e^{-2\gamma x} - 2e^{-\gamma x})$ | $C_{zz} = -\gamma^2,\ c_- = 2\sqrt{A}\gamma,$ $c_z = -2\gamma\left(\sqrt{A} + (J+\frac{1}{2})\gamma\right)$ |
| Pöschl-Teller potential | $-\frac{U}{\cosh^2\gamma x}$ | $C_{zz} = -\gamma^2,\ C_{--} = -\gamma^2$ $c_z = \gamma^2\left[(1+4U/\gamma^2)^{1/2} - -(2J+1)\right]$ |
| Periodic Potential | $a^2\cos^2 x +$ $+a(4J+1)\sin x$ | $C_{zz} = 1, C_{--} = -1, c_z = 2J$ $c_- = 2a, c_+ = 2a$ |
| Hyperbolic Potential | $a^2\sinh^2 x +$ $+a(4J+1)\cosh x$ | $C_{zz} = -1, C_{--} = 1, c_z = -2J$ $c_- = -2a, c_+ = -2a$ |

Table 5.1.: Some examples of potentials that can be realized using partial algebraization. Examples are taken from [15] but rescaled to fit to our choice of units. All not given coefficients taken to be 0.

Which is straightforward to invert to $\xi(x) = \sin(x)$. Substituting this in the Hamiltonian we obtain

$$H = -\frac{d^2}{dx^2} + 2a\cos(x)\frac{d}{dx} + 4Ja\sin(x). \tag{5.30}$$

In a final step, we perform the gauging and obtain

$$H = -\frac{d^2}{dx^2} + (4J+1)\,a\sin(x) + a^2\cos^2(x) \tag{5.31}$$

and indeed if we shift $x \mapsto x + \pi/2$ this model coincides with the double sine model discussed in Chapter 4.

To obtain analytic solutions from these considerations we can use that in the case that $J$ is integer or half integer there is a $2J+1$ dimensional representations of the operators $L_-$, $L_+$ and $L_z$. As an example we can consider the case $J = 1/2$. In that case the finite sized representation of the operators are given by

$$L_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad L_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad L_z = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}. \tag{5.32}$$

Thus, we can evaluate Eq. (5.27) in the two dimensional basis

$$H = \begin{bmatrix} -\frac{1}{4} & 2a \\ 2a & \frac{3}{4} \end{bmatrix} \quad \text{with eigenvalues} \quad E = \frac{1}{4} \pm \sqrt{\frac{1}{4} + 4a^2}. \tag{5.33}$$

## 5.3. Degeneracy

The method introduced above allows to solve for certain states in a model exactly, yet it does not allow to make conclusions about degeneracies. Klishevich and Plyushchay [16]

worked out a relation between supersymmetry and some of the models with partial algebraization given in Tab. 5.1 which implies degeneracies in their spectrum. They use an approach in which they postulate the form of a supercharge and work out restrictions on them by demanding that they produce a Hamiltonian of Schrödinger type. This produces an equation which is in their formulation similar to Eq. (5.52) which our approach produces as well eventually. We will tackle the problem by using a purely algebraic prove that is applicable not just to Schrödinger type problems but also certain problems formulated in terms of rotational momentum operators.

We will demonstrate that this property can be derived generally for an Hamiltonian of the form

$$H = PR + n[P, R] \tag{5.34}$$

with $n \in \mathbb{N}$ and

$$[P, [P, [P, R]]] = \gamma[P, R] \tag{5.35}$$

if an operator $\Pi$ exists such that

$$PR\Pi = \Pi RP \text{ and } RP\Pi = \Pi PR. \tag{5.36}$$

The last statement can equivalently be formulated as $\{[R, P], \Pi\} = 0$ and $[\{R, P\}, \Pi] = 0$. Given Eq. (5.34), Eq. (5.35), and Eq. (5.36) we will show that the operator

$$Q_n = \left[\prod_{k=1}^{n}(P^2 - k^2\gamma^2)\right] P\Pi \tag{5.37}$$

satisfies

$$[H_n, Q_n] = 0. \tag{5.38}$$

These conditions are satisfied by some relevant Hamiltonians and operators. For example the double sine potential that served as example many times before can be brought in this form by choosing

$$P = i\frac{d}{d\phi} + i\sin\phi, \quad R = P^\dagger, \quad \Pi : \phi \mapsto \phi + \pi. \tag{5.39}$$

This formulation of the double sine potential was already introduced in Chapter 4 when discussing degeneracy in the $\alpha \gg 1$ limit, note that for $\gamma = 0$ the charge $Q_n$ found here and the charge we found in the $\alpha \gg 1$ limit are equal.

Another relevant example is given by choosing rotational momentum operators

$$P = \boldsymbol{p} \cdot \boldsymbol{L} \text{ and } R = \boldsymbol{r} \cdot \boldsymbol{L} \tag{5.40}$$

where $\boldsymbol{L} = (L_x, L_y, L_z)$ and $\boldsymbol{p}, \boldsymbol{r} \in \mathbb{C}^3$. We check that the operators $P$ and $R$ satisfy Eq. (5.35) by using the relation

$$[P, R] = i(\boldsymbol{p} \times \boldsymbol{r}) \cdot \boldsymbol{L} \tag{5.41}$$

and consecutively by vector algebra

$$\begin{aligned}[P,[P,[P,R]]] &= -i(\boldsymbol{p}\times(\boldsymbol{p}\times(\boldsymbol{p}\times\boldsymbol{r})))\cdot\boldsymbol{L}\\ &= -i\left(\boldsymbol{p}\times((\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{p}-(\boldsymbol{p}\cdot\boldsymbol{p})\boldsymbol{r})\right)\cdot\boldsymbol{L}\\ &= i(\boldsymbol{p}\cdot\boldsymbol{p})(\boldsymbol{p}\times\boldsymbol{r})\cdot\boldsymbol{L} = (\boldsymbol{p}\cdot\boldsymbol{p})[P,R].\end{aligned} \tag{5.42}$$

The nature of the operator $\Pi$ is more intriguing since $\boldsymbol{r}$ and $\boldsymbol{p}$ are not necessarily real. The operator $\Pi$ may be composed from a rotation $O$ and the complex conjugation $K$ with the property

$$K^\dagger(\boldsymbol{v}\cdot\boldsymbol{L})K = \boldsymbol{v}^*\cdot\begin{bmatrix}L_x & -L_y & L_z\end{bmatrix}^T. \tag{5.43}$$

It turns out that it is always possible to construct a $\Pi$ as $OK$ such that

$$\Pi P\Pi^\dagger = R\frac{|\boldsymbol{p}|}{|\boldsymbol{r}|} \quad\text{and}\quad \Pi R\Pi^\dagger = P\frac{|\boldsymbol{r}|}{|\boldsymbol{p}|}. \tag{5.44}$$

From which obviously follows Eq. (5.36).

We are going to show that $[H_n, Q_n] = 0$ by induction over $n$. To get started we evaluate the base case

$$[H_0, Q_0] = [PR, P\Pi] = PRP\Pi - P\Pi PR = PRP\Pi - PRP\Pi = 0. \tag{5.45}$$

To close the induction we have to show that $[H_{n+1}, Q_{n+1}] = 0$ from $[H_n, Q_n] = 0$. To do so we evaluate

$$\begin{aligned}[H_{n+1}, Q_{n+1}] =& [H_n + [P,R], (P^2-(n+1)^2)Q_n]\\ =& (P^2-(n+1)^2)[H_n,Q_n] + [H_n, P^2-(n+1)^2]Q_n\\ &+ [[P,R],(P^2-(n+1)^2)Q_n].\end{aligned} \tag{5.46}$$

The first step is to use that in the expansion the expression $[H_n, Q_n]$ vanishes due to the induction assumption. So we are left with showing that

$$[H_{n+1}, Q_{n+1}] = [H_n, P^2]Q_n + [[P,R],(P^2-(n+1)^2)Q_n] = 0. \tag{5.47}$$

Since this derivation is somewhat lengthy and involves a lot of algebraic manipulations using Eq. (5.35) and Eq. (5.36) it is done in the Appendix.

Equipped with this theorem we can proceed to construct other models that have the property that was found in the double sine potential. The most straightforward approach is to construct a Witten model and enforcing Eq. (5.35). To obtain a Witten model we can set

$$P = -i\partial_x - i\sigma_z q(x), \quad R = P^\dagger \tag{5.48}$$

and check that the Hamiltonian

$$H_0 = PR = (-i\partial_x - i\sigma_z q(x))(-i\partial_x + i\sigma_z q(x)) = -\partial_x^2 + q^2(x) + q'(x)\sigma_z, \tag{5.49}$$

is a valid Witten model. The commutator $[P,R]$ that enters in the Hamiltonians $H_n$ with $n \neq 0$ is given by

$$[P,R] = 2\sigma_z q'(x) \tag{5.50}$$

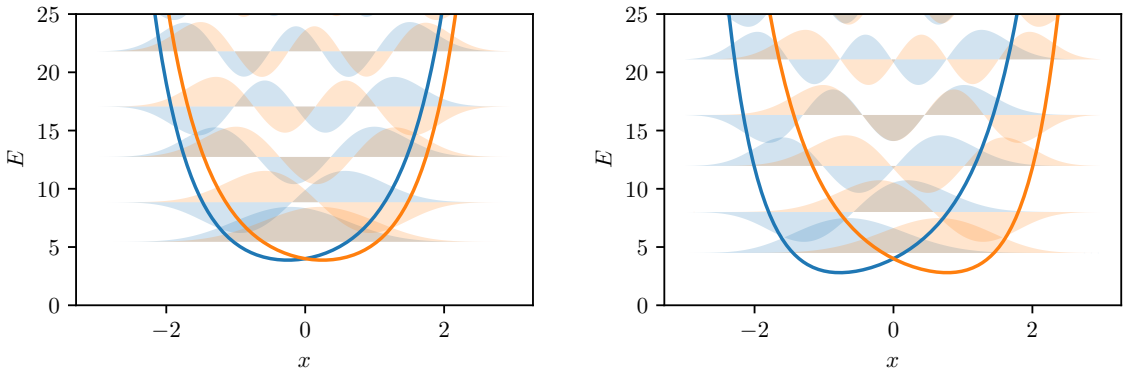


Figure 5.4.: On the right the potential and the supersymmetric partner potential obtained by setting the superpotential $q(x) = \cosh(x) + 1$. On the left the two potentials given by $q(x)^2 \pm 3q'(x)$.

and thus $H_n = -\partial_x^2 + q^2(x) + q'(x)(2n+1)\sigma_z$.

In order to obtain a model that is handled by the theorem above we have to make a choice for $q(x)$ such that it satisfies Eq. (5.35). It can be done by evaluating the triple commutator

$$[P,[P,[P,R]]] = [-i\partial_x, [-i\partial_x, 2\sigma_z q'(x)]] = -2\sigma_z q'''(x) \tag{5.51}$$

and equating it to Eq. (5.50)

$$2\sigma_z q'(x) = -2\gamma^2 \sigma_z q'''(x). \tag{5.52}$$

To find a valid choice for $q(x)$ we have to solve the differential equation $q'(x) = -\gamma^2 q'''(x)$. The same constraint that was found in [16]. The most general solution to this equation is

$$q(x) = \alpha \exp(\gamma x) + \beta \exp(-\gamma x) + c. \tag{5.53}$$

We want to look exemplary on the hyperbolic potentials obtained by setting $\alpha = (a+b)/2$ and $\beta = (a-b)/2$. First, let us look at systems with $b = 0$ they are of the form

$$q = a\cosh(\gamma x) + c, \quad V_n = (b\cosh(\gamma x) + c)^2 + (2n+1)\sigma_z \gamma b \sinh(\gamma x). \tag{5.54}$$

The supersymmetric system $H_0$ exhibits broken supersymmetry, since the superpotential can be bound from neither below nor above. Also, when considering $H_n$ with $n \neq 0$ there is no splitting of energies since the two pair potentials considered here are always related by inversion and are thus isospectral. See Figure 5.4. More interesting are problems with $a = 0$ which are of the form

$$q = b\sinh(\gamma x) + c, \quad V_n = (b\sinh(\gamma x) + c)^2 + (2n+1)\sigma_z \gamma b \cosh(\gamma x) \tag{5.55}$$

shown in figure Figure 5.5. This class of problems allows, in similarity to the double sine potential, to construct low lying non degenerate state while the higher excited

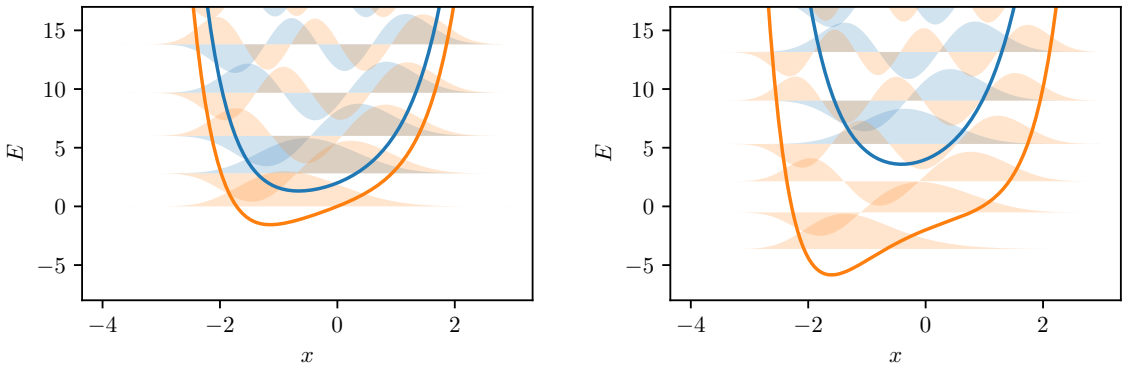


Figure 5.5.: On the right the potential and the supersymmetric partner potential obtained by setting the superpotential $q(x) = \sinh(x) + 1$. On the left the two potentials given by $q(x)^2 \pm 3q'(x)$.

states all have a partner state. It is not surprising that this model is also in the class of models that can be constructed using the $\mathfrak{sl}(2)$ algebra. The relevant coefficients can be found in Tab. 5.1.

# 6. Nonlinear Supersymmetry

In regular supersymmetry the Hamiltonian is formed as the square of the conserved supercharge this gets relaxed for nonlinear supersymmetric systems. Nonlinear supersymmetry only demands that $\{Q, Q^\dagger\}$ and $H$ can be related by some function, not neccessarily a linear one. Thus, nonlinear supersymmetry is a more general algebraic structure allowing to cover more systems, at the cost of weaker possible statements. But we will still be able to make conclusions about degeneracy properties. We have already discussed the double sine potential, which is an example of nonlinear supersymmetry.

In this chapter, we will construct a general class of periodic problems that have a spectrum that contains multiple non degenerate states while still having the property that almost all states come in degenerate pairs. This shows that the double sine potential is not a rare exception but that the class of problems is just comparable with the class of periodic supersymmetric models. In addition, we will solve for the energy of the states that have no degenerate partner. We will use the idea of nonlinear supersymmetry but without deriving the actual relation between $\{Q, Q^\dagger\}$ and $H$.

## 6.1. Intertwining Relation

We want to start with a general model in which we have an explicit fermionic and a bosonic sector that can easily be distinguished. We give a Hamiltonian for the fermionic part $H_f$ and a separate Hamiltonian for the bosonic part $H_b$. An approach which can be considered as a generalization of the Witten model in which we have the two pair potentials.

Instead, demanding that $H_b$ and $H_f$ are constructed in a particular way we will only demand that there is an operator $q$ such that the three operators satisfy the intertwining relation

$$H_f q = q H_b. \tag{6.1}$$

The intertwining relation can be understood as a weaker form of a linear basis transformation. If $q$ would be invertible, we could bring the equation in form of standard basis transformation

$$q^{-1} H_f q = H_b \tag{6.2}$$

which would imply complete isospectralility between $H_f$ and $H_b$. We want to consider the more general case in which $q$ may be non invertible because it leads to interesting properties in the model.

Given this relation we can cast both parts of the model in a single Hamiltonian

$$H = \begin{bmatrix} H_f & 0 \\ 0 & H_b \end{bmatrix} \tag{6.3}$$

and construct a charge $Q$ and an involution $K$ similar to $n = 1$ SUSY QM as

$$Q = \begin{bmatrix} 0 & q \\ q^\dagger & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{6.4}$$

From the intertwining relation we can conclude that $[H, Q] = 0$ and by the matrix structure we have $[H, K] = 0$ and $\{K, q\} = 0$. This is exactly the structure we used in the last chapter to prove the degeneracy in the double sine potential.

Using the intertwining relation, we can make conclusions about the spectrum of $H_f$ and $H_b$ that are similar to the conclusions we can make in a supersymmetric setup. Given an eigenstate $|\psi\rangle$ with energy $E$ of $H_b$ we can apply Eq. (6.1)

$$H_f q |\psi\rangle = q H_b |\psi\rangle = E q |\psi\rangle . \tag{6.5}$$

Therefore, $q|\psi\rangle$ is either vanishing, in that case we can make no statement about its relation to $H_b$, or it is not vanishing, in that case we can be sure that the state $q|\psi\rangle$ is an eigenstate of $H_f$ with the same energy. However, in contrast to supersymmetry we can not conclude that a state that gets annihilated by $q$ must have vanishing energy. In contrast, about states that are annihilated by $q$ we can explicitly make no statements at all, consider for example the trivial choice $q = 0$ annihilating all states. This choice allows conclusions about neither $H_f$ nor $H_b$.

## 6.2. Pair Potentials

In the following we will construct models that are similar to the pair potentials we discussed when studying the Witten model. Therefore, we use the ansatz

$$H_f = \left(\frac{d}{dx} + c\right)\left(\frac{d}{dx} - c\right), \quad H_b = \left(\frac{d}{dx} + g\right)\left(\frac{d}{dx} - g\right), \tag{6.6}$$

where $c$ and $g$ are functions. Note that in the case we let $g = -c$ we would obtain exactly the pair potentials from the Witten model as discussed earlier. In that case $q$ would be given by

$$q = \left(\frac{d}{dx} - c\right). \tag{6.7}$$

Instead, we want to study a more general model. Therefore, we assume a third order intertwining operator

$$q = \left(p + q\frac{d}{dx} + \frac{d^2}{dx^2}\right)\left(\frac{d}{dx} - c\right). \tag{6.8}$$

We have factored $d/dx - c$ in the operator to simplify the evaluation. Explicitly writing down the intertwining relation yields

$$\begin{aligned} H_b q &= \left(\frac{d}{dx}+\tilde{c}\right)\left(\frac{d}{dx}-\tilde{c}\right)\left(p+q\frac{d}{dx}+\frac{d^2}{dx^2}\right)\left(\frac{d}{dx}-c\right) = \\ &= \left(p+q\frac{d}{dx}+\frac{d^2}{dx^2}\right)\left(\frac{d}{dx}-c\right)\left(\frac{d}{dx}+c\right)\left(\frac{d}{dx}-c\right) = qH_f. \end{aligned} \tag{6.9}$$

We can cancel $d/dx - c$ and we obtain the operator equation

$$\begin{aligned} &\left(\frac{d}{dx}+g\right)\left(\frac{d}{dx}-g\right)\left(p+q\frac{d}{dx}+\frac{d^2}{dx^2}\right) = \\ &= \left(p+q\frac{d}{dx}+\frac{d^2}{dx^2}\right)\left(\frac{d}{dx}-c\right)\left(\frac{d}{dx}+c\right). \end{aligned} \tag{6.10}$$

To find a particular model that obeys this intertwining relation we will proceed by commuting all differentiation operators to the right hand side. This will produce an expression with different orders of differentiation operators together with coefficients that are functions of $c, p, q$, and $g$. Since the differentiation operators will not vanish we have to enforce that the coefficients vanish identically. This produces differential equations that when ordered are equivalent to

$$q'' = c''' + pc'' - 2cc'' - 2cpc' - 2c'^2 + 2qp', \tag{6.11}$$

$$p'' = 2c'' - 4cc' + 2pp' - 2q', \tag{6.12}$$

$$g^2 + g' = c^2 - c' + 2p'. \tag{6.13}$$

Eq. (6.12) can be integrated to

$$p' = 2c' - 2c^2 + p^2 - 2q + \chi \tag{6.14}$$

where $\chi$ is a constant of integration. We will use Greek letters to indicate that objects are constants and not functions. This puts us in the position to solve for $q$

$$q = -\frac{p'}{2} - c^2 + \frac{p^2}{2} + c' + \frac{\chi}{2}. \tag{6.15}$$

We can substitute this expression in Eq. (6.11) to obtain

$$0 = p''' - 4cpc' - 4p'^2 - 4c^2p' - 2p''p + 2p^2p' + 4c'p' + 2\chi p' + 2pc''. \tag{6.16}$$

At this point it becomes necessary to reformulate the problem. We rewrite $c$ and $g$ as a part $c_s$ that is symmetric under the exchange $c \leftrightarrow g$ and a part $c_a$ that is antisymmetric, i.e.,

$$c = c_s + c_a \quad \text{and} \quad g = c_s - c_a. \tag{6.17}$$

We will do the same for the function $p$ writing it as $p = p_s + p_a$ such that under the exchange $c \leftrightarrow g$ the transformed $p$ would read $p = p_s - p_a$.

With these substitutions Eq. (6.13) turns into

$$-4c_s c_a + 2c_s' = 2p_s' + 2p_a'. \tag{6.18}$$

Given an equation which contains a mixture of terms that are symmetric and antisymmetric under the operation $c \leftrightarrow g$, we can split the equation into a symmetric and an antisymmetric part, both must be satisfied independently. Symmetrizing and anti symmetrizing Eq. (6.18) yields

$$\text{the symmetric part } \; 2c_s' = 2p_s' \; \text{ and the antisymmetric part } \; -4c_s c_a = 2p_a'. \tag{6.19}$$

Integrating the symmetric equation allows us to express $c_s$ and $c_a$ via $p_s$ and $p_a$ as

$$c_s = p_s \quad \text{and} \quad c_a = \frac{-p_a'}{2p_s}. \tag{6.20}$$

In a next step, we substitute $c_s$ and $c_a$ in Eq. (6.16) this produces an expression which contains again a mixture of symmetric and antisymmetric terms. The symmetric part of that equation is given by

$$p_a' \left(4\chi p_s^3 - 2p_a p_s p_a'' + 2p_a p_a' p_s' - 2p_s p_a'^2 + 4p_a^2 p_s^3 + 2p_s^2 p_s'' - 4p_s^5\right) = p_a' \mathrm{eq}_a = 0 \tag{6.21}$$

where we have introduced the symbol $\mathrm{eq}_a$ as a shorthand for the expression in the brackets. The anti symmetric part turns out to be expressible using $\mathrm{eq}_a$ as

$$-2p_s' \mathrm{eq}_a + p_s \frac{d}{dx} \mathrm{eq}_a = 0. \tag{6.22}$$

Therefore, it is sufficient to find a solution to $\mathrm{eq}_a = 0$.

Since all expressions of $p_a$ in $\mathrm{eq}_a$ occur quadratic, we substitute $p_a = a^{1/2}$. This transforms the problem into the equation

$$2\chi p_s^3 - \frac{1}{2} p_s a'' + \frac{1}{2} a' p_s' + 2a p_s^3 + s^2 p_s'' - 2p_s^5 = 0 \tag{6.23}$$

containing only terms linear in $a$ and an inhomogeneous contribution. We can find a solution for $a$ by first solving the homogeneous equation

$$-\frac{1}{2} p_s a'' + \frac{1}{2} a' p_s' + 2a p_s^3 = 0 \implies a = \alpha \exp[-2S] + \beta \exp[2S] \tag{6.24}$$

where $S$ is the antiderivative of $p_s$. We have to add a solution to the particular problem and we find that it is solved by $S'^2 - \chi$ yielding as the complete solution

$$a = \alpha \exp[-2S] + \beta \exp[2S] + S'^2 - \chi. \tag{6.25}$$

We successfully constructed a system with a third order intertwining operator. We explore setting $S = x$ in Figure 6.1 and obtain pair potentials that are similar to those in the Witten model.

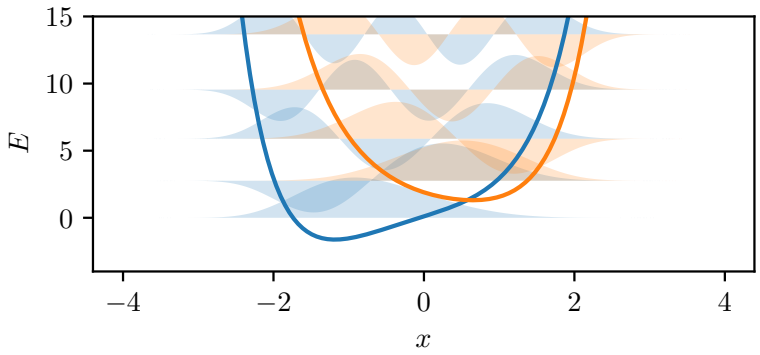


Figure 6.1.: The pair potential obtained by a third order intertwining operator as described in the text by setting $S = x$, $\alpha = 1$, $\beta = 1$, $\chi = -1$.

## 6.3. Periodic Systems

In a next step, we want to explore whether it is possible to construct periodic problems based on the approach introduced above. We are going to follow the same idea as in supersymmetry and replace the operation switching between the pair potentials $c \leftrightarrow g$ with the translation $x \mapsto x + \pi$. This makes the problem a lot more involved since we now require that the antisymmetric function $p_a$ includes a sign change but so far we have expressed it as a square root. To cure this we have to ensure that the radiant $a$ includes double roots such that we can change the branch at these points and are still left with an analytic function.

In order to construct such an $a$ we perform some rewriting. We rewrite $S = \log(u^2 + o)$ with $u$ an antisymmetric function. This restricts the functions $S$ to those that are bounded from below, yet in a periodic system this is without loss of generality. This choice has the advantage that we can express $p_a$ in the way

$$p_a = \sqrt{-\chi + \frac{\alpha}{(u^2+o)^2} + \beta(u^2+o)^2 + \frac{4u^2u'^2}{(u^2+o)^2}}. \tag{6.26}$$

Next, we must search for the minima of the radiant: they are assumed when $u = 0$. Therefore, we have to ensure that the radiant vanishing whenever $u = 0$. Evaluating the expression at $u = 0$ yields

$$p_a|_{u=0} = \sqrt{-\chi + \frac{\alpha}{o^2} + \beta o^2}. \tag{6.27}$$

Thus, the choice $\chi = \alpha/o^2 + \beta o^2$ will satisfy that constraint. Plugging that back into the expression of $p_a$ yields

$$p_a = \sqrt{u^2\left(-\alpha\frac{(u^2+2o)}{o^2(u^2+o)^2} + \beta(u^2+2o) + \frac{4u'^2}{(u^2+o)^2}\right)}. \tag{6.28}$$

It has the side effect that we can pull $u$ out of the root, this is equivalent to choosing a branch, since we are free to either choose $u$ or $-u$.

In a next step, we have to ensure that $c_a = p_a'/2p_s = p_a'(u^2+o)/4uu'$ does not get singular at points where $u$ vanishes. Therefore, we have to ensure that for $u \to 0$ also $p_a' \to 0$. We can evaluate

$$p_a' = 2u' \frac{-\alpha + \beta\left(o+u^2\right)^4 + 2\left(o+u^2\right)uu'' + 2\left(o-u^2\right)u'^2}{\frac{p_a}{u}(o+u^2)^3}. \tag{6.29}$$

Since the denominator stays finite for $u \to 0$ we have to ensure that the nominator becomes zero, this can be obtained by fixing

$$\alpha = \beta o^4 + 2o\, u'^2\big|_{u=0}. \tag{6.30}$$

This expression has two downsides. On the one hand we now require that $u$ crosses 0 always with the same slope in order for $\alpha$ to be well defined. On the other this choice of $\alpha$ leads to $p_a$ vanishing not just linearly for $u \to 0$ but quadratically. In order to do so we have to fix the nominator in Eq. (6.29) that it also vanishes quadratically. Therefore, we make the ansatz $u = \gamma x + \delta x^2 + \epsilon x^3 + \mathcal{O}(x)$ and expand $c_a$ in $x$. The leading term is given by the divergence

$$\sqrt{\frac{9\delta}{8\gamma x}}. \tag{6.31}$$

Thus we have to enforce that $\delta$ is vanishing, which implies $u''|_{u=0} = 0$. With this choice $c_a$ stays finite, yet it is not continuous. Since $c_a'$ enters the potential and we want to avoid potentials containing $\delta$-distributions we have to remove the discontinuity. We use the ansatz $u = \gamma x + \epsilon x^3$ and evaluate the limits

$$\lim_{x\to 0^+} c_a = \frac{o\sqrt{\beta\gamma^2 - \frac{\gamma^4 - 2\gamma o\epsilon}{2o^3}}}{\gamma} \quad \text{and} \quad \lim_{x\to 0^-} c_a = -\frac{o\sqrt{\beta\gamma^2 - \frac{\gamma^4 - 2\gamma o\epsilon}{2o^3}}}{\gamma}. \tag{6.32}$$

To avoid that they differ we can set

$$\beta = \left.\frac{u'^3 - 2ou'''}{2o^3u'}\right|_{u=0} \tag{6.33}$$

and obtain a continuous function. In addition, we can find for every periodic potential constructed in the way introduced above another isospectral system by considering $c \mapsto -c$.

With this construction we have shown that it is possible to create other analytical models which have the property that almost every eigenstate has a degenerate partner state except for a finite number of states. While this is easily possible for a single non degenerate state as in periodic supersymmetry models we showed here that this can also be done exactly for three states without partners. To show that this algorithm produces reasonable potentials we evaluate it exemplary with the setting $u = \sin x$ and $o = 1$ the resulting potentials can be found in Figure 6.2.

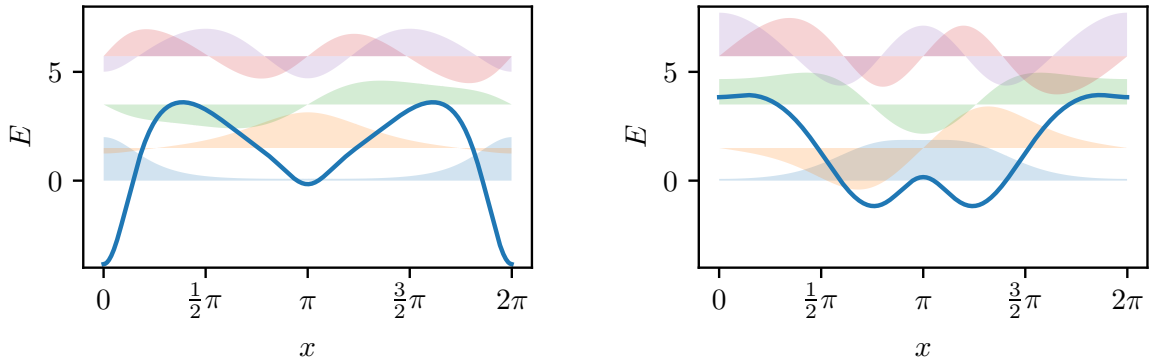


Figure 6.2.: The two potentials obtained by following the algorithm described in the text starting with $u = \sin x$ and $o = 1$. Both periodic potentials have the exact eigenvalues 0, 3/2, and 7/2 and all higher excited states are doubly degenerate.

## 6.4. Eigenenergies

In the last chapter we have seen that some systems that show degeneracies also obtain algebraic eigenvalues. We will show that this is also the case for the class of models we constructed above. We will proceed by solving for the kernel states of $q$. Since the intertwining relation ensures that the nullspace of $q$ will be mapped by $H$ back into the same space. This space is separated from the rest of the Hilbertspace and we can solve for eigenstates within that subspace. Because $q$ is a third order differential operator this space is three dimensional.

To find eigenstates in that subspace, we search for simultaneous solutions to the equations

$$-\left(\frac{d}{dx}+c\right)\left(\frac{d}{dx}-c\right)\psi = E\psi, \tag{6.34}$$

$$q\psi = \left(q + p\frac{d}{dx} + \frac{d^2}{dx^2}\right)\left(\frac{d}{dx}-c\right)\psi = 0. \tag{6.35}$$

We have constructed the Hamiltonian in Eq. (6.6) such that we can already identify the ground state of the system similar to the ground state in supersymmetric models and we constructed $q$ such that this ground state is also incorporated in the the null space of $q$.

We can make use of this construction when we are facing to solve this system of differential equation by employing the substitution

$$\psi = \left(\frac{d}{dx}-c\right)^{-1}\phi \tag{6.36}$$

followed by multiplication of Eq. (6.34) with $(d/dx - c)$ from the left to bring it in the

form

$$-\left(\frac{d}{dx}-c\right)\left(\frac{d}{dx}+c\right)\phi = E\phi, \quad \left(q+p\frac{d}{dx}+\frac{d^2}{dx^2}\right)\phi = 0. \tag{6.37}$$

Bringing $E$ to the other side and adding both equations gives us the first order equation

$$\left(c^2 - c' - E + q + p\frac{d}{dx}\right)\phi = 0. \tag{6.38}$$

If we substitute the expression for $q$ from Eq. (6.15) we obtain

$$\left(-E + \frac{p^2}{2} - \frac{p'}{2} + \frac{\chi}{2} + p\frac{d}{dx}\right)\phi = 0. \tag{6.39}$$

For this equation we can give the formal solution

$$\phi = \alpha \exp\left[\int dx\, \frac{p' + 2E - k - p^2}{2p}\right], \tag{6.40}$$

yet we must take care since this expression might be ill defined at points where $p = 0$. But fortunately we do not require the hole solution $\phi$, but can only solve for the energy $E$ locally. By plugging $\phi$ in the equation Eq. (6.37) and substituting the expression for $c$ from Eq. (6.14) we obtain the constraint

$$\frac{\phi}{4p^2}\left((\chi - 2E)^2 + 2pp'' + p^2\left(4q - 2p'\right) - p'^2 - p^4\right) = 0 \tag{6.41}$$

Solving for $E$ gives two solutions

$$E = \frac{1}{2}\left(\chi \pm \sqrt{-2pp'' + 2p^2p' + p'^2 - 4p^2q + p^4}\right) \tag{6.42}$$

which can be evaluated for any $x$ and mus be valid for all. By substituting in the solution for $p$ and $q$ we derived in terms of $u$ in the last section we obtain the explicit energy

$$E = -\frac{u'''}{u'} + \left.\frac{(3 \pm 2)u'^2}{2o}\right|_{u=0}. \tag{6.43}$$

In Figure 6.2 we study the model that is obtained by setting $u = \sin x$ and $o = 1$. With the derivation we did above it is straightforward to solve for the three low lying eigenvalues. The lowest is given by construction as 0, and the next two are obtained by Eq. (6.43) as 3/2 and 7/2.

# 7. Conclusion and Outlook

In this thesis, we discussed algebraic methods to obtain exact results about the eigenvalue spectrum of the one dimensional Schrödinger equation. In our discussion we focus on two properties: degeneracies and eigenvalues that are exactly solvable. We started our discussion by reviewing the mathematical results obtained in the domain of Sturm-Liouville theory, the mathematical description of the eigenvalue problem of the one dimensional Schrödinger equation. We defined the scope of the theory and reviewed theorems concerning the structure of the spectrum.

Next, we focussed on supersymmetry an algebraic structure that allows the construction and discussion of models that have essentially all states degenerate. We introduced both Witten models and supersymmetric periodic models, the two classes of models on which we focussed during the hole thesis. We summarized the important concept of shape invariance that can be used to discuss models that are exactly solvable.

The central problem we studied is the double sine potential. We found that for certain parameters almost all states have degenerate partner states. We showed that this property is present in all limits of the coefficients and used this insight to motivate the class of problems with partial algebraization. We were able prove that a certain subset of these models, as well as a certain class of operators build from rotational momentum operators, have for almost every state a degenerate partner state. Finally, we showed that the double sine potential is not unique in the space of periodic potentials that have an almost complete degenerate spectrum by constructing an algorithm that maps continuous function, with some additional requirements, onto periodic systems that have three exactly solvable states while all other states are doubly degenerate.

Future work could include models that have only a finite number of degenerate states since in this work we only covered models in which all except for a finite number of states are degenerate. Another interesting aspect for future work could be to include isospectral models that are obtained by evolution under the Kortweg-de-Vries equation. Possible application of the methods discussed in this thesis might be found in all fields in which the Sturm-Liouville equations plays a role. We think especially of fields where Sturm-Liouville problems with certain properties have to be designed such as quantum information, acoustics, or electrodynamics.

# A. Prove

To close the induction we have to show that

$$\begin{aligned}[H_{n+1}, Q_{n+1}] =& [H_n + [P, R], (P^2 - \gamma(n+1)^2)Q_n] \\ =& (P^2 - \gamma(n+1)^2)[H_n + [P, R], Q_n] + \\ &+ [H_n + [P, R], P^2 - \gamma(n+1)^2]Q_n\end{aligned} \tag{A.1}$$

is vanishing. We can use the induction assumption

$$[H_n, Q_n] = 0 \tag{A.2}$$

to eliminate one appearance of $H_n$ from Eq. (A.1) to show that is equal to

$$(P^2 - \gamma(n+1)^2)[[P, R], Q_n] + [H_n + [P, R], P^2]Q_n. \tag{A.3}$$

The tricky part in evaluating this expression is commuting the operator $[P, R]$ with $Q_n$ since we have no commutation relations that helps braking down the commutator. But we will use a shortcut and expand $Q_n$ another time

$$(P^2 - \gamma(n+1)^2)[[P, R], (P^2 - \gamma n^2)Q_{n-1}] + [H_n + [P, R], P^2](P^2 - \gamma n^2)Q_{n-1}. \tag{A.4}$$

Concentrating on the expression $[[P, R], (P^2 - \gamma n^2)Q_{n-1}]$ alone we can expand the commutator

$$[[P, R], (P^2 - \gamma n^2)Q_{n-1}] = (P^2 - \gamma n^2)[[P, R], Q_{n-1}] + [[P, R], (P^2 - \gamma n^2)]Q_{n-1}. \tag{A.5}$$

We want to bring the operator $Q_{n-1}$ out of this expression. We use the induction assumption with $n-1$ instead of $n$, i.e., $[H_{n-1}, Q_{n-1}] = 0$ which is equivalent to the statement that Eq. (A.4) with $n$ replaced with $n-1$ is vanishing. Therefore, we can equate

$$(P^2 - \gamma n^2)[[P, R], Q_{n-1}] = -[H_{n-1} + [P, R], P^2]Q_{n-1}. \tag{A.6}$$

and transform Eq. (A.5) to

$$[[P, R], (P^2 - \gamma n^2)Q_{n-1}] = -[H_{n-1} + [P, R], P^2]Q_{n-1} + [[P, R], (P^2 - \gamma n^2)]Q_{n-1}. \tag{A.7}$$

The consequence of using the induction assumption with both $n$ and $n-1$ forces us, in order to close the induction, to show that the assumptions hold for two base cases $n = 0$ and $n = 1$.

In the expression Eq. (A.7) we can factor $Q_{n-1}$ giving

$$[[P, R], (P^2 - \gamma n^2)Q_{n-1}] = \left(-[H_{n-1} + [P, R], P^2] + [[P, R], (P^2 - \gamma n^2)]\right) Q_{n-1}. \tag{A.8}$$

substituting that equation back into Eq. (A.4) yields

$$\begin{aligned}\big[(P^2-\gamma(n+1)^2)\big(-[H_{n-1}+[P,R],P^2]+[[P,R],(P^2-\gamma n^2)]\big)+\\ +[H_n+[P,R],P^2](P^2-\gamma n^2)\big]\,Q_{n-1}.\end{aligned} \tag{A.9}$$

Since $Q_{n-1}$ does not vanish in general we have to show that the expression in front of it vanishes. The expression does not contain $Q$ and therefore also not $\Pi$, from which we assumed that it may contain the time inversion operator which could possibly not commute with $\gamma$. Therefore, we can rescale $P \mapsto \gamma^{1/2}P$ and factor out $\gamma$ leaving us with the term

$$\begin{aligned}A=(P^2-(n+1)^2)\big(-[H_{n-1}+[P,R],P^2]+[[P,R],(P^2-n^2)]\big)+\\ +[H_n+[P,R],P^2](P^2-n^2)\end{aligned} \tag{A.10}$$

that has to vanish.

At this point we can brute force expand the expression and collect the different orders of $n$. Since this relation shall hold for all $n$, we must show that each operators that comes with a different order of $n$ is vanishing independently. The expansion is given by

$$\begin{aligned}A=&-(P^2-(n+1)^2)[H_0+n[P,R],P^2]+\\ &+(P^2-(n+1)^2)[[P,R],P^2]+[H_0+(n+1)[P,R],P^2](P^2-n^2)\end{aligned} \tag{A.11}$$

$$\begin{aligned}=&-P^2[H_0,P^2]+(2n+1)[H_0,P^2]-(n-1)P^2[[P,R],P^2]-\\ &-(n+1)[[P,R],P^2]+[H_0,P^2]P^2+(n+1)[[P,R],P^2]P^2.\end{aligned} \tag{A.12}$$

Since we are going to show that this expression vanishes for all $n$ we can shift $n \mapsto n-1$ in order to simplify the appearing terms

$$\begin{aligned}A=&-P^2[H_0,P^2]+(2n-1)[H_0,P^2]-(n-2)P^2[[P,R],P^2]\\ &-n[[P,R],P^2]+[H_0,P^2]P^2+n[[P,R],P^2]P^2\end{aligned} \tag{A.13}$$

$$\begin{aligned}=&n\big(2[H_0,P^2]-[[P,R],P^2]-P^2[[P,R],P^2]+[[P,R],P^2]P^2\big)\\ &-P^2[H_0,P^2]-[H_0,P^2]+2P^2[[P,R],P^2]+[H_0,P^2]P^2.\end{aligned} \tag{A.14}$$

In the next step, we show that the two operators (the one that comes with the coefficient $n$ and the one that does not) vanish. The operators are composed of terms which are of different order in $P$ and $R$. We only have the commutation relation $[P,[P,[P,R]]]=[P,R]$ to change the order of the operators and want to use it to show that they cancel each other. We start by bringing the 4th order operator into a form in which we can transform them into 6th order expressions

$$\begin{aligned}[H_0,P^2]&=PRPP-PPPR=P[R,P]P+PP[R,P]\\ &=-P[P,[P,[P,R]]]P-PP[P,[P,[P,R]]]\\ &=-PPPPPR+2PPPPRP-2PPRPPP+PRPPPP\end{aligned} \tag{A.15}$$

and

$$\begin{aligned}[[P,R],P^2] =&[[P,[P,[P,R]]]],P^2] \\ =&PPPPPR - 3PPPPRP + 2PPPRPP \\ &+ 2PPRPPP - 3PRPPPP + RPPPPP.\end{aligned} \tag{A.16}$$

All that is left to do is to verify that the arrangement of 6th order operators indeed exactly cancel each other.

We made use of the induction cases $n$ and $n-1$ to prove the $n+1$ case. Therefore, we have to prove as induction anchor not just that the assumption is true in the $n=0$ case but also in the $n=1$ case, i.e, we have to show that

$$\begin{aligned}[H_1,Q_1] &= [H_0+[P,R],(P^2-\gamma)P\Pi] \\ &= (H_0+[P,R])(P^2-\gamma)P\Pi - (P^2-\gamma)P\Pi(H_0+[P,R])\end{aligned} \tag{A.17}$$

vanishes. To reduce the equation we try to pull $\Pi$ to the right of the equation in a similar fashion we did with $Q_n$ earlier. We can use that $[H_0, P\Pi] = 0$ and $\{\Pi,[P,R]\} = 0$ to reduce the expression to

$$[H_1,Q_1] = (H_0+[P,R])(P^2-\gamma)P\Pi - (P^2-\gamma)(H_0P - P[P,R])\Pi. \tag{A.18}$$

Here we can factor $\Pi$ and are left with

$$H_0P^3 + [P,R]P^3 - \gamma[P,R]P - P^2H_0P + P^3[P,R] + \gamma P[P,R] \tag{A.19}$$

for which we can again substitute $\gamma[P,R] \mapsto [P,[P,[P,R]]]$ and use a full expansion of the expression to show that it vanishes.

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